19. If $T: H \to H$ is any bounded linear operator on the Hilbert space H, show that T^*T and TT^* are self-adjoint.
20. Show that if T is normal, then $\lambda I - T$ is normal for all complex numbers λ.
21. Let $L = {}_s\lim L_n$ and L_n be bounded linear operators on a Hilbert space H.
    (a) Show that if L_n is self-adjoint for all n, then L is self-adjoint. [*Hint*: Study Theorem 5.23.4.]
    (b) Show that if L_n is normal for all n, then L is normal. [*Hint*: Study Theorem 5.23.12.]
22. Do the relationships $A \leq B$ or $A < B$ define a partial ordering (refer to Appendix C) on the collection of all self-adjointed operators on a Hilbert space H? Is it ever a total ordering?
23. Let ξ be a random variable with range in a Hilbert space H and such that $E(\|\xi\|^2) < \infty$. Define the *covariance operator* A by
$$E((x,\xi)\overline{(y,\xi)}) = (Ax,y),$$
where $x, y \in H$. Show that A is a bounded, positive, self-adjoint linear operator.
24. Calculate $\|A\|$, where
    (a) $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$,
    (b) $A = \begin{bmatrix} a & d & f \\ d & b & e \\ f & e & c \end{bmatrix}$,

    when the entries are all real.
25. Let A be a self-adjoint operator on a Hilbert space H, and let
$$U = e^{iA} = \sum_{n=0}^{\infty} \frac{(iA)^n}{n!}.$$
    (a) Show that U is a unitary operator.
    (b) Show that $U^n = e^{inA}$ for every integer n.
26.[19] Let A be a bounded self-adjoint operator on a Hilbert space H and let
$$U_t = e^{itA} = \sum_{n=0}^{\infty} \frac{(itA)^n}{n!}.$$
    (a) Show that for each real number t, U_t is a unitary operator.
    (b) Show that $U_t U_s = U_{t+s}$.
    (c) Show that the mapping $t \to U_t$ is continuous, where the space of operators has the usual operator norm.
    (d) Discuss the meaning of the equality
$$\left.\frac{dU_t}{dt}\right|_{t=0} = \lim_{t\to 0} \frac{U_t - I}{t} = iA$$
    in terms of topologies on the space of operators. (See Section 8.)

[19] In this exercise, one constructs a continuous group U_t of unitary operators in terms of a given self-adjoint operator A. It is possible to turn this around, that is, given the continuous group U_t of unitary operators one can construct the "infinitesimal generator" A by means of $dU_t/dt = iAU_t$, and show that $U_t = e^{itA}$, see Dunford and Schwartz [1].

27. (Scattering operators.) Let A and B be two bounded linear self-adjoint operators on a separable Hilbert space H. Define U_t and V_t by $U_t = e^{-itA}$ and $V_t = e^{-itB}$. Assume that the limits
$$\lim_{t\to+\infty} V_t^* U_t x = x_- = \Omega_- x$$
and
$$\lim_{t\to-\infty} V_t^* U_t x = x_+ = \Omega_+ x$$
exist for each $x \in H$. Let $R_\pm$ denote the range of $\Omega_\pm$ and assume that $R_+ = R_-$. The *scattering operators* are defined by
$$S = \Omega_-^* \Omega_+$$
and
$$T = \Omega_+ \Omega_-^*.$$
(Compare with Jauch [1].) The object here is to show that S is a unitary operator.
    (a) Show that $\|\Omega_\pm x\| = \|x\|$ and that $\|\Omega_\pm^* x\| = \|x\|$ for all $x \in H$. (Explain why this fact alone does *not* show that S and T are unitary.)
    (b) Show that $\Omega_+^*\Omega_+ = I$ and $\Omega_-^*\Omega_- = I$.
    (c) Show that $\Omega_+\Omega_+^* = \Omega_-\Omega_-^* = P$, where P is the orthogonal projection onto $R_+ = R_-$.
    (d) Show that $SS^* = S^*S = I$.
    (e) Show that $TT^* = T^*T = P$.
28. Let $y = Kx$ be a positive self-adjoint operator on $L_2[a,b]$ that is given by $y(t) = \int_a^b k(t,\tau)x(\tau)\,d\tau$, where $k(t,\tau)$ is real-valued and continuous.
    (a) Show that $k(t,t) \geq 0$ for $a \leq t \leq b$.
    (b) Show that the converse need not be true. That is, construct a kernel $k(t,\tau)$ that satisfies $k(t,t) \geq 0$ for $a \leq t \leq b$ such that the corresponding operator K is self-adjoint but not positive.

24. COMPACT OPERATORS

The compact operators form another important class of linear operators. As we shall see below they are operators with finite- or, in a meaningful sense, almost finite-dimensional ranges. They are neither included in nor include the class of normal operators or, for that matter, the class of self-adjoint operators. The situation (for infinite-dimensional spaces) is illustrated in Figure 5.24.1. As we shall see in the next chapter, operators that are both normal and compact yield about the closest thing to a finite-dimensional structure that one can have on an infinite-dimensional space.

Since the elementary properties of compact operators are not dependent on the presence of an inner product, we shall abandon Hilbert space structure for this section and return to Banach spaces.[20]

[20] Compact operators can be defined on normed linear spaces, but many results require completeness, so we just assume it at the outset.

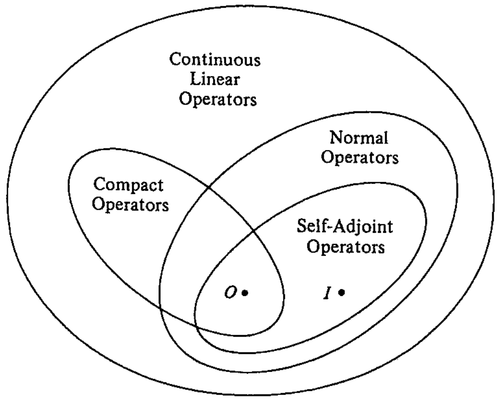

Figure 5.24.1.

5.24.1 DEFINITION. Let X and Y be two Banach spaces and let $L: X \to Y$ be a linear transformation. L is said to be *compact*[21] if $L(D)$ lies in a compact subset of Y, where $D = \{x \in X: \|x\| \leq 1\}$.

The following theorem states that a compact operator is continuous; however, there are plenty of continuous operators that are not compact. For example, consider the identity mapping I on any infinite-dimensional space.

5.24.2 THEOREM. *Let $L: X \to Y$ be a compact linear transformation of a Banach space X into a Banach space Y. Then L is continuous.*

Proof: The validity of this theorem follows almost immediately from Definition 5.24.1. Since $L(D)$ lies in a compact set, it is totally bounded, hence bounded (Lemma 3.16.3). That is, there exists an $M < \infty$ such that $\sup\{\|x\| : x \in L(D)\} \leq M$. It follows that $\|Lx\| \leq M\|x\|$ for all $x \in X$; therefore, L is bounded (that is, continuous). ∎

An important example of a compact operator is described in the next theorem.

5.24.3 THEOREM. *Let $L: X \to Y$ be a linear operator where the range $\mathscr{R}(L)$ is finite-dimensional. Then L is compact.*

Proof: By Theorem 5.10.7 we see that the unit ball in $\mathscr{R}(L)$ is compact. Therefore a ball of any radius is compact, from which it is easily seen that L is compact (Compare with Example 3, Section 10.) ∎

As we said, any compact operator comes close to having a finite-dimensional range.

[21] Some authors use the phrase "completely continuous" instead.

5.24.4 THEOREM. *Let $L: X \to Y$ be a compact linear transformation, where X and Y are Banach spaces. Then given any $\varepsilon > 0$, there exists a finite-dimensional subspace M of $\mathscr{R}(L)$ such that*

$$\inf\{\|Lx - m\| : m \in M\} \leq \varepsilon\|x\|.$$

In other words, the finite-dimensional subspace M comes within ε (in the above sense) of being the range of L. Presumably, the smaller ε is, the larger the dimension of M must be.

Proof: Let $\varepsilon > 0$ be given. Since $L(D)$ is contained in a compact set, where D is the closed unit ball in X, there is an ε-net in $\mathscr{R}(L) \cap L(D)$. Let M be the linear subspace of Y generated by this ε-net. It follows that M is finite dimensional. Moreover, $\operatorname{dist}(Lz, M) \leq \varepsilon$ for all $z \in D$. Then if x is any point in X it follows that

$$\inf\left\{\left\| L\frac{x}{\|x\|} - m\right\| : m \in M\right\} \leq \varepsilon$$

so

$$\inf\{\|Lx - m'\| : m' \in M\} \leq \varepsilon\|x\|,$$

where $m' = \|x\|m$. ∎

The following theorem presents a number of equivalent formulations for compactness of an operator.

5.24.5 THEOREM. *Let $L: X \to Y$ be a linear operator, where X and Y are Banach spaces. Then the following statements are equivalent:*

(a) *L is compact.*
(b) *If B is any bounded set in X, then $L(B)$ lies in a compact subset of Y.*
(c) *If B is any bounded set in X, then $L(B)$ lies in a sequentially compact subset of Y.*
(d) *If $\{x_n\}$ is any bounded sequence in X, then $\{Lx_n\}$ contains a convergent subsequence in Y.*
(e) *If B is any bounded set in X, then $L(B)$ is a totally bounded set in Y.*

Proof: Since the equivalence of (b), (c), (d), and (e) follows from the characterization of compactness in Section 3.17, we shall prove only that (a)$\Leftrightarrow$(b).

It is obvious that (b)$\Rightarrow$(a). Let us show that (a)$\Rightarrow$(b). Let B be any bounded set in X. Then there is a real number $k > 0$ such that

$$\|x - 0\| = \|x\| \leq k, \qquad \text{for all } x \in B.$$

Let $D = \{x \in X: \|x\| \leq 1\}$. Then $B \subset kD$, where

$$kD = \{kx \in X: \|x\| \leq 1\} = \{x \in X: \|x\| \leq k\}.$$

Since $L(B) \subset L(kD) = kL(D)$ and since $\overline{L(kD)} = \overline{kL(D)}$, it follows that $L(kD)$ lies in a compact set in Y. Hence, $L(B)$ lies in a compact set in Y. ∎

Let us now consider some examples of compact and noncompact operators.

EXAMPLE 1. Let $\phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi_n$ be elements of $L_2(I)$ and let

$$k(t,s) = \sum_{i=1}^{n} \alpha_i \phi_i(t)\psi_i(s),$$

where the α_i's are scalars. Define $y = Kx$ by

$$y(t) = \int_I k(t,s)x(s)\, ds.$$

Since every point y in $\mathscr{R}(K)$ is given by

$$y(t) = \sum_{i=1}^{n} \beta_i \phi_i(t),$$

where $\beta_i = \alpha_i \int_I \psi_i(s)x(s)\, ds$, we see that $\mathscr{R}(K)$ has dimension less than or equal to n. Hence K is compact. ∎

EXAMPLE 2. Every linear operator defined on a finite-dimensional normed linear space is compact. ∎

EXAMPLE 3. Consider the multiplication operator

$$F: x(t) \to f(t)x(t)$$

on $L_2(I)$, where f is a bounded measurable function. We have seen elsewhere (Example 2, Section 7) that F is a bounded linear operator and $\|F\| \le \|f\|_\infty$. We will now show that F is compact if and only if $f(t) = 0$ almost everywhere, that is, $\|f\|_\infty = \|F\| = 0$.

It is clear that $\|f\|_\infty = 0$ implies that F is the zero operator and, therefore, compact. Going the other way now, assume on the contrary that $\|f\|_\infty \ne 0$.

If f is continuous, then there are positive numbers α, β such that

$$|f(t)| \ge \alpha, \qquad t \in J, \tag{5.24.1}$$

where J is an interval of length β.

If $x \in L_2(I)$ and $x(t) = 0$ for $t \notin J$ one has

$$\begin{aligned}\|Fx\|_2{}^2 = \int_I |f(t)|^2 |x(t)|^2\, dt &\ge \int_J |f(t)|^2 |x(t)|^2\, dt \\ &\ge \alpha^2 \|x\|_2{}^2.\end{aligned}$$

Now choose an orthonormal sequence $\{x_n\}$ in $L_2(I)$ such that $x_n(t) = 0$ for $t \notin J$. (Why can we do this?) One then has $\|x_n - x_m\| = \sqrt{2}$ for $n \ne m$ and

$$\|Fx_n - Fx_m\| \ge \sqrt{2}\alpha, \qquad n \ne m$$

by the above. Hence $\{Fx_n\}$ cannot contain a convergent subsequence. Therefore, F is not compact.

If f is not continuous, then for some integer n the set

$$A_n = \left\{t: |f(t)| \ge \frac{1}{n}\right\}$$

has positive measure. For this n, let $\alpha = 1/n$ and let J be a subset of A_n with measure $\beta > 0$. One can then repeat the above argument and show that F is not compact. ∎

EXAMPLE 4. Let $H = l_2$, and let K denote the linear transformation of H into itself defined by

$$y_n = \alpha_n x_n, \qquad n = 1, 2, 3, \ldots,$$

where $y = Kx$, $x = \{x_1, x_2, x_3, \ldots\}$, $y = \{y_1, y_2, y_3, \ldots\}$, and the α_n's are scalars.

We claim that K is compact if and only if the α_n's satisfy the condition

$$\lim_{n\to\infty} |\alpha_n| = 0. \tag{5.24.2}$$

First assume that K is compact and that $|\alpha_n| \ge \varepsilon > 0$ for all n. Then, let

$$e_m = \{\delta_{1m}, \delta_{2m}, \delta_{3m}, \ldots\},$$

where δ_{ij} is the Kronecker function. Then

$$Ke_m = (\alpha_1\delta_{1m}, \alpha_2\delta_{2m}, \ldots) = (0, 0, \ldots, \alpha_m, 0, \ldots),$$

and for $m \ne n$

$$\|Ke_m - Ke_n\|^2 = |\alpha_m|^2 + |\alpha_n|^2 \ge 2\varepsilon^2.$$

Hence, $\{Ke_m\}$ does not contain any subsequence that is convergent, and we contradict the fact that K is compact.

If K is compact and (5.24.2) fails, then there is an $\varepsilon > 0$ and a subsequence $\{\alpha_{n_k}\}$ with $|\alpha_{n_k}| \ge \varepsilon$. By using the sequence $\{e_{n_k}\}$ and the above argument we arrive at a similar contradiction. Hence, K compact implies that (5.24.2) holds.

On the other hand, assume that (5.24.2) holds, and then let $A = K(D)$, where $D = \{x: \|x\| \le 1\}$. We shall now show that A has compact closure by applying Exercise 1, Section 3.17. Since $\|Kx\| \le \{\max |\alpha_n|\}\|x\|$, we see that A is bounded. If $y \in A$, then

$$\begin{aligned}\sum_{n=N}^{\infty} \|y_n\|^2 &= \sum_{n=N}^{\infty} \|\alpha_n x_n\|^2 \\ &\le \max_{N \le n} |\alpha_n|^2 \left\{\sum_{n=1}^{\infty} \|x_n\|^2\right\} \\ &\le \max_{N \le n} |\alpha_n|^2 \to 0, \text{ as } N \to \infty.\end{aligned}$$

Hence, $\sum_{n=N}^{\infty} \|y_n\|^2 \to 0$ uniformly as $N \to \infty$, so K is compact. ∎

6.9.13 THEOREM. *A bounded linear operator T that is the weighted sum of projections is normal.*

We leave the proof of this theorem to the reader.

This is an important result for it shows that the only operators that we can hope to express as weighted sums of projections are normal ones.

The bulk of the remainder of this chapter will be concerned with compact operators. Let us see when weighted sums of projections are compact.

6.9.14 THEOREM. *A weighted sum of projections is compact if* (i) *for every nonzero λ_n the range of P_n, $\mathscr{R}(P_n)$, is finite dimensional and (ii) for every real number $\alpha > 0$ the number of λ_n's with $|\lambda_n| \geq \alpha$ is finite.*

Proof: Suppose that T is a weighted sum of projections satisfying (i) and (ii). We must show that T is compact. Recall that a compact transformation maps bounded sets into compact sets. Let T_N be

$$T_N = \sum_{n=1}^{N} \lambda_n P_n,$$

where $\{\lambda_1,\ldots,\lambda_N\}$ are all the λ's such that $|\lambda_n| \geq \varepsilon$, where $\varepsilon > 0$. By our hypotheses N is finite and the range of T_N is finite dimensional. Moreover, since

$$\|(T - T_N)x\|^2 = \left\| \sum_{n=N+1}^{\infty} \lambda_n P_n x \right\|^2 \leq \sup_{n \geq N+1} |\lambda_n|^2 \sum_n \|P_n x\|^2 \leq \varepsilon^2 \|x\|^2,$$

one has $\|T - T_N\| \leq \varepsilon$. It follows now from Theorems 5.24.3 and 5.24.8 that T is compact. ∎

The converse of this theorem is also true. That is, if a weighted sum of projections is compact, then (i) and (ii) follow. We will not prove this here because we plan to consider a more general case in a later section.

The rather significant result that we will obtain shortly is that *every* compact normal operator is a weighted sum of projections.

EXERCISES

1. Let $T = \sum_n \lambda_n P_n$ be a continuous weighted sum of projections on a Hilbert space H. Show that there exists an orthonormal basis of eigenvectors $\{x_1,x_2,\ldots\}$ for H. Let $\{\mu_1,\mu_2,\ldots\}$ be the corresponding eigenvalues.
   (a) How are the μ_i's related to the λ_n's?
   (b) Show that
   $$Tx = \sum_n \mu_n(x,x_n)x_n$$
   for all $x \in H$.
   (c) What happens if T is not continuous?

2. Consider the operator
   $$\Phi: (x_1,x_2,\ldots) \to (\phi(1)x_1, \phi(2)x_2,\ldots)$$
   on $l_2(0,\infty)$.
   (a) Show that Φ is a weighted sum of projections.
   (b) What is the spectrum of Φ?
   (c) Assume that $\phi(n) \neq 0$ for all n. Show that Φ^{-1} exists and that Φ^{-1} is a weighted sum of projections. What is the spectrum of Φ^{-1}?
   (d) Assume further that $|\phi(n)| \to \infty$ as $n \to \infty$. Show that Φ^{-1} is compact.
   (e) Assume that $\phi(n) \to \lambda_0$ as $n \to \infty$, where λ_0 is finite. Show that $(\Phi - \lambda_0 I)$ is compact.
3. (Continuation of Exercise 2.) Let $L = T + \Phi = S_r + S_i + \Phi$. (See Exercises 13 and 14 of Section 6.) Assume that $|\phi(n)| \to +\infty$ as $n \to \infty$. Show that L^{-1} exists and is compact. [*Hint*: Use Exercise 10, Section 7 with $S = \Phi + \lambda I$, for an appropriate choice of λ.]

10. SPECTRAL PROPERTIES OF COMPACT, NORMAL, AND SELF-ADJOINT OPERATORS

In this section we first investigate the spectral properties of compact operators. Then we investigate the spectral properties of self-adjoint and normal operators. In the next section we will combine the results of this and the previous section to get the Spectral Theorem.

A. Compact Operators

The following theorems state the spectral properties of compact linear operators which we will need later.

6.10.1 THEOREM. *Let T be a compact linear transformation of a Hilbert space H into itself and let $\lambda \neq 0$. Then the null space $\mathscr{N}(\lambda I - T)$ is finite dimensional.*

Proof: The compact operator T maps $\mathscr{N}(\lambda I - T)$ into $\mathscr{N}(\lambda I - T)$. Moreover, the restriction of T to $\mathscr{N}(\lambda I - T)$ is λI. The restriction of a compact operator is a compact operator; therefore, λI is compact. It follows from Theorem 5.10.7 (or Exercise 11, Section 5.24) that $\mathscr{N}(\lambda I - T)$ is finite dimensional. ∎

6.10.2 THEOREM. *Let T be a compact linear transformation of a Hilbert space H into itself and let $\lambda \neq 0$. Then λ is either an eigenvalue of T or λ is in the resolvent set $\rho(T)$. [That is $\lambda \neq 0$ is never in the continuous spectrum $C\sigma(T)$ or the residual spectrum $R\sigma(T)$.]*

Proof[5]: Choose λ with $\lambda \neq 0$. Suppose $\lambda \in \sigma(T)$. First let us show that λ cannot be in the continuous spectrum. We shall do this by assuming that $(\lambda I - T)$ is

[5] This proof is long and technical and the reader may wish to skip it on his first reading.

one-to-one and then show that $\lambda I - T$ is bounded below, that is, there exists a constant $m > 0$ such that $\|(\lambda I - T)x\| \geq m\|x\|$ for all x. [This shows that any time $(\lambda I - T)$ has an inverse, this inverse is continuous, and the scalar λ cannot be in the continuous spectrum.]

We argue by contradiction. Suppose there is a sequence of unit vectors $\{x_n\}$ such that $\|\lambda x_n - Tx_n\| \to 0$ as $n \to \infty$. Since T is compact, $\{Tx_n\}$ contains a convergent subsequence, which we shall also denote by $\{Tx_n\}$. Let $z = \lim_{n\to\infty} Tx_n$.

Since

$$z - \lambda x_n = (z - Tx_n) + (Tx_n - \lambda x_n)$$

we have

$$\|z - \lambda x_n\| \leq \|z - Tx_n\| + \|Tx_n - \lambda x_n\|.$$

But both sequences on the right converge to zero. Hence

$$z = \lim_{n\to\infty} \lambda x_n,$$

or, using the fact that $\lambda \neq 0$, one has

$$\frac{1}{\lambda} z = \lim_{n\to\infty} x_n.$$

Since, $\|x_n\| = 1$ we have $\|z\| = |\lambda|$, thus $z \neq 0$. Since T is continuous one has

$$T\left(\frac{1}{\lambda} z\right) = \lim_{n\to\infty} T(x_n) = z.$$

In other words, z is an eigenvector of T. But this is a contradiction, for we have assumed $(\lambda I - T)$ is one-to-one. Hence we have shown that there does exist a $m > 0$ such that $\|(\lambda I - T)x\| \geq m\|x\|$ for all x, and $(\lambda I - T)^{-1}$ must be continuous. [*Note*: $\lambda \neq 0$ was important.]

Next let us show that λ is not in the residual spectrum of T. Recall that λ is in the residual spectrum of T if $(\lambda I - T)$ is one-to-one and the range of $(\lambda I - T)$ is not dense in H. We will again argue by contradiction. We will suppose that $(\lambda I - T)$ is one-to-one and $\overline{\mathscr{R}(\lambda I - T)} \neq H$. Let $X_0 = H$, $X_1 = (\lambda I - T)X_0$, $X_2 = (\lambda I - T)X_1$, and $X_{n+1} = (\lambda I - T)X_n$. It can be seen that $X_0 \supset X_1 \supset X_2 \supset X_3 \supset \cdots$. The rest of this proof depends on the fact that $X_1 \neq X_0$ implies that X_{n+1} is a proper closed linear subspace of X_n for all n. For the moment let us assume that this has been shown. It follows, then, that there is an $x \in X_0$ such that $\|x_0\| = 1$ and $x_0 \perp X_1$, by Corollary 5.14.5. Furthermore, there is an $x_1 \in X_1$ such that $\|x_1\| = 1$ and $x_1 \perp X_2$. In fact, there is an $x_n \in X_n$ such that $\|x_n\| = 1$ and $x_n \perp X_{n+1}$ for all n. It can be seen that $\{x_n\}$ is an orthonormal sequence. Let $n > m$, then

$$\frac{1}{\lambda}(Tx_m - Tx_n) = x_m + \left\{-x_n - \left[\frac{(\lambda I - T)x_m - (\lambda I - T)x_n}{\lambda}\right]\right\}.$$

But the term

$$\left\{-x_n - \left[\frac{(\lambda I - T)x_m - (\lambda I - T)x_n}{\lambda}\right]\right\}$$

is a point in X_{m+1}, call it $-x$; therefore,

$$\frac{1}{\lambda}(Tx_m - Tx_n) = x_m - x.$$

Since $\|x_m\| = 1$ and $x_m \perp X_{m+1}$, one has

$$\|Tx_m - Tx_n\| \geq |\lambda|,$$

which shows that the sequence $\{Tx_n\}$ cannot contain a convergent subsequence. This contradicts the assumption that T is compact. Hence, $\overline{\mathscr{R}(\lambda I - T)} = H$ and λ is not in the residual spectrum of T.

We are not finished yet with the proof. We still have to show $X_1 \neq X_0$ implies that X_{n+1} is a proper closed linear subspace of X_n for all n. First, let us show that $\mathscr{R}(\lambda I - T)$ is closed for all $\lambda \neq 0$.

6.10.3 LEMMA. *The range of $(\lambda I - T)$ is a closed linear subspace of H for all $\lambda \neq 0$.*

Proof: Let $\{y_n\}$ be any convergent sequence in $\mathscr{R}(\lambda I - T)$, and let $y_0 = \lim_{n\to\infty} y_n$. We want to show that $y_0 \in \mathscr{R}(\lambda I - T)$. Since $\{y_n\} \in \mathscr{R}(\lambda I - T)$, there is at least one sequence $\{x_n\}$ in H such that $(\lambda I - T)x_n = y_n$ for all n. Let us show that the sequence $\{x_n\}$ is bounded. Since $\lambda I - T$ is continuous, its null space $\mathscr{N}(\lambda I - T)$ is closed. Then $H = \mathscr{N}(\lambda I - T) + \mathscr{N}(\lambda I - T)^{\perp}$. With no loss in generality we can assume that $\{x_n\}$ is in $\mathscr{N}(\lambda I - T)^{\perp}$. (Why?) Now $(\lambda I - T)$ restricted to the closed subspace $\mathscr{N}(\lambda I - T)^{\perp}$ is one-to-one. So, repeating the argument used to prove the first part of Theorem 6.10.2, we know that there exists a constant $m > 0$ such that $\|(\lambda I - T)x\| \geq m\|x\|$ for all $x \in \mathscr{N}(\lambda I - T)^{\perp}$. Since $\{y_n\}$ is convergent, there is a bound $M > 0$ such that $\|y_n\| \leq M$ for all n. Then $M \geq \|(\lambda I - T)x_n\| \geq m\|x_n\|$ or $\|x_n\| \leq M/m$ for all n, showing that $\{x_n\}$ is bounded. Since T is compact, $\{x_n\}$ contains a subsequence, which we denote by $\{x_n\}$, such that $\{Tx_n\}$ is convergent. Then

$$\lambda x_n = y_n + Tx_n. \tag{6.10.1}$$

Since $\{y_n\}$ converges to y_0, both sequences on the right of (6.10.1) are convergent and $\lambda \neq 0$, so $\{x_n\}$ is convergent. Let $x_0 = \lim_{n\to\infty} x_n$. Since $(\lambda I - T)$ is continuous one has

$$(\lambda I - T)\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} (\lambda I - T)x_n$$

or

$$(\lambda I - T)x_0 = y_0.$$

Thus $y_0 \in \mathscr{R}(\lambda I - T)$ and $\mathscr{R}(\lambda I - T)$ is closed. ∎

The above lemma shows that the space X_1 constructed in Theorem 6.10.2 is closed. A slight variation on it shows that X_n is closed for all n. Thus we do not have to distinguish between $\bar{X}_n$ and X_n.

We are now ready to finish the proof of Theorem 6.10.2. We want to show that X_{n+1} is a proper closed linear subspace of X_n for all n. We argue by induction. By our hypotheses, X_1 is a proper closed linear subspace of X_0. Assume that X_k is a proper closed linear subspace of X_{k-1} for $1 \le k \le n$ and we will now show that this implies that X_{n+1} is a proper closed linear subspace of X_n. In any event, we have that $X_{n+1} \subset X_n$. So if X_{n+1} is not a proper closed linear subspace of X_n, we have $X_{n+1} = X_n$. That is, $(\lambda I - T)X_n = X_n$. Since $(\lambda I - T)$ is one-to-one, we have $X_n = (\lambda I - T)^{-1}X_n = X_{n-1}$ which is a contradiction. Therefore, if $X_1 \ne X_0$, then X_{n+1} is a proper linear subspace of X_n. This completes the proof of Theorem 6.10.2. ▮

The next theorem shows that $\lambda = 0$ is the only possible point of accumulation for the spectrum of a compact operator.

6.10.4 THEOREM. *Let T be a compact linear transformation of a Hilbert space into itself, and let $\alpha > 0$. Then the number of eigenvalues λ with $|\lambda| \ge \alpha$ is finite.*

Proof: We argue by contradiction. Suppose that there is an $\alpha_0 > 0$ such that the number of eigenvalues λ with $|\lambda| \ge \alpha_0$ is infinite. It follows (Why?) that the spectrum of T must contain at least one nonzero point of accumulation, call it λ_0. So there must be a sequence $\{\lambda_n\}$ of eigenvalues such that $\lim_{n\to\infty} \lambda_n = \lambda_0$. Let x_n be an eigenvector associated with λ_n, $n = 1, 2, \ldots$. The set $\{x_1, x_2, \ldots\}$ is linearly independent (see Exercise 3, Section 5). Let X_n be the finite dimensional and, therefore, closed linear subspace spanned by $\{x_1, x_2, \ldots, x_n\}$. We know from the Riesz Theorem (Theorem 5.5.4) that there is a sequence $\{y_n\}$ with $y_n \in X_n$, $\|y_n\| = 1$, and $\operatorname{dist}(y_n, X_{n-1}) \ge \frac{1}{2}$ $(n = 2,3,\ldots)$. If $n > m$, then

$$\frac{1}{\lambda_n} Ty_n - \frac{1}{\lambda_m} Ty_m = y_n + \left(-y_m - \frac{\lambda_n y_n - Ty_n}{\lambda_n} + \frac{\lambda_m y_m - Ty_m}{\lambda_m}\right)$$

$$= y_n - z,$$

where $z \in X_{n-1}$. (Why?) Therefore,

$$\left\| \frac{1}{\lambda_n} Ty_n - \frac{1}{\lambda_m} Ty_m \right\| \ge \operatorname{dist}(y_n, X_{n-1}) \ge \tfrac{1}{2}.$$

But we can use the above inequality together with $\lim_{n\to\infty} \lambda_n = \lambda_0 \ne 0$ to show that the sequence $\{Ty_n\}$ does not contain a convergent subsequence. This contradicts the fact that T is compact. Hence, the assumption about α_0 leads to a contradiction. ▮

The next corollary should be obvious.

6.10.5 COROLLARY. *Let T be a compact operator on a Hilbert space H. Then the spectrum of T is (at most) countably infinite and $\lambda = 0$ is the only possible point of accumulation.*

As far as the point $\lambda = 0$ is concerned, we cannot say too much. If T is compact, $\lambda = 0$ can be in the resolvent set or any part of the spectrum. However, if $\lambda = 0$ is in the resolvent set, H must be finite dimensional.

B. Normal and Self-Adjoint Operators

Now let us consider operators that are normal but not necessarily compact. Recall that every self-adjoint operator is normal; therefore, anything that is said about the class of normal operators applies also to self-adjoint operators.

6.10.6 THEOREM. *Let T be a normal transformation of a Hilbert space H into itself. If $x \in H$ is an eigenvector of T associated with an eigenvalue λ, then x is an eigenvector of T^*, the adjoint of T, associated with an eigenvalue $\bar{\lambda}$. Furthermore,*

$$\mathcal{N}(\lambda I - T) = \mathcal{N}(\bar{\lambda} I - T^*).$$

Proof: From Theorem 5.23.10 we know that T is normal if and only if $\|Tx\| = \|T^*x\|$ for all x. Moreover, if T is normal, then $\lambda I - T$ is normal. Hence, $\|(\lambda I - T)x\| = 0$ if and only if $\|(\bar{\lambda} I - T^*)x\| = 0$. ▮

6.10.7 THEOREM. *Let T be a normal operator mapping a Hilbert space H into itself. Then the null spaces $\mathcal{N}(\lambda I - T)$ and $\mathcal{N}(\mu I - T)$ are orthogonal to one another whenever $\lambda \ne \mu$.*

Proof: Let $x \in \mathcal{N}(\lambda I - T)$ and $y \in \mathcal{N}(\mu I - T)$. We want to show that $(x,y) = 0$. By using the last theorem and the fact that $(Tx,y) = (x,T^*y)$ we get $(\lambda x,y) = (x,\bar{\mu}y)$ or $(\lambda - \mu)(x,y) = 0$. Hence $(x,y) = 0$. ▮

Recall (Corollary 5.22.5) that a closed linear subspace M reduces a bounded linear operator T if and only if M is invariant under T and T^*. We can say more when T is normal.

6.10.8 THEOREM. *Let T be a normal transformation of a Hilbert space H into itself. Then for each complex number λ the closed linear subspace $\mathcal{N}(\lambda I - T)$ reduces T.*

Proof: Let $M = \mathcal{N}(\lambda I - T)$. Since $(\lambda I - T)$ is continuous, there is no question about M being closed. We have to show that $T(M) \subset M$ and $T(M^\perp) \subset M^\perp$. If λ is not an eigenvalue of T, then $M = \{0\}$ and $M^\perp = H$. In this case, then, the theorem is clearly true. Assume that λ is an eigenvalue of T. Since M is the eigenmanifold associated with λ, we immediately have that $T(M) \subset M$. Let $x \in M$ and $y \in M^\perp$, then $(x, Ty) = (T^*x, y)$. Theorem 6.10.6 assures us that $T^*(M) \subseteq M$, hence we get $(x,Ty) = 0$, for all $x \in M$ and $y \in M^\perp$. Continuing further, this shows that $T(M^\perp) \subset M^\perp$. ▮

6.10.9 COROLLARY. *If $\{M_n\}$ is a family of eigenmanifolds of a normal operator T, then $M = M_1 + M_2 + M_3 + \cdots$ reduces T.*

Proof: From Theorem 6.10.7 we know that the M_n's are pairwise orthogonal The rest of the proof should be obvious. ▮

6.10.10 THEOREM. *The residual spectrum of a normal operator is empty.*

Proof: Let T be a normal operator mapping a Hilbert space H into itself. We have to show that if $(\lambda I - T)$ is one-to-one, then the range $\mathscr{R}(\lambda I - T)$ is dense in H. Let y be a point in H that is orthogonal to $\mathscr{R}(\lambda I - T)$. That is,

$$(\lambda x - Tx, y) = 0 \qquad \text{for all } x \text{ in } H.$$

Since $(x, \bar{\lambda} y - T^* y) = 0$ for all x in H, it follows that $(\bar{\lambda} I - T^*)y = 0$, that is $y \in \mathscr{N}(\bar{\lambda} I - T^*)$. It now follows from Theorem 6.10.6 that $y = 0$. Therefore, since $R(\lambda I - T)^{\perp} = \{0\}$ we note that $R(\lambda I - T)$ is dense in H, see Theorem 5.15.4(c). ▮

Needless to say, it also follows that the residual spectrum of a self-adjoint operator is empty.

6.10.11 COROLLARY. *A complex number λ is in the spectrum of a normal operator T if and only if there exists a sequence $\{x_n\}$, $\|x_n\| = 1$ for all n, such that $\|(\lambda I - T)x_n\| \to 0$ as $n \to \infty$. In other words, the operator $(\lambda I - T)$ is not bounded below.*

The proof of this corollary is left to the reader as an easy but not completely trivial exercise. (Also, see Exercise 1, Section 6.5.)

As anyone familiar with the theory of Hermitian matrices would suspect, the spectrum of a self-adjoint operator is confined to the real line.

6.10.12 THEOREM. *The spectrum of a self-adjoint operator T is a subset of the real interval $[-\|T\|, \|T\|]$.*

Proof: We can use Corollary 6.10.11. Let us show that if λ is not real, then there exists a constant $m > 0$ such that $\|(\lambda I - T)x\| \geq m\|x\|$ for all x. It will follow from Corollary 6.10.11 that λ is in the resolvent set of T.

Assume that $\lambda = \rho + i\sigma$, where $\sigma \neq 0$. Then a simple calculation gives

$$\begin{aligned}\|(\lambda I - T)x\|^2 &= (\lambda x - Tx, \lambda x - Tx)\\ &= (\rho x - Tx, \rho x - Tx) + (i\sigma x, i\sigma x)\\ &\geq |\sigma|^2 \|x\|^2.\end{aligned}$$

Hence $\lambda I - T$ is bounded below and λ is in the resolvent set $\rho(T)$. Therefore the spectrum of T is real. It follows now from Theorem 6.7.4 that $\sigma(T)$ lies in the interval $[-\|T\|, \|T\|]$. ▮

C. Compact Self-Adjoint Operators

We turn now to a statement about the existence of eigenvalues for compact self-adjoint operators. Before giving this, though, let us recall that the norm of a self-adjoint operator T is given by

$$\|T\| = \sup\{|(Tx,x)| : \|x\| = 1\}. \tag{6.10.2}$$

(See Theorem 5.23.8.)

6.10.13 THEOREM. *Let T be a compact, self-adjoint operator on a nontrivial Hilbert space H. Then T has an eigenvalue λ with $|\lambda| = \|T\|$.*

Proof: It follows from (6.10.2) that there is a sequence $\{x_n\}$ in H with $\|x_n\| = 1$ and $|(Tx_n, x_n)| \to \|T\|$. Since T is compact we can find a subsequence of $\{Tx_n\}$ that converges in H; furthermore, since the sequence of complex numbers $\{(Tx_n, x_n)\}$ lies in a closed bounded set, we can find a subsequence of $\{(Tx_n, x_n)\}$ that converges in the complex plane. By calling this subsequence $\{x_n\}$, one then has

$$(Tx_n, x_n) \to \lambda \qquad \text{and} \qquad Tx_n \to x,$$

where $|\lambda| = \|T\|$ and $x \in H$.

If $\|T\| = 0$, the conclusion of the theorem is trivial. Assume now that $T \neq 0$, which implies that $\lambda \neq 0$. One then has

$$\begin{aligned}0 \leq \|Tx_n - \lambda x_n\|^2 &= \|Tx_n\|^2 + \|\lambda x_n\|^2 - \lambda\overline{(Tx_n, x_n)} - \bar{\lambda}(Tx_n, x_n)\\ &\leq (\|T\|^2 + |\lambda|^2)\|x_n\|^2 - \lambda\overline{(Tx_n, x_n)} - \bar{\lambda}(Tx_n, x_n)\\ &= 2|\lambda|^2 - \lambda\overline{(Tx_n, x_n)} - \bar{\lambda}(Tx_n, x_n).\end{aligned}$$

Since the right side tends to 0 as $n \to \infty$, we see that

$$Tx_n - \lambda x_n \to 0.$$

Hence $\lambda x_n \to x$, or $x_n \to (1/\lambda)x$. Hence $\|x\| = |\lambda| \neq 0$. Also

$$T\left(\frac{1}{\lambda} x\right) = T(\lim x_n) = \lim Tx_n = x,$$

or $Tx = \lambda x$. ▮

6.10.14 COROLLARY. *Let T be a compact, self-adjoint operator on a Hilbert space H. If T has no eigenvalues, then $H = \{0\}$.*

D. Compact Normal Operators

We have just seen that a compact self-adjoint operator on a nontrivial Hilbert space has at least one eigenvalue. Our object here is to show that the same conclusion is valid for compact normal operators.

Let T be a normal operator on a Hilbert space H. We know then (by Exercise 13, Section 5.23) that there are commuting self-adjoint operators A and B such that

$$T = A + iB \quad \text{and} \quad T^* = A - iB. \tag{6.10.3}$$

Furthermore, one has (Exercise 14, Section 5.23)

$$\max(\|A\|, \|B\|) \leq \|T\| = \|T^*\| \quad \text{and} \quad \|T\|^2 \leq \|A\|^2 + \|B\|^2.$$

We can use the Cartesian decomposition of T in (6.10.3) to determine whether T is compact.

6.10.15 LEMMA. *Let T be a normal operator on a Hilbert space H and let $T = A + iB$ be the Cartesian decomposition of T. Then T is compact if and only if both A and B are compact. Furthermore, T is compact if and only if T^* is compact.*

Proof: First we note that

$$\|Tx\|^2 = \|Ax\|^2 + \|Bx\|^2$$

for all x in H. Indeed, since $AB = BA$ one has

$$\begin{aligned}\|Tx\|^2 &= ((A + iB)x, (A + iB)x)\\ &= (Ax,Ax) + (Ax,iBx) + (iBx,Ax) + (iBx,iBx)\\ &= \|Ax\|^2 - i(Ax,Bx) + i(Bx,Ax) + \|Bx\|^2\\ &= \|Ax\|^2 - i(BAx,x) + i(ABx,x) + \|Bx\|^2\\ &= \|Ax\|^2 + \|Bx\|^2.\end{aligned}$$

It follows, then, that a sequence $\{Tx_n\}$ is a Cauchy sequence if and only if both the sequences $\{Ax_n\}$ and $\{Bx_n\}$ are Cauchy sequences. (Why?) Hence T is compact if and only if both A and B are compact.

Since $T^* = A - iB$, it follows from the above that T is compact if and only if T^* is compact. ∎

Let us now study the relationships between the eigenvalues of A and B.

Let $\lambda = \alpha + i\beta$ be an eigenvalue for T. We recall (Theorem 6.10.6) that $\bar{\lambda}$ is then an eigenvalue of T^*. In addition, one can show that α and β are eigenvalues of A and B, respectively. Indeed, if x satisfies $Tx = \lambda x$, then $T^*x = \bar{\lambda}x$ and

$$Ax = \tfrac{1}{2}(T + T^*)x = \tfrac{1}{2}(\lambda + \bar{\lambda})x = \alpha x,$$

$$Bx = \frac{1}{2i}(T - T^*)x = \frac{1}{2i}(\lambda - \bar{\lambda})x = \beta x.$$

This also shows that

$$\mathcal{N}(\lambda I - T) = \mathcal{N}(\bar{\lambda} I - T^*) \subseteq \mathcal{N}(\alpha I - A),$$

and

$$\mathcal{N}(\lambda I - T) = \mathcal{N}(\bar{\lambda} I - T^*) \subseteq \mathcal{N}(\beta I - B).$$

In order to get further information concerning the eigenvalues of T we have to study the relationship between the eigenspaces $\mathcal{N}(\alpha I - A)$ and $\mathcal{N}(\beta I - B)$. For this purpose let us now assume that T is compact and normal and that α is a nonzero eigenvalue of A. Let $x \in \mathcal{N}(\alpha I - A)$. Then

$$(\alpha I - A)Bx = B(\alpha I - A)x = 0,$$

which shows that B maps $\mathcal{N}(\alpha I - A)$ into itself. That is,

$$B\colon \mathcal{N}(\alpha I - A) \to \mathcal{N}(\alpha I - A)$$

and B is a compact self-adjoint operator on this subspace. Furthermore, it follows from Theorem 6.10.1 that $\mathcal{N}(\alpha I - A)$ is finite dimensional. Therefore, we can find an orthonormal basis of eigenvectors $\{e_1,e_2,\ldots,e_n\}$ of B in $\mathcal{N}(\alpha I - A)$ such that the mapping B can be represented by a diagonal matrix in terms of this basis

(Theorem 6.4.4). Let $\{\beta_1,\beta_2,\ldots,\beta_n\}$ be the entries in this diagonal matrix. Let us now show that the complex numbers

$$\alpha + i\beta_1, \alpha + i\beta_2,\ldots,\alpha + i\beta_n$$

are eigenvalues for T. Indeed, let e_j be a nonzero vector in $\mathcal{N}(\alpha I - A)$ that satisfies $Be_j = \beta_j e_j$. Hence

$$\begin{aligned}Te_j &= (A + iB)e_j = Ae_j + iBe_j\\ &= \alpha e_j + i\beta_j e_j = (\alpha + i\beta_j)e_j,\end{aligned}$$

for $j = 1, 2, \ldots, n$.

If we had started instead with a nonzero eigenvalue β of B, then one can show that A is a compact self-adjoint operator that maps $\mathcal{N}(\beta I - B)$ into itself. One can then construct an orthonormal basis for $\mathcal{N}(\beta I - B)$ so that the restriction of A to this subspace can be expressed as a diagonal matrix

$$\operatorname{diag}(\alpha_1,\alpha_2,\ldots,\alpha_m).$$

By the same reasoning used above one can then show that the complex numbers

$$\alpha_1 + i\beta, \alpha_2 + i\beta, \ldots, \alpha_m + i\beta$$

are eigenvalues for T.

It is easy, then, to see that if α is a nonzero eigenvalue of A, then for some β

$$\mathcal{N}(\lambda I - T) \text{ is nonempty,} \qquad \text{where } \lambda = \alpha + i\beta. \tag{6.10.4}$$

Moreover, one has

$$\mathcal{N}(\lambda I - T) = \mathcal{N}(\alpha I - A) \cap \mathcal{N}(\beta I - B). \tag{6.10.5}$$

Similarly if we start with a nonzero eigenvalue β for B, then for some α (6.10.4) and (6.10.5) are valid. Moreover, (6.10.4) and (6.10.5) are valid for every eigenvalue of T.

6.10.16 THEOREM. *Let T be a compact normal operator on a nontrivial Hilbert space H. Then T has an eigenvalue λ with*

$$\max(\|A\|, \|B\|) \le |\lambda|,$$

where $T = A + iB$ is the Cartesian decomposition of T, see Figure 6.10.1.

Proof: If $T = 0$, then $A = B = 0$ and $\lambda = 0$ is an eigenvalue satisfying the conclusion of the theorem.

Now assume that $T \neq 0$ and say that

$$\|A\| = \max(\|A\|, \|B\|) > 0.$$

Theorem 6.10.13 assures us that there is an eigenvalue α for A with the property that $|\alpha| = \|A\|$. The above discussion leads to the conclusion that there is a β such that $\lambda = \alpha + i\beta$ is an eigenvalue of T. Finally, we note that $|\lambda| \ge |\alpha| = \max(\|A\|, \|B\|)$. ∎

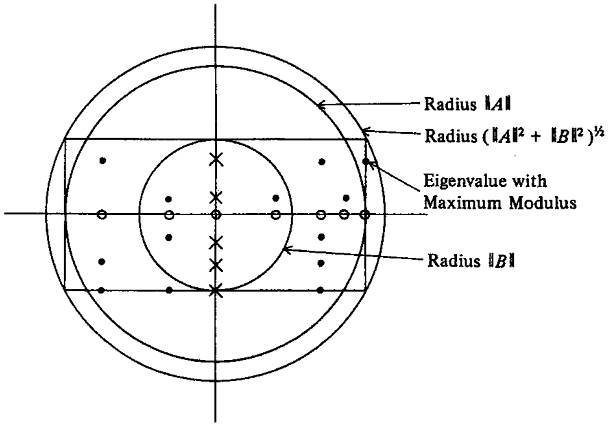

Figure 6.10.1. Eigenvalues of $T = A + iB$, T is Compact and Normal.

6.10.17 COROLLARY. *Let T be a compact normal operator on a Hilbert space H. If T has no eigenvalues then $H = \{0\}$.*

Theorem 6.10.16, then, assures us of the existence of at least one eigenvalue for T. If λ_0 is the eigenvalue of T with maximum modulus, that is,

$$|\lambda_0| = \max\{|\lambda| : \lambda \text{ is an eigenvalue for } T\},$$

then one can show that $|\lambda_0| = \|T\|$ (see Exercise 3, Section 11.)

And now for the Spectral Theorem.

EXERCISES

1. Prove Corollary 6.10.11.
2. Show that the spectrum of a unitary operator U lies on the unit circle $\{z : |z| = 1\}$.
3. (a) Construct a compact normal operator $T = A + iB$ with the property that there is at least one eigenvalue λ that satisfies $|\lambda|^2 = \|A\|^2 + \|B\|^2$.
   (b) Construct a compact normal operator $T = A + iB$ with the property that every eigenvalue λ satisfies $|\lambda|^2 < \|A\|^2 + \|B\|^2$. For your example, compute the spectral radius $r_\sigma(T)$ and the norm T.
   (c) Construct a compact normal (nonself-adjoint) operator $T = A + iB$ with the property that every eigenvalue λ satisfies $|\lambda| \le \max(\|A\|, \|B\|)$.
4. Let A be an observable (that is, a bounded self-adjoint operator) and let p be a state with associated density operator W. Assume that the spectrum of A lies in the interval $[a,b]$. Show that $E(A)$, the expected value of A with respect to the state p, satisfies

$$a \le E(A) \le b.$$

11. THE SPECTRAL THEOREM

The purpose of this section is to prove the following result:

6.11.1 THEOREM. (SPECTRAL THEOREM. FIRST VERSION.) *Let T be a compact normal operator on a Hilbert space H. Then there is a resolution of the identity $\{P_n\}$ and a sequence of complex numbers $\{\lambda_n\}$ such that*

$$T = \textstyle\sum_n \lambda_n P_n, \tag{6.11.1}$$

where the convergence in (6.11.1) is in terms of the uniform operator norm topology.

The expression (6.11.1) is sometimes called the *spectral decomposition* of T.

Proof: Let $\{\lambda_1, \lambda_2, \ldots\}$ denote the collection of *all* eigenvalues of T. This collection is at most countable by Corollary 6.10.5. Let P_n be the orthogonal projection onto $M_n = \mathcal{N}(\lambda_n I - T)$. Since $M_n \perp M_m$ for $n \neq m$, it follows that $P_n P_m = 0$ for $n \neq m$. Let

$$Q =_s \sum_n P_n.$$

Then Q is the orthogonal projection onto

$$M = M_1 + M_2 + \cdots.$$

We want to show that $Q = I$, or equivalently that $M^\perp = 0$. It follows from Corollary 6.10.9 that $T(M^\perp) \subset M^\perp$. Let S denote the restriction of T to $M^\perp$, that is, $S: M^\perp \to M^\perp$. Then S is compact and normal, and any eigenvalue of S is an eigenvalue of T. However, S has *no* eigenvalues. Therefore, it follows from Corollary 6.10.17 that $M^\perp = \{0\}$.

We have shown that $\{P_n\}$ is a resolution of the identity. Let us now show that $T = \sum_n \lambda_n P_n$. For this it will be convenient to order the eigenvalues so that $|\lambda_1| \ge |\lambda_2| \ge \cdots$. Let

$$S_N = \sum_{n=1}^{N} \lambda_n P_n.$$

Since $\{P_n\}$ is a resolution of the identity, one has

$$H = M_1 + M_2 + \cdots.$$

As a consequence of the Orthogonal Structure Theorem, every vector $x \in H$ can be written uniquely as

$$x = x_1 + x_2 + \cdots = \sum_n x_n,$$

where $x_n \in M_n$, and $\|x\|^2 = \sum_n \|x_n\|^2$. It follows that

$$Tx = \lambda_1 x_1 + \lambda_2 x_2 + \cdots = \sum_n \lambda_n x_n,$$

and

$$(T - S_N)x = \sum_{n=N+1}^{\infty} \lambda_n x_n.$$

Hence

$$\|(T - S_N)x\|^2 = \sum_{n=N+1}^{\infty} |\lambda_n|^2 \|x_n\|^2$$

$$\leq |\lambda_{N+1}|^2 \sum_{n=N+1}^{\infty} \|x_n\|^2 \leq |\lambda_{N+1}|^2 \|x\|^2.$$

Therefore $\|T - S_N\| \leq |\lambda_{N+1}| \to 0$ by Theorem 6.10.4. ▮

Actually some other versions of the Spectral Theorem are more practical in applications. Probably the most useful is the eigenvalue-eigenvector representation.

6.11.2 THEOREM. (SPECTRAL THEOREM. SECOND VERSION.) *Let T be a compact normal operator on a Hilbert space H. Then there exists a (orthonormal) basis of eigenvectors $\{e_n\}$ and corresponding eigenvalues $\{\mu_n\}$ such that if $x = \sum_n (x,e_n)e_n$ is the Fourier expansion for x, then*

$$Tx = \sum_n \mu_n (x,e_n)e_n. \tag{6.11.2}$$

Proof: We use the notation of the last theorem. With $M_n = \mathcal{N}(\lambda_n I - T)$, let $\{f_k^{(n)}\}$ be an orthonormal basis[6] for M_n. Then

$$Tf_k^{(n)} = \lambda_n f_k^{(n)}. \tag{6.11.3}$$

Let $\{f\}$ denote the union of all these $\{f_k^{(n)}\}$. Renumber the collection $\{f\}$ to get the family $\{e_1,e_2,\ldots\}$ and let $\{\mu_1,\mu_2,\ldots\}$ be the corresponding eigenvalues given by (6.11.3). The only thing we have to prove is that the family $\{f\}$, or $\{e_1,e_2,\ldots\}$ is a basis, that is, a maximal orthonormal set, in H.

First this family is orthonormal. That is, if n is fixed, then $f_i^{(n)} \perp f_j^{(n)}$ for $i \neq j$, by construction. Also, if $n \neq m$, then $f_i^{(n)} \perp f_j^{(m)}$ for any i and j, since $M_n \perp M_m$. Since each vector $f_k^{(n)}$ is a unit vector we see that $\{e_1,e_2,\ldots\}$ is an orthonormal set.

Next we claim that this family is maximal. Indeed if $x \perp e_n$ for all n, then $x \perp f_k^{(n)}$ for all n and k. That is, $x \perp M_n$ for all n, or $x \perp H$. Hence $x = 0$.

The proof of (6.11.2) is a simple adaptation of the argument of the last theorem. ▮

The last theorem admits another interpretation which can be viewed as the third version of the Spectral Theorem. For this we shall assume that the Hilbert space H is separable.[7] This means that the mapping $U: H \to l_2$ given by

$$U: x \to ((x,e_1),(x,e_2),\ldots)$$

[6] The subspace M_n is, of course, finite dimensional when $\lambda_n \neq 0$. If $\lambda = 0$ is an eigenvalue, then the corresponding null space $\mathcal{N}(T)$ may be infinite dimensional. In fact, if H is not separable, then $\lambda = 0$ is necessarily an eigenvalue and $\mathcal{N}(T)$ must have an uncountable orthonormal basis.

[7] Separability is not really necessary. It just makes things simpler.

is a unitary mapping. The operator T is then transformed into an operator Λ on l_2 by the equation

$$T = U^{-1}\Lambda U \quad \text{or} \quad \Lambda = UTU^{-1}, \tag{6.11.4}$$

see Figure 6.11.1. Also Λ is the diagonal matrix $\Lambda = \operatorname{diag}(\mu_1,\mu_2,\ldots)$. This representation Λ is sometimes called the "transfer function" of T.

In summary, then, every compact normal operator is a compact weighted sum of projections in disguise. Moreover, this fact can be used to view or represent compact normal operators in (at least) three ways: weighted sums of projections, eigenvalue-eigenvector representation, unitary equivalence to operation with a diagonal matrix or multiplication by a transfer function.

$$\begin{tikzcd} H \arrow[r, "T"] \arrow[d, "U"'] & H \\ l_2 \arrow[r, "\Lambda"'] & l_2 \arrow[u, "U^{-1}"'] \end{tikzcd}$$

Figure 6.11.1.

Now one point must be made. The Spectral Theorem presented here is not the most general one possible. This should not be surprising at all, for even the weighted sums of projections discussed in Section 9 can be used to represent some noncompact normal operators. In fact, if we generalized from weighted sums of projections to "weighted integrals of projections," we would be able to represent *all* normal operators. Likewise, the "transfer function" representation can be very successfully generalized (for example, Fourier transform and z-transform methods). However, a few mathematical difficulties arise here and there. On the other hand, the Eigenvalue-Eigenvector representation really cannot be developed much further. All this, however, is another story, beyond the scope of this book. The only generalization we will present (Section 14) concerns nonnormal compact operators.

EXAMPLE 1. (THE RAYLEIGH-RITZ METHOD.) The Rayleigh-Ritz Method, which we now describe, is a technique for finding the eigenvalues of a compact normal operator T. In this example we will assume that T is actually self-adjoint and positive, in addition to being compact. The extension of the method to arbitrary compact self-adjoint operators, or compact normal operators, is discussed in the exercises.

So then let $T: H \to H$ be a compact, self-adjoint, positive operator on a Hilbert space H. Recall that the positivity means that

$$(Tx,x) \geq 0, \qquad \text{for all } x \in H. \tag{6.11.5}$$

We see then that if μ is an eigenvalue of T, then $|\mu| \leq \|T\|$ and Equation (6.11.5) implies that $\mu \geq 0$. Now Theorem 6.10.13 tells us that

$$\mu_1 = \|T\|$$

is an eigenvalue of T. Let e_1 be an eigenvector of T associated with μ_1 and also let $M_1 = V(e_1)$ be the one-dimensional linear space spanned by e_1. Then M_1 reduces T, therefore T maps $M_1^{\perp}$ into $M_1^{\perp}$. Let T_2 denote the restriction of T to $M_1^{\perp}$. T_2 is, of course, compact and self-adjoint. So if we apply Theorem 6.10.13 again we see that

$$\mu_2 = \|T_2\|$$

is an eigenvalue of both T_2 and T. This process now continues. Let e_2 be an eigenvector associated with μ_2 and let $M_2 = V(e_1,e_2)$. Then T maps $M_2^{\perp}$ into $M_2^{\perp}$. Therefore, if we let T_3 denote the restriction of T to $M_2^{\perp}$, then

$$\mu_3 = \|T_3\|$$

is another eigenvalue of T.

It can easily be seen that if we continue in this way we can then find all the eigenvalues of T. With these preliminaries behind us, we are now prepared to give the Rayleigh-Ritz Formula for the eigenvalues, which is merely successive applications of Theorem 5.23.8, or Equation (6.10.2).

First we note that

$$\mu_1 = \sup_{(x,x)=1} (Tx,x). \tag{6.11.6}$$

Next we have

$$\mu_2 = \sup_{\substack{(x,x)=1\\(x,e_1)=0}} (Tx,x). \tag{6.11.7}$$

Indeed, the condition $(x,e_1) = 0$ in Equation (6.11.7) is precisely the condition that restricts T to the closed linear subspace $M_1^{\perp}$. Hence Equation (6.11.7) also can be written as

$$\mu_2 = \sup_{(x,x)=1} \{(Tx,x): x \in M_1^{\perp}\} = \|T_2\|.$$

In general, if the eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ are known with corresponding eigenvectors $e_1, e_2, \ldots, e_n$, then μ_{n+1} is given by

$$\mu_{n+1} = \sup_{\substack{(x,x)=1\\(x,e_1)=\cdots=(x,e_n)=0}} (Tx,x). \tag{6.11.8}$$

This formula can easily be proved by a direct application of mathematical induction.

Before the reader becomes too enamoured with this method a somewhat subtle limitation should be noted. Equation (6.11.8) does require that we know the eigenvectors $e_1, \ldots, e_n$, but the Rayleigh Ritz Method does not give any clue for determining these eigenvectors.

It is possible to circumvent this deficiency by using certain approximation techniques. We refer the reader to the work of Aronszajn [1] for more details. ▮

Example 2. (Fredholm Alternatives.) Let T be a compact normal operator on a Hilbert space H. Let y be given in H and we now seek a solution of the equation

$$x = Tx + y. \tag{6.11.9}$$

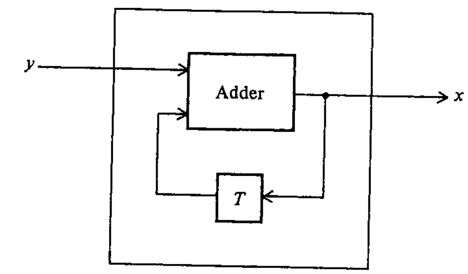

Figure 6.11.2.

This can be viewed as a black-box problem, as shown in Figure 6.11.2.

The Fredholm alternatives tell us precisely when it is possible to solve this problem.

(a) *If 1 is not an eigenvalue of T, then there is precisely one solution x for every y in H.* The solution is of course given by

$$x = (I - T)^{-1}y.$$

(See Exercise 25 for more details.)

(b) *If 1 is an eigenvalue of T, then there is a solution of* (6.11.9) *if and only if $y \perp \mathcal{N}(I - T)$. In this case, if x^* is any solution of* (6.11.9), *then every other solution is of the form*

$$x = x^* + c_1e_1 + \cdots + c_n e_n, \tag{6.11.10}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $\mathcal{N}(I - T)$.

The first alternative (a) follows from the fact that if 1 is not an eigenvalue of T, then 1 is in the resolvent set of T.

The second alternative (b) follows from the fact that Equation (6.11.9) has a solution if and only if y is in the range of $I - T$. Since $\mathcal{R}(I - T) = \mathcal{N}(I - T)^{\perp}$ we see that Equation (6.11.9) has a solution if and only if $y \perp \mathcal{N}(I - T)$.

The proof of Equation (6.11.10) is left as an exercise. ▮

EXERCISES

1. Let $K: L_2(I) \to L_2(I)$ be an integral operator $y = Kx$, where

$$y(t) = \int_I k(t,s)x(s)ds.$$

Assume that I is compact and $k(t,s)$ is continuous. Show that K is compact. Show that the eigenfunctions corresponding to nonzero eigenvalues can be chosen to be continuous. What happens to eigenfunctions corresponding to the eigenvalue $\lambda = 0$?

2. Use Mathematical Induction to prove Equation (6.11.8).

3. Let T be a compact normal operator on a Hilbert space H.
   (a) Show that there is an eigenvalue λ_0 that satisfies
   $$|\lambda_0| = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } T\}.$$
   (b) Show that $|\lambda_0| = \|T\|$.
   (c) Show that (6.10.2) holds for this T.
   (d) Let H be a two-dimensional complex Hilbert space. Show that λ_0 satisfies one of the following:
   $$|\text{Re } \lambda_0| = \|A\| \quad \text{or} \quad |\text{Im } \lambda_0| = \|B\|.$$
   What happens if H has dimension ≥ 3?
4. Let $T = A + iB$ be a compact normal operator. When is it true that
   $$\|T\|^2 = \|A\|^2 + \|B\|^2 ?$$
5. Let T be a compact self-adjoint operator on a Hilbert space H and also let $T = \sum_n \lambda_n P_n$ be the spectral decomposition of T. Then the nonzero eigenvalues $\{\lambda_n\}$ can be partitioned into two sets Λ_+ and Λ_-, the positive and the negative eigenvalues. The operators
   $$T_+ = \sum_{\lambda_n \in \Lambda_+} \lambda_n P_n, \qquad T_- = -\sum_{\lambda_n \in \Lambda_-} \lambda_n P_n$$
   are called the *positive and negative parts of T*.
   (a) Show that $T = T_+ - T_-$.
   (b) Show that $(T_+ x,x) \geq 0$ and $(T_- x,x) \geq 0$ for all x in H.
   (c) Show that $T_+T_- = T_-T_+ = 0$.
   (d) Let $|T| = T_+ + T_-$. Show that $T \leq |T|$ and $-T \leq |T|$.
6. Let T be a compact self-adjoint operator on a Hilbert space H.
   (a) Show that the positive eigenvalues of T can be found by Equations (6.11.6), (6.11.7), and (6.11.8).
   (b) Show that the negative eigenvalues of T can be found by replacing "sup" by "inf" in these three equations.
7. Use the results of Exercise 6 and Section 6.10D to discuss a method for finding the eigenvalues of a compact normal operator.
8. Let L be a compact normal operator on a Hilbert space H and let
   $$L = \sum_n \lambda_n P_n$$
   be the decomposition of L as a weighted sum of projections. Assume that $\lambda_n \neq 0$ for all n. Show that the polar decomposition of L is given by $L = RU$, where
   $$R = \sum_n |\lambda_n| P_n \qquad \text{and} \qquad U = \sum_n \lambda_n |\lambda_n|^{-1} P_n.$$
   What happens if $\lambda = 0$ is an eigenvalue of L? Show that the Cartesian decomposition of L is given by $L = A + iB$ when
   $$A = \sum_n \text{Re}(\lambda_n) P_n \qquad \text{and} \qquad B = \sum_n \text{Im}(\lambda_n) P_n.$$

9. Complete the proof of the Fredholm alternative (b) by verifying Equation (6.11.10).
10. Extend the Fredholm alternatives to compact nonnormal operators by proving the following:
    (a) If 1 is not an eigenvalue of T, then there is precisely one solution x of $x - Tx = y$ for every y in H.
    (b) If 1 is an eigenvalue of T, then $x - Tx = y$ has a solution if and only if $y \perp \mathcal{N}(I - T^*)$.
11. Let $k(t,s) \in L_2(I \times I)$ and define $y = Kx$ by
    $$y(t) = \int_I k(t,s)x(s)\, ds.$$
    Assume that $k(t,s) = \overline{k(s,t)}$.
    (a) Show that K is a compact self-adjoint operator on $L_2(I)$ and then show that $\|K\| \leq \|k\|_2$.
    (b) Let $\{e_n(t)\}$ be an orthonormal basis of eigenvectors for K with associated eigenvalues $\{\mu_n\}$. Assume that $|\mu_1| \geq |\mu_2| \geq \cdots$. Show that
    $$k(t,s) = \sum_n \mu_n e_n(t)\overline{e_n(s)}, \tag{6.11.11}$$
    where the convergence above is in $L_2(I \times I)$.
    (c) Show that
    $$\|k\|_2 = \left(\int_I \int_I |k(t,s)|^2\, dt\, ds\right)^{1/2} = \sum_n |\mu_n|^2$$
    (d) Show that $\|K\| = |\mu_1|$.
    (e) Characterize those operators K for which one has $\|K\| = \|k\|_2$.
12. (Continuation of Exercise 11.) Assume that I is closed and bounded and that $k(t,s)$ is continuous in t and s. In this exercise we will show that the series in Equation (6.11.11) converges to $k(t,s)$ uniformly in t and s provided the operator K is positive and $k(t,s)$ is real-valued.
    (a) Show that if μ_n is a nonzero eigenvalue, then the associated eigenfunction $e_n(t)$ can be chosen to be continuous and real-valued.
    (b) Let $k_N(t,s) = \sum_{n=1}^N \mu_n e_n(t)e_n(s)$ and $h_N(t,s) = k(t,s) - k_N(t,s)$. Show that $h_N(t,t) \geq 0$ for all t.
    (c) Show that there is a M such that
    $$k_N(t,t) \leq k(t,t) \leq M$$
    for all t and all N.
    (d) Show that
    $$\left|\sum_{i=m}^n \mu_i e_i(t)e_i(s)\right|^2 \leq M \sum_{i=m}^n \mu_i e_i(t)^2 \to 0$$
    as $m, n \to \infty$, uniformly in s for each fixed t. [That is, the convergence in Equation (6.11.11) is uniform in each variable separately.]
    (e) Show that the convergence in Equation (6.11.11) is pointwise.
    (f) Show that the convergence in Equation (6.11.11) is uniform in both t and s. [*Hint*: Use Dini's Theorem, from Section D.4.]

13. Let $T: H \to H$ be a compact self-adjoint operator on a Hilbert space H and let $T = \sum_n \lambda_n P_n$ be the spectral decomposition of T. For $\lambda \in (-\infty,\infty)$ let
$$Q_\lambda =_s \sum_{\lambda_n \le \lambda} P_n,$$
that is, $Q_\lambda x = \sum_{\lambda_n \le \lambda} P_n x$ for all $x \in H$.
(a) Show that for each λ, Q_λ is an orthogonal projection.
(b) Show that $Q_\lambda \le Q_\mu$ if $\lambda \le \mu$.
(c) Show that
$$0 =_s \lim_{\lambda \to -\infty} Q_\lambda, \qquad I =_s \lim_{\lambda \to +\infty} Q_\lambda.$$
(Q_λ is sometimes referred to as a *spectral family*.)

14. Let L be a self-adjoint operator on a Hilbert space H. Let $\{\phi_n\}$ be an orthonormal collection of eigenvectors of L and let M denote the closed linear subspace of H generated by $\{\phi_n\}$. Assume that every eigenvector of L lies in M.
(a) Show that if $M = H$ (that is, $\{\phi_n\}$ is an orthonormal basis for H), then L is a weighted sum of projections. Show that $\sigma(L)$ is the closure of $P\sigma(L)$.
(b) Show that if the continuous spectrum of L contains a nontrivial interval, then $M \neq H$, that is, $\{\phi_n\}$ is not a basis for H.

15. Consider $L = S_r + S_l + \Phi$ on $l_2(0,\infty)$, where S_r and S_l are the right and left shift operators, Φ is the *Coulomb perturbation*
$$\Phi(x_1,x_2,\ldots,x_n,\ldots) = 2b\left(x_1,\frac{1}{2}x_2,\ldots,\frac{1}{n}x_n,\ldots\right),$$
where $b > 0$.
(a) Show that the eigenvalues of L are
$$\lambda_k = 2\left[1 + \left(\frac{b}{k}\right)^2\right]^{1/2}, \qquad k = 1, 2, \ldots.$$
(b) Let $\{\phi_k\}$ be the associated eigenvector with $\|\phi_k\|_2 = 1$. Show that $\{\phi_k\}$ is *not* a basis for $l_2(0,\infty)$. [*Hint*: Use Exercise 14 and Exercise 14 of Section 6.]

16. Let W be a density operator, that is, W is self-adjoint with $0 \le W^2 \le W$ and $\operatorname{tr} W = 1$.
(a) Show that W is compact.
(b) Show that one can write $W = \sum_n \lambda_n W_n$, where W_n are density operators representing pure states and $\sum_n \lambda_n = 1$.
(c) Let e_n be a unit vector in $\mathscr{R}(W_n)$, and let A be an observable, that is, self-adjoint operator. Show that the expected value of A is $E(A) = \sum_n \lambda_n (Ae_n, e_n)$.

17. Let L and M be two compact normal operators on a Hilbert space H that commute, that is, $LM = ML$. Show that there is a resolution of the identity $\{P_n\}$ such that
$$L = \sum_n \lambda_n P_n, \qquad M = \sum_n \mu_n P_n$$
for appropriate choice of $\{\lambda_n\}$ and $\{\mu_n\}$.

18. Let $\{L_1,\ldots,L_k\}$ be a collection of compact normal operators on a Hilbert space H that satisfy $L_i L_j = L_j L_i$ for all i, j. Show that there is a resolution of the identity $\{P_n\}$ such that
$$L_i = \sum_n \lambda_n^{(i)} P_n, \qquad i = 1, \ldots, k,$$
where $\{\lambda_n^{(i)}\}$ depends on L_i.

19. Let $\{U\}$ be a family of unitary operators on a Hilbert space of finite dimension n. Assume that $\{U\}$ is a commutative family, that is, if U and V belong to $\{U\}$, then $UV = VU$. Show that there is an orthonormal basis of common eigenvectors. [*Hint*: Use Mathematical Induction on the dimension n.]

20. (Continuation of Exercise 19.) Let U_t, $-\infty < t < \infty$, be a commutative family of unitary operators on a finite-dimensional Hilbert space H that satisfy:
$$U_0 = I, \qquad U_s U_t = U_{s+t}, \quad \text{and} \quad U_s x \to U_t x \quad \text{as} \quad s \to t$$
for every $x \in H$. Let $\{\phi_1,\ldots,\phi_n\}$ be an orthonormal basis of eigenvectors and let $\rho_k(t)$ satisfy
$$U_t \phi_k = \rho_k(t)\phi_k, \qquad k = 1, \ldots, n.$$
(a) Show that $\rho_k(t) = \exp(iw_k t)$ for appropriate choice of w_k.
(b) Show that in terms of this basis U_t is the matrix operator $U_t = e^{itA}$, where $A = \operatorname{diag}\{w_1,\ldots,w_n\}$.

21. Show that the conclusions of Example 1, Section 4 can be extended to an infinite-dimensional Hilbert space H provided one assumed that the operator L is a compact self-adjoint strictly positive operator. What happens if one only assumes L to be compact self-adjoint and positive?

22. What conclusion could one draw in Example 1, Section 4 if one assumes L to be compact and normal?

23. Let A be a compact self-adjoint positive operator on a Hilbert space H and let $\{\mu_1,\mu_2,\ldots\}$ be an enumeration of the eigenvalues of A, including multiplicity. Show that $\operatorname{tr} A = \sum_n \mu_n$.

24. Find the eigenvectors and eigenvalues for $y(t) = \int_{-\pi}^{\pi} k(t,\tau)x(\tau)\,d\tau$, where
$$k(t,\tau) = \sum_{n=0}^{\infty} [a_n \cos nt + b_n \sin nt],$$
where $\sum_{n=0}^{\infty} (|a_n|^2 + |b_n|^2) < \infty$.

25. Consider the equation $(\lambda I - T)x = y$, where T is a compact normal operator. Let $\{e_n\}$ be an orthonormal basis of eigenvectors for T with corresponding eigenvalues $\{\lambda_n\}$. Assume that $\lambda \neq 0$.
(a) Show that if λ is not an eigenvalue of T, then for every y in the Hilbert space H there is a solution x of $(\lambda I - T)x = y$ and it is given by
$$x = \sum_{n=1}^{\infty} \frac{(y,e_n)}{\lambda - \mu_n} e_n.$$

(b) Show that if λ is an eigenvalue of T, then $(\lambda I - T)x = y$ has a solution if and only if $y \perp \mathcal{N}(\lambda I - T)$. Show that if $y \perp \mathcal{N}(\lambda I - T)$, then a solution is given by

$$x^* = \sum_{n=1}^{\infty} \frac{(y,e_n)}{\lambda - \mu_n} e_n,$$

where the terms involving eigenvectors in $\mathcal{N}(\lambda I - T)$ drop out since $(y,e_n) = 0$ for these. What is the general solution of $(\lambda I - T)x = y$ when λ is an eigenvalue of T?

12. FUNCTIONS OF OPERATORS (OPERATIONAL CALCULUS)

Let T be a compact normal operator on a Hilbert space H and express T as a weighted sum of projections

$$T = \sum_n \lambda_n P_n$$

as indicated in the Spectral Theorem. The operator T^2 is also a compact operator and, furthermore, one has

$$T^2 = \sum_n \lambda_n{}^2 P_n.$$

To see this we note that

$$\begin{aligned} T^2x = T(Tx) &= T\left(\sum_n \lambda_n P_n x\right) = \sum_m \lambda_m P_m \left(\sum_n \lambda_n P_n x\right) \\ &= \sum_{m,n} \lambda_m \lambda_n P_m P_n x \\ &= \sum_n \lambda_n{}^2 P_n x \end{aligned}$$

since $P_m P_n = 0$ when $m \neq n$ and $P_n P_n = P_n$.

Similarly one has

$$T^N = \sum_n \lambda_n{}^N P_n,$$

where N is any positive integer. In fact if

$$p(z) = \sum_{i=0}^{N} \alpha_i z^i$$

is any polynomial in z, then

$$p(T) = \sum_n p(\lambda_n) P_n,$$

where

$$p(T) = \sum_{i=0}^{N} \alpha_i T^i \quad \text{and} \quad T^0 = I.$$

We also know from Lemma 6.9.11 that

$$T^* = \sum_n \bar{\lambda}_n P_n.$$

It follows, then, that if

$$p(z,\bar{z}) = \sum_{i,j=1}^{n} \alpha_{ij} z^i \bar{z}^j$$

is a polynomial in the variables z and $\bar{z}$, then

$$p(T,T^*) = \sum_n p(\lambda_n, \bar{\lambda}_n) P_n,$$

where

$$p(T,T^*) = \sum_{i,j=1}^{n} \alpha_{ij} T^i T^{*j}.$$

We also know from Lemma 6.9.7 that the operator T is one-to-one if and only if $\lambda_n \neq 0$ for all n. In this case T^{-1} is defined on the range $\mathcal{R}(T)$ and by Lemma 6.9.10 one has

$$T^{-1} = \sum_n \lambda_n^{-1} P_n \qquad (x \in \mathcal{R}(T)). \tag{6.12.1}$$

Furthermore one has

$$T^{-N} = \sum_n \lambda_n{}^{-N} P_n \qquad (x \in \mathcal{R}(T^N)),$$

where N is a positive integer. In general, if $p(z)$ is a polynomial in z with no zeros on the spectrum of T, then one has

$$p(T)^{-1} = \sum_n p(\lambda_n)^{-1} P_n.$$

As a consequence of these observations one can easily prove the following theorem.

6.12.1 THEOREM. *Let T be a compact normal operator on a Hilbert space H and let*

$$T = \sum_n \lambda_n P_n$$

be the decomposition of T as a weighted sum of projections.

(a) *If $p(z)$ and $q(z)$ are two polynomials in Z, where $q(z)$ has no zeros on the spectrum $\sigma(T)$, and $r(z) = p(z) \cdot q(z)^{-1}$, then*

$$p(T)q(T)^{-1} = \sum_n r(\lambda_n) P_n.$$

(b) *If $p(z,\bar{z})$ and $q(z,\bar{z})$ are two polynomials in z and $\bar{z}$, where $q(z,\bar{z})$ has no zeros on $\sigma(T) \times \sigma(T^*)$, and $r = pq^{-1}$, then*

$$r(T,T^*) = \sum_n r(\lambda_n, \bar{\lambda}_n) P_n.$$

This operational calculus can be extended to discuss continuous (even discontinuous) functions of z. That is, if $f(z)$ is a continuous function defined on the spectrum $\sigma(T)$, then

$$f(T) = \sum_n f(\lambda_n)P_n.$$

The main problem here is defining $f(T)$. The reader who is interested in pursuing this further is referred to Dunford and Schwartz [1; Section 7.3], Simmons [1], and Taylor [1].

There is one more point we would like to bring up here and that is the question of the square root of a positive compact self-adjoint operator T. In this case the eigenvalues λ_n are all real and nonnegative, and therefore, the positive square root $\sqrt{\lambda_n}$ is well-defined. It should be clear that in this case one has

$$T^{1/2} = \sum_n \sqrt{\lambda_n}\, P_n. \tag{6.12.2}$$

EXERCISES

1. Let A be any bounded linear operator on a Hilbert space H.
   (a) Show that the series
   $$e^A = I + A + \frac{A^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$
   converges absolutely and represents a bounded linear operator.
   (b) Show that e^A commutes with A.
2. Let $A = \sum_n \lambda_n P_n$ be the spectral decomposition of a compact normal operator. Show that $e^A = \sum_n e^{\lambda_n} P_n$. What is $\sin A$, $\cos A$? Is it true that
   $$e^{iA} = \cos A + i \sin A?$$
3. Prove Equation (6.12.2).
4. Let A be a bounded linear operator on a Hilbert space H and assume that $\sigma(A)$ lies in the left half of the complex plane.
   (a) Show that $\|\exp At\| \to 0$ as $t \to +\infty$.
   (b) Use this to show that if u is a solution of
   $$\frac{du}{dt} = Au, \tag{6.12.3}$$
   then $\|u(t)\| \to 0$ as $t \to \infty$. (Show that $u(t) = (\exp At)u_0$ is a solution of (6.12.3) that satisfies $u(0) = u_0$.)

13. APPLICATIONS OF THE SPECTRAL THEOREM

In this section we shall present a number of applications of the Spectral Theorem.

EXAMPLE 1. (MATCHED FILTER.) Suppose that we wish to select a linear filter L so that a certain signal-to-noise ratio is maximized. In particular, let us assume that L is to be selected from among those linear filters that can be modeled mathematically in the form $y = Lx$, or

$$y(t) = \int_0^t g(t)x(t-\tau)\, d\tau, \qquad t \in [0,T],$$

where x is the input, y is the output, and the weighting function g is in $L_2[0,T]$. We assume that we are given an input signal $S(t)$ and a noise random process $N(\omega,t)$ (see Figure 6.13.1). At the final time $t = T$, the output is the sum of

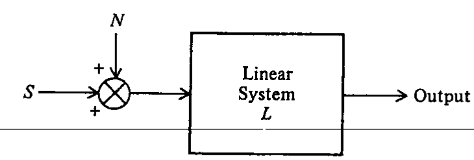

Figure 6.13.1.

$$s = \int_0^T g(\tau)S(T-\tau)\, d\tau$$

and the random variable

$$n(\omega) = \int_0^T g(\tau)N(\omega,T-\tau)\, d\tau.$$

Our problem is to pick g so as to maximize the signal-to-noise ratio

$$\frac{|s|^2}{E\{|n|^2\}},$$

where E denotes the mathematical expectation. Therefore, we assume that $S \in L_2[0,T]$. Then s can be viewed as the inner product between the points $g(\tau)$ and $S_R(\tau) = \overline{S(T-\tau)}$ in the Hilbert space $L_2[0,T]$. Furthermore, let us assume that the noise $N(\omega, t)$ satisfies

$$E\left\{\int_0^T \int_0^T g(\tau_1)\overline{g(\tau_2)N(\omega,T-\tau_2)}N(\omega,T-\tau_1)\, d\tau_1\, d\tau_2\right\}$$
$$= \int_0^T \int_0^T g(\tau_1)\overline{g(\tau_2)}E\{\overline{N(\omega,T-\tau_2)}N(\omega,T-\tau_1)\}\, d\tau_1\, d\tau_2,$$

and that the function

$$W(\tau_1,\tau_2) = E\{\overline{N(\omega,T-\tau_2)}N(\omega,T-\tau_1)\}$$

satisfies the condition

$$\int_0^T \int_0^T |W(\tau_1,\tau_2)|^2\, d\tau_1\, d\tau_2 < \infty.$$

We then obtain

$$E\{|n|^2\} = (Wg, g),$$

where W is the linear transformation of $L_2[0,T]$ into itself defined by

$$h(\tau_2) = \int_0^T W(\tau_1,\tau_2)g(\tau_1)\, d\tau_1.$$

The transformation W is, of course, by assumption, known.

We then have

$$\frac{|s|^2}{E\{|n|^2\}} = \frac{|(g,S_R)|^2}{(Wg,g)} \tag{6.13.1}$$

and we now have the problem stated entirely in terms of the Hilbert space $L_2[0,T]$. Moreover, W is a compact, positive self-adjoint transformation. Therefore it has a unique positive self-adjoint square root, say that $W = A^2$. We then have

$$(Wg,g) = (A^2g,g) = (Ag,Ag) = (\phi,\phi),$$

where $\phi = Ag$. Let us now assume that there is a function Φ in $L_2[0,T]$ with the property that $S_R = A\Phi$, that is, S_R lies in the range of A. One then has

$$\frac{|s|^2}{E\{|n|^2\}} = \frac{|(g,A\Phi)|^2}{(\phi,\phi)} = \frac{|(\phi,\Phi)|^2}{(\phi,\phi)}.$$

Then using Schwarz's Inequality we have

$$\frac{|s|^2}{E\{|n|^2\}} = \frac{|(\phi,\Phi)|^2}{(\phi,\phi)} \le (\Phi, \Phi).$$

Moreover, the equality will be taken on if and only if $\phi = k\Phi$, where k is a nonzero scalar. Thus a function g that maximizes the signal-to-noise ratio exists and is given by $k\Phi = Ag$ or equivalently $kS_R = Wg$. We can make this more explicit by using the Spectral Theorem.

Since W is a compact positive self-adjoint transformation, there exists an orthonormal system $\{w_1,w_2,w_3,\ldots\}$ and a sequence of nonnegative real numbers $\{\lambda_1{}^2,\lambda_2{}^2,\ldots\}$ with $\lambda_n{}^2 \to 0$ as $n \to \infty$ such that

$$Wx = \sum_{n=1}^{\infty} \lambda_n{}^2(x,w_n)w_n.$$

Furthermore, the square root A is given by

$$Ax = \sum_{n=1}^{\infty} \lambda_n(x,w_n)w_n.$$

The solution of the problem $kS_R = Wg$ is then given by

$$k\sum_{n=1}^{\infty} (S_R,w_n)w_n = \sum_{n=1}^{\infty} \lambda_n{}^2(g,w_n)w_n,$$

or equivalently

$$g = \sum_{n=1}^{\infty} (g,w_n)w_n = k\sum_{n=1}^{\infty} \lambda_n{}^{-2}(S_R,w_n)w_n.$$

If any of the eigenvalues $\{\lambda_n\}$ are zero, then the last equation still is valid provided the corresponding coefficient (S_R,w_n) vanishes, or equivalently, provided $S_R \perp \mathcal{N}(W)$. However, since we have assumed S_R to belong to the range of A (and ipso facto to be the range of W) we see that $S_R \perp \mathcal{N}(W)$. (Why?) ▮

EXAMPLE 2. (KARHUNEN-LOEVE EXPANSION.) Let $[a,b]$ be a finite interval. For $t \in [a,b]$ let $X(t)$ denote a random process with

$$E\{X(t)\} = 0, \qquad E\{|X(t)|^2\} < \infty, \tag{6.13.2}$$

and where the covariance function

$$r(t,s) = E\{X(t)\overline{X(s)}\} \tag{6.13.3}$$

is continuous (see Example 1, Section E.6).

Let f be a complex-valued function defined on $[a,b]$. We shall define the random variable $I = \int_a^b f(t)X(t)\,dt$ as follows: Let P: $a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a,b]$ (see Section D.2) and let $|P| = \max |t_i - t_{i-1}|$. Let $I(P)$ be the random variable given by

$$I(P) = \sum_{i=1}^{n} f(t_i)X(t_i)(t_i - t_{i-1}).$$

If it happens that $E\{|I(P) - I|^2\} \to 0$ as $|P| \to 0$, then we shall define I as

$$I = \int_a^b f(t)X(t)\,dt.$$

In the exercises the reader is asked to show that if $f(t)$ is continuous and if the covariance function $r(t,s)$ is continuous, then the integral $\int_a^b f(t)X(t)dt$ exists and that

$$E\left\{\int_a^b f(t)X(t)\,dt\right\} = 0. \tag{6.13.4}$$

Furthermore, if g is also continuous, then one can show that

$$\begin{aligned} E\left\{\int_a^b f(t)X(t)\,dt \int_a^b \overline{g(s)X(s)}\,ds\right\} &= \int_a^b \int_a^b f(t)\overline{g(s)}E\{X(t)\overline{X(s)}\}\,dt\,ds \\ &= \int_a^b \int_a^b f(t)\overline{g(s)}r(t,s)\,dt\,ds \end{aligned} \tag{6.13.5}$$

and

$$E\left\{\int_a^b f(t)X(t)\,dt \cdot \overline{X(s)}\right\} = \int_a^b f(t)r(t,s)\,dt. \tag{6.13.6}$$

6.13.1 THEOREM. *Let $X(t)$ be a random process defined on a finite interval $[a,b]$ satisfying* (6.13.2). *Assume that the covariance function $r(t, s)$ given by* (6.13.3) *is continuous. Then one can write*

$$X(t) = \sum_{n=1}^{\infty} Y_n\phi_n(t), \qquad a \le t \le b, \tag{6.13.7}$$

where $\{\phi_n\}$ is an orthonormal family of eigenfunctions of the integral operator R given by

$$y(t) = \int_a^b r(t,s)x(s)\,ds \tag{6.13.8}$$

and moreover $\{\phi_n\}$ forms a basis for $\mathcal{N}(R)^\perp$. The random variables Y_n in (6.13.7) are given by $Y_n = \int_a^b \overline{\phi_n(t)}X(t)\,dt$ and satisfy $E\{Y_n\} = 0$ and $E\{Y_n \overline{Y}_m\} = \delta_{nm}\lambda_m$, where λ_m is the eigenvalue associated with ϕ_m. Finally the series in (6.13.7) converges in the mean square sense to $X(t)$, that is,

$$E\left\{\left|X(t) - \sum_{n=1}^{N} Y_n \phi_n(t)\right|^2\right\} \to 0$$

as $N \to \infty$ for all t in $[a,b]$.

Proof: We note that the integral operator R given by (6.13.8) is compact since $r(t,s)$ is continuous, see Example 6, Section 5.24. Also $r(t,s) = \overline{r(s,t)}$, so R is self-adjoint. Let $\{\phi_n\}$ be an orthonormal collection of eigenfunctions of R associated with the nonzero eigenvalues $\{\lambda_n\}$. Then $\phi_n(t)$ is continuous and real-valued (see Exercise 12, Section 11) and the random variable $Y_n = \int_a^b \overline{\phi_n(t)}X(t)dt$ exists and by (6.13.4) one has $E\{Y_n\} = 0$. Furthermore, (6.13.5) implies that

$$\begin{aligned}E\{Y_n \overline{Y}_m\} &= \int_a^b \int_a^b \overline{\phi_n(t)}\phi_m(s)r(t,s)\,ds\,dt \\ &= \int_a^b \overline{\phi_n(t)}\lambda_m \phi_m(t)\,dt = \lambda_m \delta_{nm}.\end{aligned}$$

Next let $S_N(t) = \sum_{n=1}^N Y_n \phi_n(t)$. Then by a straightforward application of (6.13.5) and (6.13.6), together with the fact that the eigenvalues of R are real, we get

$$\begin{aligned}E\{|X(t) - S_N(t)|^2\} &= E\{(X(t) - S_N(t))(\overline{X(t)} - \overline{S_N(t)})\} \\ &= r(t,t) - \sum_{n=1}^{N} \lambda_n \phi_n(t)\overline{\phi_n(t)}.\end{aligned}$$

It is shown in Exercise 12, Section 11 that

$$r(t,s) = \lim_{N\to\infty} \sum_{n=1}^{N} \lambda_n \phi_n(t)\overline{\phi_n(s)},$$

therefore, we conclude that

$$E\{|X(t) - S_N(t)|^2\} \to 0$$

as $N \to 0$. ∎

EXAMPLE 3. (THE KARHUNEN-LOEVE EXPANSION FOR DISCRETE RANDOM PROCESSES.) The expansion described in the last example is also valid when the interval $[a,b]$ is replaced by a discrete countable set say $t = 1, 2, \ldots$ In this case, a somewhat different notation is customarily employed.

Let $\{X_n : n = 1,2,\ldots\}$ be a discrete random process with

$$E\{X_n\} = 0 \quad\text{and}\quad E\{|X_n|^2\} < \infty. \tag{6.13.9}$$

Define the covariance matrix $\Gamma = (\gamma_{nm})$ by

$$\gamma_{nm} = E\{X_n \overline{X}_m\}, \qquad n, m = 1, 2, \ldots,$$

and assume that

$$\sum_{n,m} |\gamma_{nm}|^2 < \infty. \tag{6.13.10}$$

6.13.2 THEOREM. *Let $\{X_n : n = 1,2,\ldots\}$ be a discrete random process satisfying (6.13.9) and assume that the covariance matrix Γ satisfies (6.13.10). Then one can write*

$$X = \{X_1, X_2, \ldots\} = \sum_{k=1}^{\infty} Y_k \phi_k, \tag{6.13.11}$$

where $\phi_k = \{\phi_k^{(1)}, \phi_k^{(2)}, \ldots\}$ is an element of l_2 and the collection of $\{\phi_k\}$ is an orthonormal family of eigenvectors for the matrix operator Γ given by $y = \Gamma x$, and, moreover, $\{\phi_k\}$ forms a basis for $\mathcal{N}(\Gamma)^\perp$. Furthermore the random variables Y_k in (6.13.11) are given by $Y_k = \sum_{n=1}^{\infty} \phi_k^{(n)} X_n$ and satisfy $E\{Y_k\} = 0$ and $E\{Y_k \overline{Y}_l\} = \delta_{kl}\lambda_k$, where λ_k is the eigenvalue associated with ϕ_k. Finally, the series in (6.13.11) converges to $X = \{X_1, X_2, \ldots\}$ in the mean-square sense, that is,

$$E\left\{\left|X_n - \sum_{k=1}^{K} Y_k \phi_k^{(n)}\right|^2\right\} \to 0$$

as $K \to \infty$, for all $n = 1, 2, \ldots$.

The proof of this theorem, which we shall leave as an exercise, follows the argument used in Theorem 6.13.1. The only noteworthy difference is to show that the series

$$Y_k = \sum_{n=1}^{\infty} \phi_k^{(n)} X_n$$

converges to a random variable Y_k. ∎

EXERCISES

1. This exercise will lead to a proof that the integral $\int_a^b f(t)X(t)\,dt$ is defined when f and X are continuous. We use the notation of Example 2.
   (a) Let $P = a = t_0 < t_1 < \cdots < t_n = b$ and $P' = a = t_0' < t_1' < \cdots < t_m' = b$ be two partitions of $[a,b]$. Show that

$$\begin{aligned}E(I(P)\overline{I(P')}) &= \sum_{i,j} f(t_i)f(t_j')R(t_i,t_j')(t_i - t_{i-1})(t_j' - t_{j-1}') \\ &\to \int_a^b \int_a^b f(t)f(t')R(t,t')\,dt\,dt'\end{aligned}$$

   as $|P|, |P'| \to 0$.

(b) Show that $E(|I(P) - I(P')|^2) \to 0$ as $|P|, |P'| \to 0$.
(c) Use the completeness of $L_2(\Omega,\mathscr{F},P)$, where $(\Omega,\mathscr{F},P)$ is the underlying probability space, to conclude that $I(P)$ has a limit in L_2 as $|P| \to 0$.

2. Using the notation of Example 2, show that if $E(X(t)) = 0$ for all t, then $E(\int_a^b f(t)X(t)\,dt) = 0$, when f and X are continuous.
3. Using the notation of Example 2, show that if f, g, and X are continuous, then

$$E\left(\int_a^b f(t)X(t)\,dt \int_a^b \overline{g(s)X(s)}\,ds\right) = \int_a^b \int_a^b f(t)\overline{g(s)}r(t,s)\,dt\,ds$$

$$E\left(\int_a^b f(t)X(t)\,dt \cdot \overline{X(s)}\right) = \int_a^b f(t)r(t,s)\,dt.$$

4. Prove Theorem 6.13.2.
5. Let $Y(t)$ be a random process defined on a finite interval $[a,b]$ with $E\{|Y(t)^2\} < \infty$, where $E(Y(t))$ and $E(Y(t)\overline{Y(s)})$ are continuous functions. Show that

$$Y(t) = E(Y(t)) + \sum_{n=1}^{\infty} Y_n\,\phi_n(t),$$

where Y_n and ϕ_n have structure similar to that defined in Theorem 6.13.1.

6. Let $x(\omega,t)$ be a complex-valued function defined for $\omega \in [0,W]$ and $t \in [a,b]$ and satisfying:

$$\int_0^W x(\omega,t)\,d\omega = 0, \qquad \int_0^W |x(\omega,t)|^2\,d\omega < \infty,$$

for all $t \in [a,b]$. Also assume that

$$k(t,s) = \int_0^W x(\omega,t)\,\overline{x(\omega,s)}\,d\omega$$

is continuous. Show that one can express x in the form

$$x(\omega,t) = \sum_{n=1}^{\infty} Y_n(\omega)\,\phi_n(t),$$

where

$$Y_n(\omega) = \int_a^b x(\omega,t)\,\overline{\phi_n(t)}\,dt.$$

7. Use Exercises 5 and 6 to study the function $x(\omega,t) = \exp(i\omega t)$.

14. NONNORMAL OPERATORS

So far we have been concentrating on compact normal operators. But suppose we have a compact operator that is not normal. What can we do? Clearly we cannot expect to express it as a weighted sum of projections, for all weighted sums of (orthogonal) projections are normal. Equivalently, we cannot expect the eigenvectors to form an orthonormal set. As a matter of fact, all linear operators on finite-dimensional spaces are compact and it is well known that even there, the ones that are not normal can lead to difficulties. (The reader may be familiar with the Jordan canonical form.) Not too surprisingly, things can be more difficult in

the case of infinite-dimensional spaces. For example, one may be able to show that a nonnormal compact operator is similar to the operation of multiplication by a (transfer) function, but it is impossible for it to be unitarily equivalent to such an operator, for then it would be normal. In any event, we shall avoid all of these difficulties by taking a slightly different approach. The two main advantages of this approach are that (1) it is applicable to all compact operators, normal or not, and (2) it involves only orthonormal sets of vectors. In fact, we shall show in this section the every compact operator T can be represented in the form

$$Tx = \sum_{n=1}^{\infty} \mu_n(x,x_n)y_n,$$

where the μ_n's are nonnegative real numbers and $\{x_n\}$ and $\{y_n\}$ are orthonormal sets.

6.14.1. THEOREM. *Let T be compact transformation of a Hilbert space H into itself. Then there exist two orthonormal systems $\{x_n\}$ and $\{y_n\}$ and a sequence of nonnegative real numbers $\{\mu_1,\mu_2,\mu_3,\ldots\}$ such that*

$$Tx = \sum_n \mu_n(x,x_n)y_n, \tag{6.14.1}$$

where convergence is in terms of the uniform topology, that is, $\|T - S_N\| \to 0$ as $N \to \infty$, where

$$S_N x = \sum_{n=1}^{N} \mu_n(x,x_n)y_n. \tag{6.14.2}$$

Proof: Whether T is normal or not, the operator T^*T is compact (Theorem 5.24.7) and self-adjoint. Moreover, T^*T is nonnegative, that is

$$(x,T^*Tx) = (Tx,Tx) \geq 0$$

for all $x \in H$. Therefore, it follows that the eigenvalues of T^*T are real and nonnegative. Let $\{\mu_1{}^2,\mu_2{}^2,\ldots\}$ denote these eigenvalues, where $\mu_n \geq 0$ for all n. For convenience we assume that $\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots$. Then, using the eigenvalue-eigenvector representation for T^*T, we have

$$T^*Tx = \sum_{n=1}^{\infty} \mu_n{}^2(x,x_n)x_n,$$

where a given eigenvalue is repeated according to its multiplicity and $\{x_n\}$ is an orthonormal basis of eigenvectors. This operator T^*T has a unique nonnegative square root R given by

$$Rx = \sum_{n=1}^{\infty} \mu_n(x,x_n)x_n.$$

For $\mu_n \neq 0$, let

$$y_n = \frac{1}{\mu_n} Tx_n.$$