that we now obtain by using (12.45) in (12.37) is a function of the two travelling wave coordinates, $x-4\kappa_1^2 t$ and $x-4\kappa_2^2 t$. It is now instructive to consider the behaviour of the solution for $t \gg 1$. Unless $x-4\kappa_1^2 t = O(1)$ or $x-4\kappa_2^2 t = O(1)$, we have

$$u(x,t) \sim -\frac{16\kappa_1^3(\kappa_1+\kappa_2)^2}{c_1^2(\kappa_1-\kappa_2)^2} e^{-16\kappa_1^3 t} \quad \text{as } t \to \infty,$$

assuming that the bound state eigenvalues are ordered so that $\kappa_1 < \kappa_2$. The solution is therefore exponentially small away from an $O(1)$ neighbourhood of the two points $x = 4\kappa_1^2 t$ and $x = 4\kappa_2^2 t$.

When $x - 4\kappa_2^2 t = O(1)$, we have $x - 4\kappa_1^2 t \sim 4(\kappa_2^2-\kappa_1^2)t \gg 1$, and

$$u(x,t) \sim -2\kappa_2^2 \operatorname{sech}^2\left\{\kappa_2(x-4\kappa_2^2 t)+\delta\right\},$$

where δ is a constant (see exercise 12.5). This is just the single soliton solution corresponding to the bound state eigenvalue κ_2, translated by an $O(1)$ distance in the x-direction. Similarly, when $x - 4\kappa_1^2 t = O(1)$ we obtain the solution for a single soliton with eigenvalue κ_1 at leading order. Analogous results hold for t large and negative. The solution corresponding to the reflectionless potential $u(x,0) = -6\operatorname{sech}^2 x$ is shown in figure 8.19. The solution for $\kappa_1 = 1.25$, $\kappa_2 = 1.75$, $c_1 = \sqrt{6}$ and $c_2 = 2\sqrt{3}$, with reflectionless potential $u(x,0)$, which is plotted in figure 12.1, is shown as a grey scale plot in figure 12.2. In each case, there are two solitary waves, widely spaced for t large and negative. The larger, faster wave catches the shorter, slower wave, there is a *nonlinear* interaction, and the waves separate as $t \to \infty$, whilst retaining their identities and emerging unchanged by the interaction. It is this particle-like behaviour that characterises these solitary waves as solitons. Figure 12.2 shows that there is a phase change due to the interaction, by which we mean that the faster wave emerges from the interaction displaced further to the right than it would have been in the absence of the slower wave, whilst the slower wave is displaced to the left. This phase change is a consequence of the nonlinearity of the interaction, and always occurs when solitons collide. When two *linear* waves interact, for example two counter-propagating solutions of the one-dimensional wave equation, $f(x-ct)+g(x+ct)$, although the waves emerge unchanged by the interaction, there is no phase change.

Example 7: General Reflectionless Potentials – the Interaction of N Solitons. The general reflectionless potential with N distinct bound state eigenvalues is given by (12.35) and (12.36). From the discussion in the

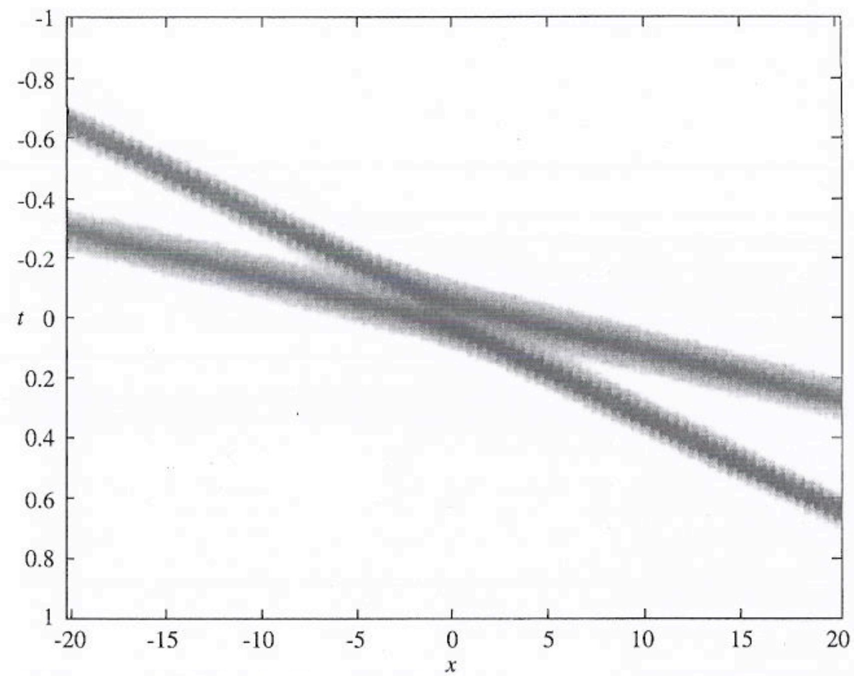

Fig. 12.2. The solution of the KdV equation corresponding to two bound states with $\kappa_1 = 1.25$, $\kappa_2 = 1.75$, $c_1(0) = \sqrt{6}$ and $c_2(0) = 2\sqrt{3}$. The larger $-u$, the darker the plot. The profile $u(x,0)$ is shown in figure 12.1.

previous subsection, it is clear that to obtain $u(x,t)$ for $t \neq 0$ we simply need to make the substitution

$$\kappa_n x \mapsto \kappa_n(x-4\kappa_n^2 t). \tag{12.46}$$

For t large and negative the solution consists of N solitons, which interact nonlinearly as t increases, eventually separating as $t \to \infty$. The reflectionless potential $u(x,0) = -N(N+1)\operatorname{sech}^2 x$ is a special case where all of the solitons combine at $t = 0$ into a profile with a single local minimum, before emerging again, all with phase shifts. A more typical interaction between three solitons is plotted in figure 12.3 for the arbitrarily chosen values $\kappa_1 = 1.25$, $\kappa_2 = 1.75$, $\kappa_3 = 2.25$, $c_1 = \sqrt{6}$, $c_2 = 2\sqrt{3}$ and $c_3 = 1$. We have used (12.35) and (12.36) to determine $u(x,t)$, with the aid of a computer algebra package. Note that, since the KdV equation, (12.1), is unchanged by the transformation $x \mapsto -x$, $t \mapsto -t$, the solution must satisfy $u(x,t) = u(-x,-t)$. This symmetry is evident in figures 8.19, 12.2

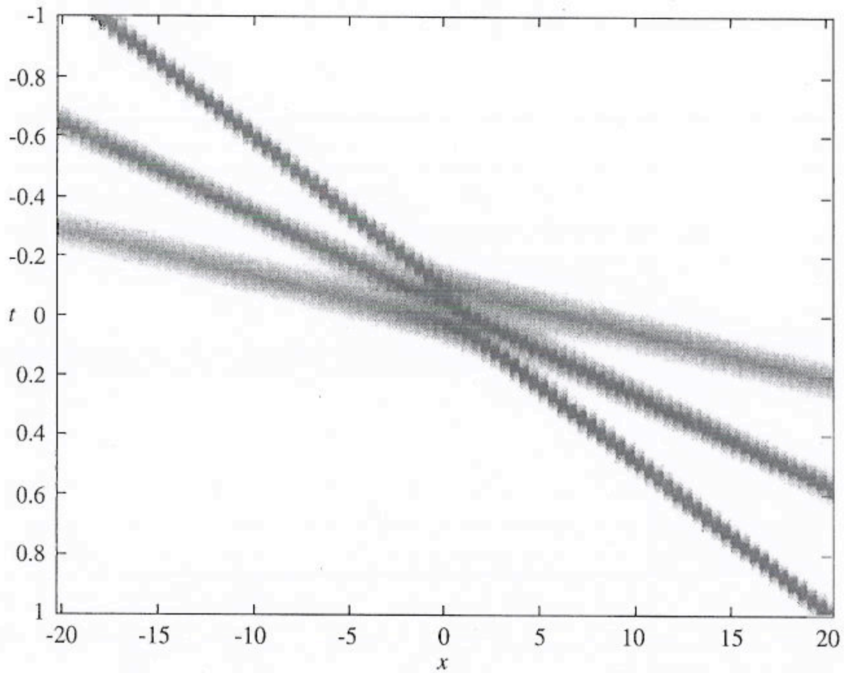

Fig. 12.3. The solution of the KdV equation corresponding to three bound states with $\kappa_1 = 1.25$, $\kappa_2 = 1.75$, $\kappa_3 = 2.25$, $c_1(0) = \sqrt{6}$, $c_2(0) = 2\sqrt{3}$ and $c_3(0) = 1$. The larger $-u$, the darker the plot.

and 12.3. The curious reader can consult the book by Shen (1993), where the first seven N soliton solutions are written out in all their glory. The $N = 7$ solution occupies nine printed pages!

Example 8: The Delta Function Potential, $u(x,0) = -U_0\delta(x)$ – the Generation of Cnoidal Waves. We have seen in example 1 that, provided $U_0 > 0$, the initial scattering data for $u(x,0) = -U_0\delta(x)$ is a single bound state eigenvalue, $\kappa = \frac{1}{2}U_0$, with normalisation constant $c(0) = \sqrt{\kappa}$ and reflection coefficient

$$\frac{b(\xi,0)}{a(\xi,0)} = -\frac{U_0}{U_0 + 2i\xi}.$$

The evolving scattering data retains its single bound state eigenvalue, whilst

$$c(t) = \sqrt{\kappa}e^{4\kappa^3 t}, \quad \frac{b(\xi,t)}{a(\xi,t)} = -\frac{U_0}{U_0 + 2i\xi}e^{8i\xi^3 t}. \tag{12.47}$$

We must now try to solve the inverse scattering problem to find the solution $u(x,t)$. The function $B(X,t)$, which is given by (12.30) and appears in the GLM equation, (12.31), is

$$B(X,t) = \kappa e^{8\kappa^3 t - \kappa X} - \frac{U_0}{2\pi}\int_{-\infty}^{\infty} \frac{e^{8i\xi^3 t + i\xi X}}{U_0 + 2i\xi}\,d\xi. \tag{12.48}$$

Now, since the GLM equation is linear, we can identify the first term in this expression, associated with the bound state eigenvalue, with a single soliton. Specifically, if we write

$$u(x,t) = -2\kappa_1^2 \operatorname{sech}^2\left\{\kappa\left(x - x_0 - 4\kappa^2 t\right)\right\} + u_c(x,t), \tag{12.49}$$

then $u_c(x,t)$ is determined in the usual way from the GLM equation with

$$B(X,t) = B_c(X,t) = -\frac{U_0}{2\pi}\int_{-\infty}^{\infty} \frac{e^{8i\xi^3 t + i\xi X}}{U_0 + 2i\xi}\,d\xi. \tag{12.50}$$

Moreover, if $U_0 < 0$, the solution is given by $u = u_c(x,t)$ alone. We would now like to know how u_c behaves for $t \gg 1$. If we consider B_c evaluated at $X = vt$, then

$$B_c(vt,t) = -\frac{U_0}{2\pi}\int_{-\infty}^{\infty} \frac{e^{it(8\xi^3 + \xi v)}}{U_0 + 2i\xi}\,d\xi,$$

and the integral can be approximated using the method of stationary phase. We find that for $v > 0$, and hence $x > 0$, the integral is exponentially small as $t \to \infty$, since there are no real points of stationary phase, whilst for $v < 0$ the integral is of $O(t^{-1/2})$, with two real points of stationary phase at $\xi = \pm\sqrt{-v/24}$. This means that B_c is uniformly small, and we can neglect the integral involving the product of K_c and B_c in the GLM equation, and find that

$$u_c(x,t) \sim 2\frac{d}{dx}B_c(2x,t) = -\frac{2iU_0}{\pi}\int_{-\infty}^{\infty} \frac{e^{8i\xi^3 t + 2i\xi x}}{U_0 + 2i\xi}\,d\xi \text{ as } t \to \infty. \tag{12.51}$$

We can use the method of stationary phase to analyse this integral, and find that u_c is exponentially small for $x > 0$, and

$$u_c(x,t) \sim \frac{4U_0}{U_0^2 + 4\xi_0^2}\sqrt{\frac{\xi_0}{24\pi t}} \times \left\{U_0 \sin\left(\frac{4}{3}\xi_0 vt + \frac{\pi}{4}\right) - 2\xi_0 \cos\left(\frac{4}{3}\xi_0 vt + \frac{\pi}{4}\right)\right\} \text{ for } t \gg 1, \tag{12.52}$$

when $x = vt < 0$, with $\xi_0 = \sqrt{-v/12}$. We conclude that the component of the solution due to the reflection coefficient, $b(\xi,t)/a(\xi,t)$, represents a cnoidal wave, whose amplitude is of $O(t^{-1/2})$ for $t \gg 1$, when it decays