INTRODUCTION TO HOMOGENIZATION

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1. Background

1.1. L^p and Sobolev spaces.

1.1.1. *Weak convergence.*

Definition 1.1. Let X be a real Banach space, X^* its dual and $\langle \cdot, \cdot \rangle$ the canonical pairing over $X^* \times X$.

i. The sequence x_h in X *converges weakly* to $x \in X$ and we write
$$x_h \rightharpoonup x \text{ in } X$$
if $\langle x^*, x_h \rangle \to \langle x^*, x \rangle$ for every $x^* \in X^*$.

ii. The sequence x_h^* in X^* *converges weak** to $x^* \in X^*$ and we write
$$x_h^* \rightharpoonup^* x^* \text{ in } X^*$$
if $\langle x_h^*, x \rangle \to \langle x^*, x \rangle$ for every $x \in X$.

Theorem 1.2. *Let X be a Banach space. Let (x_h) and (x_h^*) be two sequences in X and in X^* respectively.*

i. *Let $x_h \rightharpoonup x$, then there exists a constant $k > 0$ such that $\|x_h\| \leq k$; furthermore $\|x\| \leq \liminf_{h\to\infty} \|x_h\|$.*

ii. *Let $x_h^* \rightharpoonup^* x^*$, then there exists a constant $k > 0$ such that $\|x_h^*\|_{X^*} \leq k$; furthermore $\|x\|_{X^*} \leq \liminf_{h\to\infty} \|x_h^*\|_{X^*}$.*

iii. *If $x_h \to x$, then $x_h \rightharpoonup x$.*

iv. *If $x_h^* \to x^*$, then $x_h^* \rightharpoonup^* x^*$.*

v. *If $x_h \rightharpoonup x$ and $x_h^* \to x^*$, then $\langle x_h^*, x_h \rangle \to \langle x^*, x \rangle$.*

Theorem 1.3. *Let X be a reflexive Banach space. Let (x_h) be a sequence in X and k a positive constant such that $\|x_h\| \leq k$. Then there exist $x \in X$ and a subsequence $(x_{\sigma(h)})$ of (x_h) such that $x_{\sigma(h)} \rightharpoonup x$ in X.*

Theorem 1.4. *Let X be a separable Banach space. Let (x_h^*) be a sequence in X^* and k a positive constant such that $\|x_h^*\|_{X^*} \leq k$. Then there exist $x^* \in X^*$ and a subsequence $(x_{\sigma(h)}^*)$ of (x_h^*) such that $x_{\sigma(h)}^* \rightharpoonup^* x^*$ in X^*.*

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1.1.2. *L^p spaces.*

Definition 1.5. Let Ω be an open subset of $\mathbb{R}^n$.

i. Let $1 \le p < \infty$. $L^p(\Omega, \mathbb{R}^n)$ is the set of all measurable functions $f : \Omega \to \mathbb{R}^n$ such that

$$\|f\|_{L^p(\Omega,\mathbb{R}^n)} \equiv \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p} < \infty.$$

We have $\|\cdot\|_{L^p(\Omega,\mathbb{R}^n)}$ is a norm.

ii. Let $p = \infty$. A measurable function $f : \Omega \to \mathbb{R}^n$ is said to be in $L^\infty(\Omega, \mathbb{R}^n)$ if

$$\|f\|_{L^\infty(\Omega,\mathbb{R}^n)} \equiv \inf \{\alpha : |f(x)| \le \alpha \text{ a.e. in } \Omega\} < \infty.$$

We have that $\|\cdot\|_{L^\infty(\Omega,\mathbb{R}^n)}$ is a norm.

iii. $L^p_{loc}(\Omega, \mathbb{R}^n)$ denotes the linear space of measurable functions u such that $u \in L^p(\Omega', \mathbb{R}^n)$ for every $\Omega' \subset\subset \Omega$ (note that $u_h \to u$ in $L^p_{loc}(\Omega, \mathbb{R}^n)$ if $u_h \to u$ in $L^p(\Omega', \mathbb{R}^n)$ for every $\Omega' \subset\subset \Omega$).

If $n = 1$, $L^p(\Omega, \mathbb{R}^1) = L^p(\Omega)$.

Remark 1.6. Note that

a. Let $1 \le p \le \infty$. We denote by q the conjugate exponent of p, i.e.,

$$\frac{1}{p} + \frac{1}{q} = 1,$$

where it is understood that if $p = 1$ then $q = \infty$ and reciprocally.

b. Let $1 \le p < \infty$. Then the dual space of $L^p(\Omega, \mathbb{R}^n)$ is $L^q(\Omega, \mathbb{R}^n)$. We point out also that the dual space of $L^\infty(\Omega, \mathbb{R}^n)$ contains strictly $L^1(\Omega, \mathbb{R}^n)$.

c. The notion of weak convergence in $L^p(\Omega, \mathbb{R}^n)$ becomes then as follows: If $1 \le p < \infty$, then $f_h \rightharpoonup f$ weakly in $L^p(\Omega, \mathbb{R}^n)$ if

$$\int_\Omega (f_h(x), g(x)) \, dx \to \int_\Omega (f(x), g(x)) \, dx$$

for every $g \in L^q(\Omega, \mathbb{R}^n)$. For the case $p = \infty$, $f_h \rightharpoonup^* f$ in $L^\infty(\Omega, \mathbb{R}^n)$ weak* if

$$\int_\Omega (f_h(x), g(x)) \, dx \to \int_\Omega (f(x), g(x)) \, dx$$

for every $g \in L^1(\Omega, \mathbb{R}^n)$.

Theorem 1.7. *For every $1 \le p \le \infty$, $L^p(\Omega, \mathbb{R}^n)$ is a Banach space. It is separable if $1 \le p < \infty$ and reflexive if $1 < p < \infty$. Moreover, $L^2(\Omega, \mathbb{R}^n)$ turns out to be a Hilbert space with the scalar product defined by $(f, g)_{L^2(\Omega,\mathbb{R}^n)} = \int_\Omega (f(x), g(x))\, dx$.*

1.1.3. *Sobolev spaces.*

Definition 1.8. Let Ω be an open subset $\mathbb{R}^n$ and $1 \le p \le \infty$. The Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega, \mathbb{R}^n)\},$$

where $\nabla u = (\nabla_1 u, \nabla_2 u, ..., \nabla_n u) = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}\right)$ denotes the first order distributional derivative of the funtion u.

On $W^{1,p}(\Omega)$ we define the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega,\mathbb{R}^n)}\right)^{1/p}.$$

Definition 1.9. Let $1 \le p < \infty$. $W^{1,p}_0(\Omega)$ denotes the closure of $C^\infty_0(\Omega)$ in $W^{1,p}(\Omega)$. $W^{-1,q}(\Omega)$ indicates the dual space of $W^{1,p}_0(\Omega)$.

Remark 1.10. If $p = 2$, the notation $H^{1,2}(\Omega)$ or $H^1(\Omega)$ are very common for $W^{1,2}(\Omega)$. Moreover, $H^{1,2}_0(\Omega)$ or $H^1_0(\Omega)$ stand for $W^{1,2}_0(\Omega)$. The spaces $H^{1,2}(\Omega)$ and $H^{1,2}_0(\Omega)$ are naturally endowed with the scalar product

$$(u, v)_{H^{1,2}(\Omega)} = (u, v)_{L^2(\Omega)} + \sum_{i=1}^n (\nabla_i u, \nabla_i v)_{L^2(\Omega)}$$

which induces the norm $\|u\|_{H^{1,2}(\Omega)}$.

Theorem 1.11. *The space $W^{1,p}(\Omega)$ is a Banach space for $1 \le p \le \infty$. $W^{1,p}(\Omega)$ is separable if $1 \le p < \infty$ and reflexive if $1 < p < \infty$.*

Moreover, the space $W^{1,p}_0(\Omega)$ endowed with the norm induced by $W^{1,p}(\Omega)$ is a separable Banach space; it is reflexive if $1 < p < \infty$.

The spaces $H^{1,2}(\Omega)$ and $H^{1,2}_0(\Omega)$ are separable Hilbert spaces.

We now quote the Sobolev and Rellich-Kondrachov imbedding theorems.

Theorem 1.12. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary.*

i. *If $1 \leq p < \infty$, then*
$$W^{1,p}(\Omega) \subset L^q(\Omega) \text{ for every } 1 \leq q \leq \frac{np}{n-p}$$
and the imbedding is compact for every $1 \leq q < \frac{np}{n-p}$.

ii. *If $p = n$, then*
$$W^{1,p}(\Omega) \subset L^q(\Omega) \text{ for every } 1 \leq q < \infty$$
and the imbedding is compact.

iii. *If $p > n$, then*
$$W^{1,p}(\Omega) \subset C(\overline{\Omega})$$
and the imbedding is compact.

Remark 1.13. We have that

a. The regularit of the boundary $\delta\Omega$ in the theorem can be weakened. Note that if the space $W^{1,p}(\Omega)$ is replaced by $W_0^{1,p}(\Omega)$, then no regularity of the boundary is required.

b. The compact imbedding can be read in the following way. Let
$$u_h \rightharpoonup u \text{ in } W^{1,p}(\Omega).$$

I. If $1 \leq p < n$, then $u_h \to u$ in $L^q(\Omega)$, $1 \leq q < \frac{np}{n-p}$;

II. If $p = n$, then $u_h \to u$ in $L^q(\Omega)$, $1 \leq q < \infty$;

III. If $p > n$, then $u_h \to u$ in $L^\infty(\Omega)$.

Let us state two important inequalities.

Theorem 1.14. *We have the following*

i. *(Poincaré Inequality) Let Ω be a bounded open set and let $1 \leq p < \infty$. Then there exists a constant $k > 0$ such that*
$$\|u\|_{L^p(\Omega)} \leq k \|\nabla u\|_{L^p(\Omega;\mathbb{R}^n)}$$
for every $u \in W_0^{1,p}(\Omega)$.

ii. *(Poincaré-Wirtinger Inequality) Let Ω be a bounded open convex set and let $1 \leq p < \infty$. Then there exists a constant $k > 0$ such that*
$$\left\| u - \frac{1}{|\Omega|} \int_\Omega u(x)dx \right\|_{L^p(\Omega)} \leq k \|\nabla u\|_{L^p(\Omega;\mathbb{R}^n)}$$
for every $u \in W^{1,p}(\Omega)$.

Remark 1.15. From the previous theorem it follows that $\|\nabla u\|_{L^p(\Omega;\mathbb{R}^n)}$ defines a norm on $W_0^{1,p}(\Omega)$, denoted by $\|u\|_{W_0^{1,p}(\Omega)}$, which is equivalent to the norm $\|u\|_{W^{1,p}(\Omega)}$.

1.1.4. *Extension and convergence lemmas for periodic functions.* Let $Y = (0,1)^n$ be the unit cube in $\mathbb{R}^n$ and let $1 < p < \infty$. By $W^{1,p}_{per}(Y)$ we denote the subset of $W^{1,p}(Y)$ of all the functions u with mean value zero which have the same trace on the opposite faces of Y. In the case $p = 2$ we use the notation $H^{1,2}_{per}(Y)$.

Lemma 1.16. *Let $f \in W^{1,p}_{per}(Y)$. Then f can be extended by periodicity to an element of $W^{1,p}_{loc}(\mathbb{R}^n)$.*

Lemma 1.17. *Let $g \in L^q(Y;\mathbb{R}^n)$ such that $\displaystyle\int_Y (g,\nabla v) = 0$ for every $v \in W^{1,p}_{per}(Y)$. Then g can be extended by periodicity to an element of $L^q_{loc}(\mathbb{R}^n;\mathbb{R}^n)$, still denoted by g such that $-div g = 0$ in $D'(\mathbb{R}^n)$.*

Theorem 1.18. *Let $f \in L^p(Y)$. Then f can be extended by periodicity to a function (still denoted by f) belonging to $L^p_{loc}(\mathbb{R}^n)$. Moreover, if (ϵ_h) is a sequence of positive real numbers converging to 0 and $f_h(x) = f\left(\frac{x}{\epsilon_h}\right)$, then*

$$f_h \rightharpoonup \langle f\rangle = \frac{1}{|Y|}\int_Y f(y)dy \ \textit{weakly in } L^p_{loc}(\mathbb{R}^n)$$

if $1 \le p < \infty$, and

$$f_h \rightharpoonup^* \langle f\rangle \ \textit{ in } L^\infty(\mathbb{R}^n) \ \textit{weak*}$$

if $p = \infty$.

Remark 1.19. Let us point out some features of the weak convergence. To this aim, let us consider $Y = (0, 2\pi)$ and let $f(x) = \sin x$. Let (ϵ_h) be a sequence of positive real numbers converging to 0. By the previous theorem we have that $f_h(x) = f\left(\frac{x}{\epsilon_h}\right)$ converges to 0 in $L^\infty(Y)$ weak* (hence weakly in $L^2(Y)$).

In particular

$$\int_0^{2\pi} f_h(x)dx \to \frac{1}{2\pi}\int_0^{2\pi} \sin y dy = 0,$$

i.e., the mean values of f_h converges to 0. On the other hand, we have that (f_h) does not converge a.e. on Y. Furthermore,

$$\|f_h - 0\|^2_{L^2(Y)} = \int_0^{2\pi} \sin^2\left(\frac{x}{\epsilon_h}\right) dx \to \left(\frac{1}{\pi}\int_0^{\pi} \sin^2 y dy\right) 2\pi = \pi \neq 0,$$

which shows that we do not have convergence of (f_h) to f in the strong topology of $L^2(Y)$.

This example shows also another mathematical difficulty one meets by handling with weak convergent sequences. More precisely, it two sequences and their product converge in the weak topology, the limit of the product is not equal, in general, to the product of the limits. Indeed the remark above proves that $f_h^2 = f_h \times f_h$ does not converge weakly in $L^2(Y)$ to 0.

1.2. **A Compensated Compactness Lemma.**

Lemma 1.20. *Compensated Compactness Lemma*

Let $1 < p < \infty$. Let (u_h) be a sequence converging to u weakly in $W^{1,p}(\Omega)$, and let (g_h) be a sequence in $L^q(\Omega;\mathbb{R}^n)$ converging weakly to g in $L^q(\Omega;\mathbb{R}^n)$. Moreover assume that $(-divg_h)$ converges to $-divg$ strongly in $W^{-1,q}(\Omega)$. Then

$$\int_\Omega (g_h, \nabla u_h)\,\varphi dx \to \int_\Omega (g, \nabla u)\,\varphi dx$$

for every $\varphi \in C_0^\infty(\Omega)$.

Proof. The lemma is a simple case of compensated compactness. It can be proved by observing that

$$\int_\Omega (g_h, \nabla u_h)\,\varphi dx = \langle -divg_h, u_h\varphi\rangle - \int_\Omega u_h\,(g_h, \nabla\varphi)\, dx$$

for every $\varphi \in C_0^\infty(\Omega)$. □

Note that $(g_h, \nabla u_h)$ is the product of two sequences which converge only in the weak topology, and that by passing to the limit we get the product of the limits. This fact is known as the phenomenon of "compensated compactness".

1.3. **Abstract existence theorems.**

1.3.1. *Lax-Milgram Theorem.* Let H be a Hilbert space. A bilinear form a on H is called *continuous* (or bounded) if there exists a positive constant k such that

$$|a(u,v)| \leq k \|u\| \|v\| \text{ for every } u,v \in H$$

and *coercive* if there exists a positive constant α such that

$$a(u,u) \geq \alpha \|u\|^2 \text{ for every } u \in H.$$

A particular example of a continuous, coercive bilinear form is the scalar product of H itself.

Lemma 1.21. *Let a be a continuous, coercive bilinear form on a Hilbert space H. Then for every bounded linear functional $f \in H^*$ there exists a unique element $u \in H$ such that*

$$a(u,v) = \langle f, v \rangle \text{ for every } v \in H.$$

1.3.2. *Maximal Monotone Operators.* Let X be a Banach space and X^* its dual space. Let A be a single-valued operator from $D(A)$ to X^*, where $D(A)$ is a linear subspace of X and is called the domain of A. The range $R(A)$ of A is the set of all points f of X^* such that there exists $x \in D(A)$ with $Ax = f$. Then

a. A is said to be *monotone* if

$$\langle Ax_1 - Ax_2, x_1 - x_2 \rangle \geq 0 \text{ for every } x_1, x_2 \in D(A).$$

b. A is said to be *strictly monotone* if for every $x_1, x_2 \in D(A)$

$$\langle Ax_1 - Ax_2, x_1 - x_2 \rangle = 0 \text{ implies } x_1 = x_2.$$

c. A is said to be *maximal monotone* if for every par $[x,y] \in X \times X^*$ such that

$$\langle y - A\xi, x - \xi \rangle \geq 0 \text{ for every } \xi \in D(A)$$

if follows that $y = Ax$.

d. A is said to be *hemicontinuous* if

$$\lim_{t \to 0} A(x + ty) = Ax \text{ weakly in } X^*$$

for any $x \in D(A)$ and $y \in X$ such that $x + ty \in D(A)$ for $0 \leq t \leq 1$.

Theorem 1.22. *Let X be a Banach space and let $A : X \to X^*$ be everywhere defined (i.e., $D(A) = X$), monotone and hemicontinuous. Then A is maximal monotone. In addition, if X is reflexive and A is coercive, i.e.,*

$$\lim_{\|x\| \to \infty} \frac{\langle Ax, x \rangle}{\|x\|} = \infty,$$

then $R(A) = X^*$.

2. Introduction

The *Theory of Homogenization* dates back to the late sixties, it has been very rapidly developed during the last two decades, and it is now established as a distinct discipline within mathematics.

Composites are materials that have inhomogeneities on length scales that are much larger than the atomic scale (which allows us to use the equations of classical physics at the lenght scales of the inhomogeneities) but which are essentially homogeneous at macroscopic lenght scales.

Composite materials (e.g. fibred, stratified, crystalline, porous,...) play an important role in many branches of Mechanics, Physics, Chemistry and Engineering.

The main problem is to determine macroscopic effective properties (for example heat transfer, elasticity, electric conductivity, magnetic permeability, flow, etc.) of strongly heterogeneous multiphase materials. A common feature in such problems is that the governing equations involve rapidly oscillating functions due to the heterogeneity of the underlying material, i.e. the physical parameters (such as conductivity, elasticity coefficients,...) are discontinuous and oscillate very rapidly between the different values characterizing each of the components.

These rapid oscillations render a direct numerical treatment very hard or even impossible. Therefore one has to do some kind of averaging or asymptotic analysis. We may think to get a good approximation of the macroscopic behaviour of such a heterogeneous material by letting the parameter ϵ_h , which describes the fineness of the microscopic structure, tend to zero in the equations governing phenomena such as heat conduction and elasticity. It is the purpose of homogenization theory to describe these limit processes, when ϵ_h tends to zero.

More precisely, homogenization deals with the asymptotic analysis of Partial Differential Equations of Physics in heterogeneous materials with a periodic structure, when the characteristic length ϵ_h of the period tends to zero.

3. Basic Ideas

Suppose we would like to know the stationary temperature distribution in an homogeneous body $\Omega \subset \mathbb{R}^3$ with an internal heat source f, heat conductivity a (which describes the relation between the heat current and the temperature gradient) and zero temperature on the boundary $\delta\Omega$. Or maybe we are interested in the deflection of a membrane of shape $\Omega \subset \mathbb{R}^2$ made of an homogeneous material with stiffness a, loaded by a vertical force f and clamped on the boundary $\delta\Omega$ (has deflection 0 on the boundary). A good model to describe these problems is given by the following boundary value problem: Find $u \in W_0^{1,p}(\Omega)$, $1 < p < \infty$, such that

$$-div\,(a\,(\nabla u)) = f \text{ on } \Omega, \tag{3.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, f is a given function on Ω, and $a : \mathbb{R}^n \to \mathbb{R}^n$ satisfies suitable continuity and monotonicity conditions that allows the existence and uniqueness of the solution of (3.1).

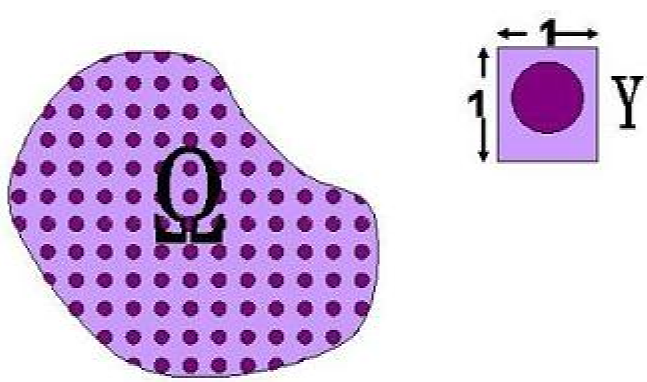
Here $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$, or the space of funtions u with boundary value 0 such that $u \in L^p(\Omega)$ and $\nabla u \in L^p(\Omega, \mathbb{R}^n)$.

Now, suppose that we would like to be able to model the case when the underlying material is heterogeneous, that is, Ω consists of a material with different properties in different positions of Ω. Then we replace a in (3.1) with a map $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ to get the equation

$$\begin{cases} -div\,(a\,(x, \nabla u)) = f & \text{on } \Omega \\ u \in W_0^{1,p}(\Omega). \end{cases} \tag{3.2}$$

Since (3.2) depends on x, this is much more difficult to handle than (3.1).

An interesting special case is a two-phase composite where one material is periodically distributed in the other. In this case the underlying periodic inclusions are often microscopic with respect to Ω. By periodicity, we can divide Ω into periodic cells Y (the microstructure of a given periodic material can be described by several different period cells). This is usually described by maps of the form $a_h(x, \xi) = a\left(\frac{x}{\epsilon_h}, \xi\right)$,



where $a(\cdot,\xi)$ is assumed to be Y-periodic and ϵ_h is the fineness of the periodic structure ($a_h(\cdot,\xi)$ is $\epsilon_h Y$-periodic). Equation (3.2) becomes

$$\begin{cases} -div\left(a_h\left(x,\nabla u_h\right)\right) = -div\left(a\left(\frac{x}{\epsilon_h},\nabla u_h\right)\right) = f \quad \text{on } \Omega \\ u_h \in W_0^{1,p}(\Omega). \end{cases} \tag{3.3}$$

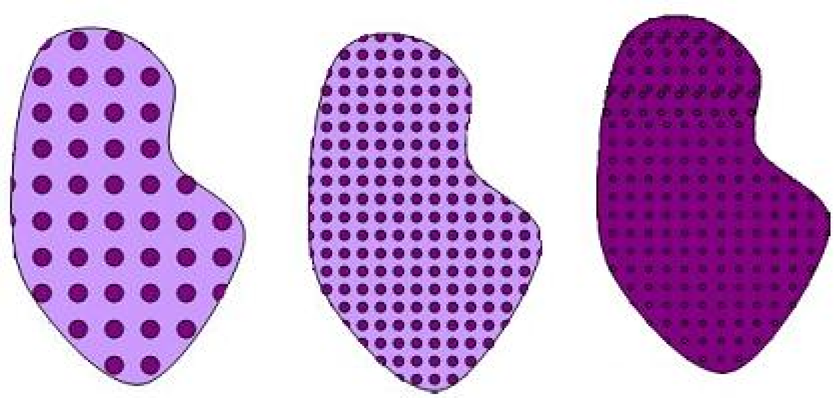
(The function u_h can be interpreted as the electric potential, magnetic potential, or the temperature and a_h describes the physical properties of the different materials constituting the body (they are the dielectric coefficients, the magnetic permeability and the thermic conductivity coefficients, respectively)).

Remark 3.1. The partial differential equations in these notes should be interpreted in the weak sense. The model problem (3.3), for instance, should be read as

$$\begin{cases} \displaystyle\int_\Omega \left\langle a_h\left(x,\nabla u_h\right),\nabla\phi\right\rangle = \int_\Omega f\phi \quad \text{for every } \phi \in W_0^{1,p}(\Omega), \\ u_h \in W_0^{1,p}(\Omega), \end{cases} \tag{3.4}$$

where $\langle\cdot,\cdot\rangle$ denotes de Euclidean inner product.

Let ϵ_h be a sequence of positive real numbers such that $\epsilon_h \to 0$ as $h \to \infty$. In this way we get a sequence of problems, one for each value of h. The larger h gets, the finer the microstructure becomes.



The natural question arises as if there is some type of convergence of the solutions u_h.

Assume that we can establish convergence in some appropiate sense, that is

$$u_h \to u, \text{ as } h \to \infty.$$

Now, do we get that u satisfies an equation of a similar type as the one u_h satisfies?

$$\begin{cases} -div\,(b\,(x, \nabla u)) = f, & \text{on } \Omega \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

If this is the case, how do we find b? For large values of h, the material macroscopically appears to behave like a homogeneous material, even though the material is strongly heterogeneous on the microscopic level. This makes it reasonable to assume that b is independent of x, which means that u satisfies an homogenized equation of the form

$$\begin{cases} -div\,(b\,(\nabla u)) = f, & \text{on } \Omega \\ u \in W_0^{1,p}(\Omega). \end{cases} \tag{3.5}$$

The "homogenized" b may be interpreted as the physical parameters of a homogeneous body, whose behaviour is equivalent, from a "macroscopic" point of view, to the behaviour of the material with the given periodic microstructure, described by (3.3).

The subject that deals with these types of questions is known as Homogenization. In particular, the convergence of partial differential operators of the type above is an important case of G-convergence of monotone operators (Introduced by Spagnolo in 1967).

Another approach to study different physical phenomena in heterogeneous materials is by using the fact that the state of the material u can be often found as the solution of a minimization problem of the form

$$E_h = \min_{u \in W_0^{1,p}(\Omega)} \left\{ \int_\Omega g\left(\frac{x}{\epsilon_h}, \nabla u(x)\right) dx - \int_\Omega f u dx \right\},$$

where the local *energy density function* $g(\cdot, \xi)$ is periodic and is assumed to satisfy the so called natural growth conditions. The convergence of this type of integral functionals is called Γ-convergence (Introduced by DeGiorgi). From the theory of Γ-convergence it follows that $E_h \to E_{hom}$, as $h \to \infty$, where

$$E_{hom} = \min_{u \in W_0^{1,p}(\Omega)} \left\{ \int_\Omega g_{hom}\left(\nabla u(x)\right) dx - \int_\Omega f u dx \right\}.$$

Here the homogenized energy density function g_{hom} is given by

$$g_{hom}(\xi) = \frac{1}{|Y|} \min_{u \in W_{per}^{1,p}(Y)} \int_Y g(x, \xi + \nabla u) dx,$$

where $W^{1,p}_{per}(Y)$ is the set of all functions $u \in W^{1,p}_{per}(Y)$ which are Y-periodic and have mean value zero. Again we note that the limit problem does not depend on x, that is, g_{hom} is the energy density function of a homogeneous material.

4. Example

To demonstrate some of the techniques and difficulties encountered in the homogenization procedure, we consider homogenization of the one dimensional Poisson equation. The homogenization process is much simpler in $\mathbb{R}$ than in higher dimensions, however it reveals the main difficulty.

Let $\Omega = (0,1)$, $f \in L^2(0,1)$, and $a \in L^\infty(0,1)$ be a measurable and periodic function with period 1 satisfying

$$0 < \beta_1 \leq a(x) \leq \beta_2 < \infty, \text{ for a.e. } x \in \mathbb{R}. \tag{4.1}$$

Moreover, we define $a_h = a\left(\frac{x}{\epsilon_h}\right)$. Then equation (3.4) takes the form

$$\begin{cases} \displaystyle\int_0^1 a_h(x) \frac{du_h}{dx}\frac{d\phi}{dx}dx = \int_0^1 f\phi dx \quad \text{for every } \phi \in W_0^{1,2}(0,1), \\ u_h \in W_0^{1,2}(0,1), \end{cases} \tag{4.2}$$

$$\begin{cases} -\dfrac{d}{dx}\left(a_h(x)\dfrac{du_h(x)}{dx}\right) = f \quad \text{in } (0,1), \\ u_h \in W_0^{1,2}(0,1), \qquad\qquad u_h(0) = u_h(1) = 0. \end{cases} \tag{4.3}$$

By a standard result in the existence theory of partial differential equations, there exists a unique solution of these problems for each h. By choosing $\phi = u_h$ in (4.2) and taking (4.1) into account, we obtain by Hölder's inequality that

$$\begin{aligned} \beta_1 \left\| \frac{du_h}{dx} \right\|^2_{L^2(0,1)} &\leq \int_0^1 a_h(x) \left| \frac{du_h(x)}{dx} \right|^2 dx \\ &= \int_0^1 f(x) u_h(x) dx \\ &\leq \|f\|_{L^2(0,1)} \|u_h\|_{L^2(0,1)} . \end{aligned}$$

The Poincaré inequality for functions with zero boundary values states that there is a constant k only depending on $\Omega = (0,1)$ such that

$$\|u_h\|_{L^2(\Omega)} \leq k \left\| \frac{du_h}{dx} \right\|_{L^2(\Omega)}.$$

This implies that

$$\|u_h\|^2_{W_0^{1,2}(\Omega)} \leq C, \tag{4.4}$$

where C ia s constant independent of h. Since $W_0^{1,2}(\Omega)$ is reflexive, there is a subsequence, still denote (u_h), such that

$$u_h \rightharpoonup u_* \text{ in } W_0^{1,2}(\Omega). \tag{4.5}$$

Since $W_0^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$, we have by Rellich embedding theorem that

$$u_h \to u_* \text{ in } L^2(\Omega).$$

In general, however, we only have that

$$\frac{du_h}{dx} \rightharpoonup \frac{du_*}{dx} \text{ in } L^2(\Omega).$$

Moreover, since a is 1-periodic, we have that (a_h) converges weakly* to its arithmetic mean $\langle a \rangle$, that is

$$a_h \rightharpoonup^* \langle a \rangle = \int_0^1 a(x)dx \text{ in } L^\infty(\Omega). \tag{4.6}$$

From (4.2), (4.5), and (4.6), it is then reasonable to assume that, in the limit, we have

$$\begin{cases} \displaystyle\int_0^1 \langle a \rangle \frac{du_*(x)}{dx} \frac{d\phi(x)}{dx} dx = \int_0^1 f(x)\phi(x)dx \quad \text{for every } \phi \in W_0^{1,2}(0,1), \\ u_* \in W_0^{1,2}(0,1). \end{cases}$$

However, this is not true in general, since $a_h \frac{du_h}{dx}$ is the product of two sequences which only converges weakly. *This is the main difficulty in the limit process.* To obtain the correct answer we proceed in the following way: first we note that, according to (4.6) and (4.4), $a_h \frac{du_h}{dx}$

is bounded in $L^2(\Omega)$ and that (4.2) implies that $-\frac{d}{dx}(a_h(x)\frac{du_h}{dx}) = f$. Hence there is a constant C independent of h such that

$$\left\| a_h \frac{du_h}{dx} \right\|_{W^{1,2}(\Omega)} \leq C.$$

Since $W^{1,2}(\Omega)$ is reflexive, there exists a subsequence, still denoted $(a_h \frac{du_h}{dx})$ and a $\gamma \in L^2(\Omega)$ such that

$$a_h \frac{du_h}{dx} \to \gamma \text{ in } L^2(\Omega).$$

This combined with the fact that $\left(\frac{1}{a_h}\right)$ converges to $\langle \frac{1}{a} \rangle$ weakly* in $L^\infty(\Omega)$ (hence weakly in $L^2(\Omega)$) gives us

$$\frac{du_h}{dx} = \frac{1}{a_h} a_h \frac{u_h}{dx} \rightharpoonup \left\langle \frac{1}{a} \right\rangle \gamma \text{ in } L^2(\Omega). \tag{4.7}$$

Thus in view of (4.5) and (4.7), we see that

$$\gamma = \frac{1}{\langle \frac{1}{a} \rangle} \frac{du_*}{dx}.$$

Now, by passing to the limit in (4.2) we obtain that

$$\begin{cases} \displaystyle\int_0^1 b \frac{du_*}{dx} \frac{d\phi}{dx} dx = \int_0^1 f(x)\phi(x) dx \quad \text{for every } \phi \in W_0^{1,2}(0,1), \\ u_* \in W_0^{1,2}(0,1). \end{cases}$$

where the homogenized operator is given by $b = \frac{1}{\langle \frac{1}{a} \rangle}$, the harmonic mean of a. Now since

$$\frac{1}{\beta_2} \leq \left\langle \frac{1}{a} \right\rangle \leq \frac{1}{\beta_1},$$

we conclude that the homogenized equation has a unique solution and thus that the whole sequence (u_h) converges.

One shall refrain from drawing the conclusion that the weak* convergence in $(L^\infty(\Omega))^{n\times n}$ of the inverse matrices A_h^{-1} of A_h is the key to the understanding of the problem in higher dimensions. In this case, the problem of passing to the limit is rather delicate and requires introduction of new techniques. One of the main tools to overcome this difficulty is the *compensated compactness* method introduced by

Murat and Tartar. This method shows that under some additional assumptions, the product of two weakly convergent sequences in $L^2(\Omega)$ converges in the sense of distributions to the product of their limits.

Remark 4.1. The corresponding homogenization problem for the one-dimensional Poisson equation

$$\begin{cases} \int_\Omega a_h(x) \left|\nabla u_h\right|^{p-2} \nabla u_h \nabla \phi dx = \int_\Omega f(x)\phi(x)dx \quad \text{for every } \phi \in W_0^{1,p}(\Omega), \\ u_h \in W_0^{1,p}(\Omega) \end{cases}$$

gives the homogenized operator $b = \left\langle a^{\frac{1}{1-p}} \right\rangle^{1-p}$.

5. Homogenization in $\mathbb{R}^n$

Assume that a satisfies suitable structure conditions.

Remark 5.1. A common assumption is that $a(x,\xi)$ satisfies the conditions

$$|a(x,\xi_1) - a(x,\xi_2)| \leq c_1\lambda(x)\left(1 + |\xi_1| + |\xi_2|\right)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha ,$$

$$(a(x,\xi_1) - a(x,\xi_2), \xi_1 - \xi_2) \geq c_2\lambda(x)\left(1 + |\xi_1| + |\xi_2|\right)^{p-\beta} |\xi_1 - \xi_2|^\beta ,$$

for c_1,$c_2 > 0$, $0 \leq \alpha \leq \min(1, p-1)$ and $\max(p, 2) \leq \beta < \infty$.

These conditions are for instance satisfied by the p-Poisson operator

$$a(x,\xi) = \lambda(x) \left|\xi\right|^{p-2} \xi,$$

where the constants c_1 and c_2 can be chosen as $c_1 = \max\left(p-1, \left(2\sqrt{2}\right)^{2-p}\right)$, $c_2 = \min\left(p-1, \left(2\sqrt{2}\right)^{2-p}\right)$.

Theorem 5.2. *Let $1 < p < \infty$ and q such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The main result for the homogenization problem connected to the Dirichlet problems

$$\begin{cases} -div\left(a_h\left(x, \nabla u_h\right)\right) = f \qquad on\ \Omega, \\ u_h \in W_0^{1,p}(\Omega), \end{cases}$$

is that

$$u_h \rightharpoonup u \ \text{in} \ W_0^{1,p}(\Omega), \tag{5.1}$$

$$a_h\left(x, \nabla u_h\right) \rightharpoonup b(\nabla u) \ \text{in} \ L^q(\Omega, \mathbb{R}^n),$$

where u is the solution of the homogenized equation

$$\begin{cases} -div\left(b\left(\nabla u\right)\right) = f & \text{on } \Omega, \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

The homogenized operator $b : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$b(\xi) = \frac{1}{|Y|}\int_Y a(x, \xi + \nabla\omega^\xi(x))dx, \tag{5.2}$$

where ω^ξ is the solution of the local problem on Y

$$\begin{cases} \displaystyle\int_Y \left\langle a\left(x, \xi + \nabla\omega^\xi\right), \nabla\phi\right\rangle dx = 0 & \text{for every } \phi \in W^{1,p}_{per}(Y), \\ \omega^\xi \in W^{1,p}_{per}(Y). \end{cases} \tag{5.3}$$

A common technique to prove this theorem is *Tartar's method of oscillating test functions* related to the notion of compensated compactness mentioned above. Another technique is the *two-scale convergence method*.

6. Correctors

Returning to (5.1), we see that $u_h - u$ converges to 0 weakly in $W_0^{1,p}(\Omega)$. The Rellich embedding theorem then implies that $u_h - u$ converges to 0 strongly in $L^p(\Omega)$. Unfortunately, in general, we do not have strong convergence of $\nabla u_h - \nabla u$ to 0 in $L^p(\Omega, \mathbb{R}^n)$.

However, it is possible to express ∇u_h in terms of ∇u up to a rest which strongly converges to 0 in $L^p(\Omega, \mathbb{R}^n)$. Results of this type are called *corrector results*.

Indeed, let $P_h : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$P_h(x, \xi) = \xi + \nabla\omega_\xi\left(\frac{x}{\epsilon_h}\right),$$

where ω_ξ is the periodic extension of the solution to the local problem (5.3).

Now we define the step function $M_h\phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$(M_h\phi)(x) = \sum_{i\in I_h} \chi_{Y_h^i}(x)\frac{1}{|Y_h^i|}\int_{Y_h^i}\phi(y)dy$$

for every $\phi \in L^p(\Omega, \mathbb{R}^n)$, where

$$Y_h^i = \epsilon_h(i + Y)$$

and

$$I_h = \left\{ i \in \mathbb{Z}^n : Y_h^i \subset \Omega \right\}.$$

Then

$$\nabla u_h - P_h\left(\cdot, M_h\nabla u\right) \to 0 \text{ in } L^p(\Omega, \mathbb{R}^n).$$

This result was proved by DalMaso and Defranceschi.

7. Some special cases with closed form expressions for the homogenized operator b

As we saw before, the homogenized operator (5.2) depends on the solution of a cell problem (5.3). There are, however, some special cases when we can get form expressions for b. We give a few of them below.

Some of these special cases are intimately connected to the concept of bounds. For simplicity, we concentrate on linear heat conductivity in the plane, that is, we consider the case $p = 2$ and $a(y, \xi) = \lambda(y)\xi$ where $\xi \in \mathbb{R}^2$.

7.1. **The Hashin structure.** In the early sixties, Hashin and Shtrikman investigated bounds for the effective properties of isotropic three dimensional mixtures with arbitrary phase geometry. Hashin and Rosen later showed that these bounds were optimal, meaning that one can find phase geometries such that they are obtained. The so called Hashin structure, which consists of coated spheres, is one such case.

We study a three-phase composite consisting of three isotropic materials, let us call them material 1,2, and 3, with conductivity

$$\lambda(x)I = \left[\alpha_1\chi_{\Omega_1}(x) + \alpha_2\chi_{\Omega_2}(x) + \alpha_3\chi_{\Omega_3}(x)\right] I$$

where χ_{Ω_i} is the characteristic function for the set Ω_i and I the unit matrix.

Let the unit cell geometry be described by

$$\Omega_1 = \{x : |x| \le r_1\}\,,\ \Omega_2 = \left\{x : r_1 \le |x| \le r_2 < \frac{1}{2}\right\},$$
$$\Omega_3 = \left\{x : |x_i| < \frac{1}{2} \wedge |x| \ge r_2, i = 1,2\right\}.$$

In order to compute the homogenized coefficients (5.2), we need to solve the cell problem (5.3)

$$-div\left(\lambda(x)\nabla\phi^{\xi}(x)\right) = 0 \text{ on } Y, \tag{7.1}$$

where $\phi^{\xi}(x) = \langle \xi, x\rangle + \omega^{\xi}(x)$ and $\omega^{\xi}(x)$ is Y-periodic.

In the case $\xi = e_1 = [1\ 0]^T$, we look for a solution of the type

$$\phi^{e_1}(x) = \begin{cases} C_1 x_1, & x \in \Omega_1, \\ x_1\left(C_2 + \frac{K_2}{|x|^2}\right), & x \in \Omega_2, \\ x_1, & x \in \Omega_3. \end{cases} \tag{7.2}$$

It is easily seen that (7.2) satisfies (7.1) on Y.

By physical reasons, the solution $\phi^{\xi}(x)$ as well as the flux $\lambda(x)\frac{\partial\phi^{\xi}}{\partial n}$ must be continuous over the boundaries $\Omega_1 \cap \Omega_2$ and $\Omega_2 \cap \Omega_3$. This gives four equations to solve for the three unknowns C_1, C_2, and K_2. In order to get a consistent solution, we get that α_3 must be

$$\alpha_3 = \alpha_2\left(C_2 - \frac{K_2}{r_2^2}\right) = \frac{\alpha_2\left(1 + \frac{\alpha_1}{\alpha_2} + m_1\left(\frac{\alpha_1}{\alpha_2} - 1\right)\right)}{1 + \frac{\alpha_1}{\alpha_2} - m_1\left(\frac{\alpha_1}{\alpha_2} - 1\right)}, \tag{7.3}$$

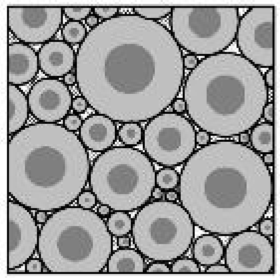
where $m_1 = \frac{r_1^2}{r_2^2}$, the volume fraction of material 1 in material 2. Since we now know the solution $\omega^{e_1}(y) = \phi^{e_1}(y) - \langle e_1, y\rangle$ of the cell problem, we can compute the homogenized coefficients

$$b(e_1) = \int_Y \lambda(x)(e_1 + \nabla\omega^{e_1})dx = [\alpha_3\ 0]^T$$

and similarly

$$b(e_2) = \int_Y \lambda(x)(e_2 + \nabla\omega^{e_2})dx = [0\ \alpha_3]^T\,.$$

This means that we can put the coated cylinder consisting of material 1 coated by material 2 into the homogeneous isotropic material 3 without changing the effective properties. By filling the whole cell with such homothetically coated cylinders, we get an isotropic two-phase composite with conductivity α_3.



We also mention the fact that if $\alpha_1 \geq \alpha_2$, then α_3 above is the lower Hashin-Shtrikman bound. By letting material 1 and 2 change places and solving a similar problem as above, we get the upper Hashin-Shtrikman bound.

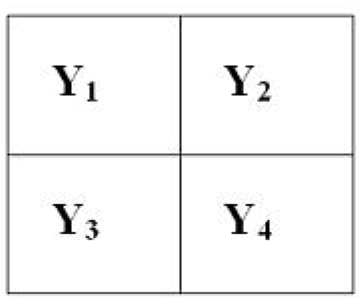
7.2. **The Mortola-Steffe structure.** We define our unit cell $Y = (0,1)^2$ and divide it into four equal parts

$$Y_1 = \left(0, \frac{1}{2}\right) \times \left(\frac{1}{2}, 1\right), \; Y_2 = \left(\frac{1}{2}, 1\right) \times \left(\frac{1}{2}, 1\right),$$
$$Y_3 = \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right), \; Y_4 = \left(\frac{1}{2}, 1\right) \times \left(0, \frac{1}{2}\right).$$

We study a four-phase composite consisting of four isotropic materials, let us call them materials 1, 2, 3, and 4, with conductivity

$$\lambda(x)I = \left[\alpha\chi_{Y_1}(x) + \beta\chi_{Y_2}(x) + \gamma\chi_{Y_3}(x) + \delta\chi_{Y_4}(x)\right] I,$$

where $\chi_{Y_i}(x)$ is the characteristic function for the set Y_i and I the unit matrix.



In 1985, Mortola and Steffe conjectured that the homogenized conductivity coefficients of this structure are

$$(\overline{\lambda}_{ij}) = \begin{pmatrix} \overline{\lambda}_{11} & 0 \\ 0 & \overline{\lambda}_{22} \end{pmatrix},$$

where

$$\overline{\lambda}_{11} = \sqrt{\frac{\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta}{\alpha+\beta+\gamma+\delta} \frac{(\alpha+\gamma)(\beta+\delta)}{(\alpha+\beta)(\gamma+\delta)}},$$

$$\overline{\lambda}_{22} = \sqrt{\frac{\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta}{\alpha+\beta+\gamma+\delta} \frac{(\alpha+\beta)(\gamma+\delta)}{(\alpha+\gamma)(\beta+\delta)}}.$$

This conjecture was proven to be true by Craster and Obnosov and independently by Milton in 2000.

If we let $\delta = \alpha$ and $\gamma = \beta$, we get the so called checkerboard structure. We immediately see that the homogenized conductivity coefficients for the checkerboard structure are

$$\overline{\lambda}_{11} = \overline{\lambda}_{22} = \sqrt{\alpha\beta},$$

the geometric mean. This was proved already in 1970 by Dykhne, but Schulgasser(1977) showed that this was a corollary of Keller's phase interchange identity in 1963.

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