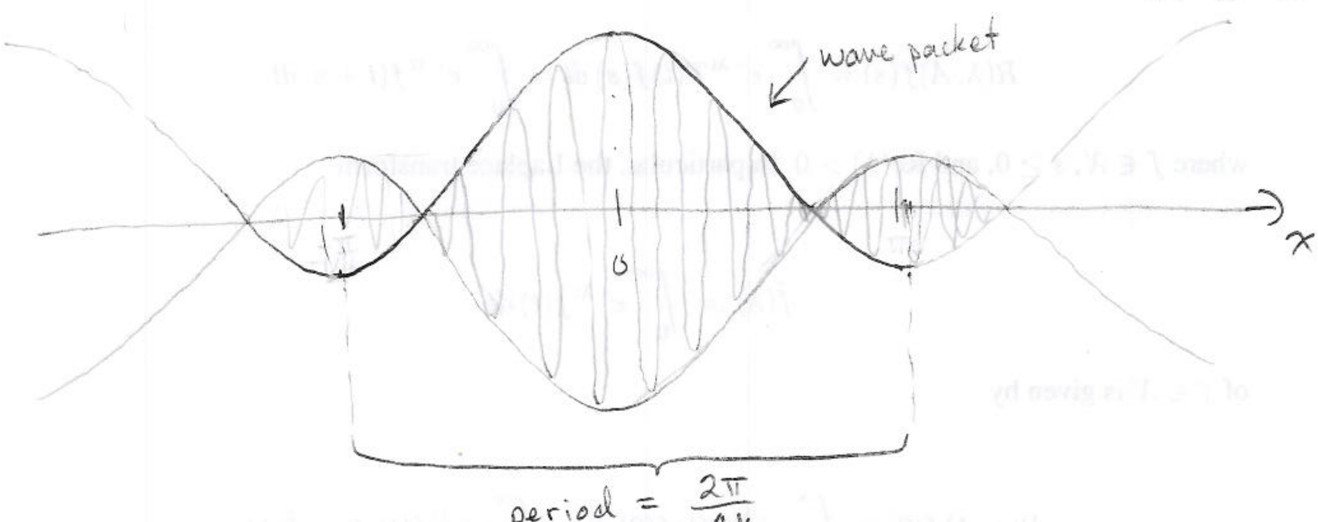
Propagation of wave packets and dispersion

Let us begin with a superposition of three sinusoidal 1D waves with equally spaced wavenumbers and frequencies, centered at wavenumber k_0 and (circular) frequency ω_0:

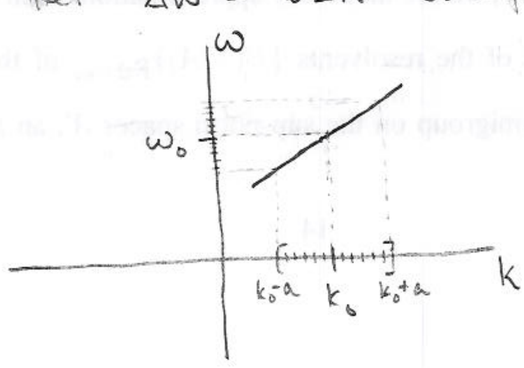
$$e^{i((k_0-\Delta k)x-(\omega_0-\Delta\omega)t)} + e^{i(k_0x-\omega_0t)} + e^{i((k_0+\Delta k)x-(\omega_0+\Delta\omega)t)}$$

$$= \underbrace{e^{i(k_0x-\omega_0t)}}_{v_{ph}=\frac{\omega_0}{k_0}}\underbrace{\left[1+2\cos(\Delta k x-\Delta\omega t)\right]}_{v_{gr}=\frac{\Delta\omega}{\Delta k}\ \text{(speed of envelope)}}$$

at $t=0$



Now let us take many sinusoidal waves of equal amplitude, centered at wavenumber k_0 and frequency ω_0, and separated by Δk and $\Delta\omega$. We take $2N+1$ wavenumbers in the interval $[k_0-a, k_0+a]$ separated by $\Delta k = \frac{a}{N}$ and let $\Delta\omega = b\Delta k$ (b a fixed real constant):



$$u(x,t) = e^{i(k_0x-\omega_0t)}\sum_{m=-N}^{N} e^{i(m\Delta k x - m\Delta\omega t)} = e^{i(k_0x-\omega_0t)}\sum_{m=-N}^{N} e^{i(m\Delta k(x-bt))}$$

$$= e^{i(k_0x-\omega_0t)}\left[1 + 2\sum_{m=1}^{N}\cos(m\Delta k(x-bt))\right] \qquad \left\{\text{set } \underline{\theta := \Delta k(x-bt)},\ \Delta k = \frac{a}{N}\right\}$$

$$= e^{i(k_0x-\omega_0t)}\left[1 + 2\sum_{m=1}^{N}\cos m\theta\right] \doteq e^{i(k_0x-\omega_0t)}\left[\cos N\theta + \frac{\sin\theta}{1-\cos\theta}\sin N\theta\right]$$

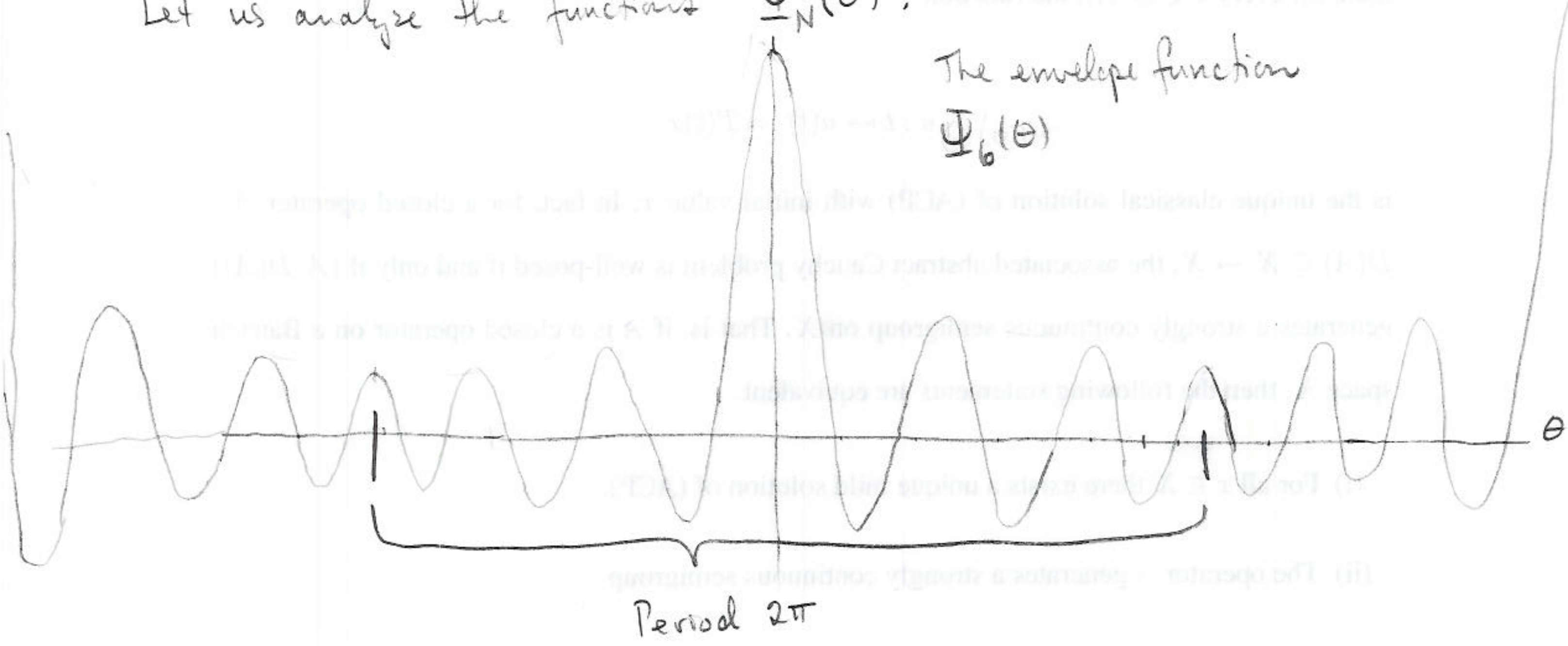
$$= e^{i(k_0x-\omega_0t)}\underbrace{\left[\cos(a(x-bt)) + \frac{\sin\left(\frac{a(x-bt)}{N}\right)}{1-\cos\left(\frac{a(x-bt)}{N}\right)}\sin(a(x-bt))\right]}_{\text{envelope function, which travels at the group velocity } \frac{\Delta\omega}{\Delta k} = b.}$$

$$= e^{i(k_0x-\omega_0t)}\, E(x,t;a,N)$$

If we define $\Psi_N(\theta) = 1 + 2\sum_{m=1}^{N}\cos m\theta = \cos N\theta + \frac{\sin\theta}{1-\cos\theta}\sin N\theta$,

then $E(x,t;a,N) = \Psi_N\left(\frac{a(x-bt)}{N}\right)$.

Let us analyze the functions $\Psi_N(\theta)$.



$\Rightarrow$ $E(x,t;a,N)$ has period $\underline{\underline{\frac{2\pi N}{a}}}$

(a) $\Phi_N(\theta)$ has prime period 2π

(b) It has N peaks in one period, with a maximum at $\theta = 2\pi M$

(c) On $[-\pi, \pi]$, $\Phi_N(\theta)$ tends to the delta-function $\delta_0(\theta)$ as $N \to \infty$

(d) $\frac{1}{N}\Phi_N\left(\frac{\theta}{N}\right) = \frac{1}{N}\cos\theta + \frac{\frac{1}{N}\sin\frac{\theta}{N}}{1-\cos\frac{\theta}{N}}\sin\theta \longrightarrow 2\frac{\sin\theta}{\theta} = 2\operatorname{sinc}\theta$ as $N \to \infty$.

The third statement means that, for each test function $\phi \in C_c^\infty(-\pi,\pi)$,

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}\Phi_N(\theta)\,\phi(\theta)\,d\theta = \phi(0).$$

This can be proved according to the following sketch:

1. $\int_a^b \Phi_N(\theta)\,d\theta \longrightarrow 0$

2. $\int_a^b \Phi_N(\theta)\,\phi(\theta)\,d\theta \longrightarrow 0 \quad \forall\, \phi \in C_c^\infty(-\pi,\pi)$

   } as $N \to \infty$ if $0 \notin [a,b]$.

   The Riemann-Lebesgue Lemma is used to prove these.

3. $\int_{-\pi}^{\pi}\Phi_N(\theta)\,d\theta = 1$

4. Given $\phi \in C_c^\infty(-\pi,\pi)$ and $\eta > 0$, there exist $\varepsilon > 0$ and M s.th., if $N > M$, then

   $$\left|\int_{-\varepsilon}^{\varepsilon}\Phi_N(\theta)\,\phi(\theta)\,d\theta - \phi(0)\right| < \eta/2 \qquad \text{(use (1), (3), and continuity of } \phi\text{)}$$

5. Now choose $M' \geq M$ s.th. for all $N > M'$,

   $$\left|\int_{-\pi}^{-\varepsilon} + \int_{\varepsilon}^{\pi}\Phi_N(\theta)\,\phi(\theta)\,d\theta\right| < \eta/2 \qquad \text{(use (2))}$$

Analysis of the envelope $E(x,t;a,N) = \Psi_N\left(\frac{a(x-bt)}{N}\right)$.

Let us set $t=0$: $E(x,0;a,N) = \Psi_N\left(\frac{ax}{N}\right) = \Psi_N(\Delta k\, x)$

<u>Case 1</u> Fixed wavenumber interval $[k_0-a, k_0+a]$ with increasingly fine collocation of k-values:

$$a \text{ fixed}$$
$$N \to \infty$$

The period of the envelope is $\frac{2\pi N}{a} = \frac{2\pi}{\Delta k}$, inversely proportional to the smallest increment in wavenumber. The wavelength of the N oscillations remains about the same: In fact, according to (d) of the previous page, if the amplitudes are scaled by $\Delta k = \frac{a}{N}$, we have the limit

$$\frac{a}{N} E(x,0;a,N) = \frac{a}{N}\Psi_N\left(\frac{ax}{N}\right) \longrightarrow 2\frac{\sin ax}{x} = 2a \operatorname{sinc} ax.$$

<u>Case 2</u> Fixed increment between wavenumbers, but increasing interval of k-values:

$$\Delta k = \frac{a}{N} \text{ fixed}$$
$$N \to \infty$$
$$a \to \infty$$

The period remains fixed at $\frac{2\pi}{\Delta k} = \frac{2\pi N}{a}$, but as $N \to \infty$, the envelope approaches a periodic array of δ-functions, one at each $x = \frac{2\pi}{\Delta k} j$, $j \in \mathbb{Z}$. Of course, the word "envelope" is no longer reasonable.

We have seen that the idea of a wave packet traveling in an envelope at a group velocity makes sense for small bands of wavenumber with closely spaced values of k. As $\Delta k \to 0$, with $[k_0 - a, k_0 + a]$ fixed, we arrive at integral superpositions of sinusoidal waves. Let us set

$$\omega = W(k) = \omega_0 + b(k - k_0)$$

$$u(x,t) = e^{i(k_0 x - \omega_0 t)} \int_{-a}^{a} e^{i(k' x - b k' t)}\, dk' = e^{i(k_0 x - \omega_0 t)} \int_{-a}^{a} \cos(k' x - b k' t)\, dk'$$

$$= e^{i(k_0 x - \omega_0 t)} \cdot 2\,\frac{\sin a(x - bt)}{x - bt} = e^{i(kx - \omega t)} \underbrace{2a \operatorname{sinc} a(x - bt)}_{\text{envelope}}$$

This envelope is, of course, the limit of the "Riemann sums" we calculated in Case 1. Notice that as a becomes large, the "envelope" becomes more oscillatory and ceases to act as a true envelope.

<u>Nonconstant amplitudes</u>

$$u(x,t) = \int_{k_0 - a}^{k_0 + a} c(k)\, e^{i(kx - W(k)t)}\, dk \quad \text{with} \quad W(k) = \omega_0 + b(k - k_0)$$

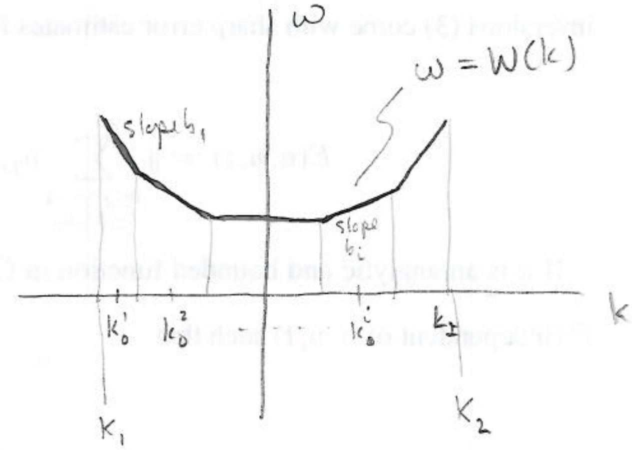
$$= e^{i(k_0 x - \omega_0 t)} \underbrace{\int_{-a}^{a} c(k_0 + k)\, e^{i k'(x - bt)}\, dk'}_{\substack{\text{envelope traveling at} \\ \text{group velocity } b.}}$$

Study case for dispersion

Suppose that the "dispersion relation" $\omega = W(k)$ is piecewise linear.

$$u(x,t) = \int_{k_1}^{k_2} c(k)\, e^{i(kx + W(k)t)}\, dk =$$

$$\sum_{i=1}^{I} e^{i(k_0^i x - \omega_0^i t)} \int_{-k_i'}^{k_i'} c(k_0^i + k')\, e^{ik'(x - b_i t)}\, dk'$$



The function $u(x,t)$ is a linear integral superposition of wavepackets traveling at the speeds b_i given by the slopes of the pieces of the dispersion relation.

Gaussian wave packets

If the wavenumbers are distributed in a Gaussian way about k_0,

$$c(k) = e^{-a|k-k_0|^2},$$

we obtain a Gaussian envelope:

$$u(x,t) = \int_{-\infty}^{\infty} e^{-a|k-k_0|^2} e^{i(kx - (\omega_0 + b(k-k_0))t)}\, dk \qquad (k' = k - k_0)$$

$$= e^{i(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} e^{-a|k'|^2} e^{ik'(x-bt)}\, dk'$$

$$= e^{i(k_0 x - \omega_0 t)} \left(\frac{\pi}{a}\right)^{1/2} e^{-\frac{|x-bt|^2}{4a}}$$

Rule: $\left\{ \begin{array}{l} \text{narrow distribution of wavenumbers} \rightsquigarrow \text{wide spatial envelope} \\ \text{wide distribution of wavenumbers} \rightsquigarrow \text{narrow spatial envelope.} \end{array} \right.$

Stationary-phase analysis for long-time asymptotics of the solution to $\{ u_t + a u_x + b u_{xxx} = 0, \quad u(x,0) = u_0(x) \}$.

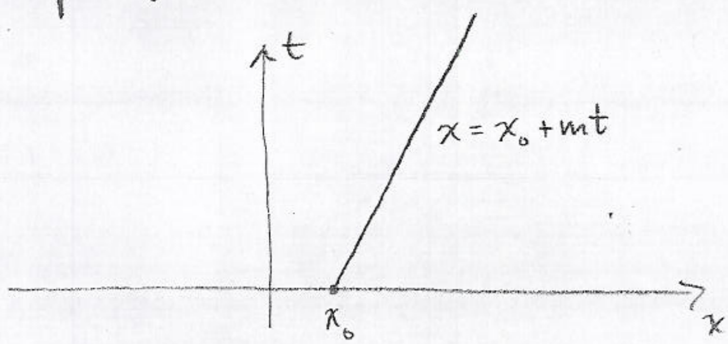
The solution is

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}_0(k)\, e^{i(kx - W(k)t)}\, dk,$$

where $\hat{u}_0$ is the Fourier transform of u_0 and

$$W(k) = ak - bk^3$$

is the dispersion relation.



We examine the long-time asymptotics of u along a path traveling at a velocity m, that is, we invesigate

$$u(x_0 + mt, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}_0(k)\, e^{ikx_0}\, e^{i(km - W(k))t}\, dk$$

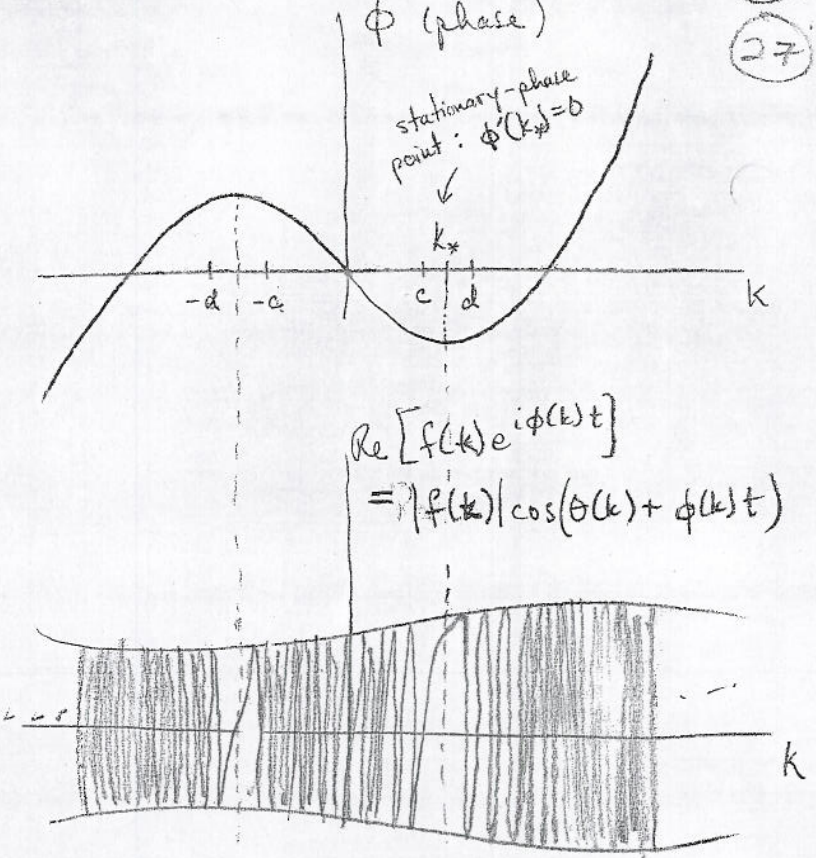
as $t \to \infty$.

Letting $f(k) = \frac{1}{\sqrt{2\pi}} \hat{u}_0(k) e^{ikx_0}$ and $\phi(k) = km - W(k) = (m-a)k + bk^3$, the integral becomes

$$\int_{-\infty}^{\infty} f(k)\, e^{i\phi(k)t}.$$

$$\int_{-\infty}^{\infty} f(k)e^{i\phi(k)t}\,dk =$$

$$= \left[\int_{-\infty}^{-d} + \int_{-d}^{-c} + \int_{-c}^{c} + \int_{c}^{d} + \int_{d}^{\infty}\right] f(k)e^{i\phi(k)t}\,dk$$



The integrals $\int_{-\infty}^{-d}$, $\int_{-c}^{c}$,

and $\int_{d}^{\infty}$ are treated similarly.

We will do $\int_{d}^{\infty}$:

$$\int_{d}^{\infty} f(k)e^{i\phi(k)t}\,dk = \int_{d}^{\infty}\left(\frac{f(k)}{\phi'(k)}\right)\phi'(k)e^{i\phi(k)t}\,dk =$$

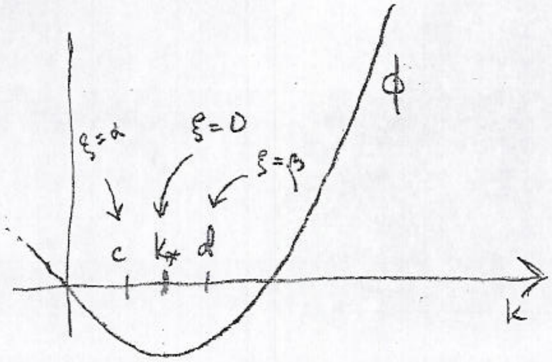
$$= \frac{1}{it}e^{i\phi(k)t}\left(\frac{f(k)}{\phi'(k)}\right)\Bigg|_{k=d}^{\infty} - \frac{1}{it}\int_{d}^{\infty}\frac{d}{dk}\left(\frac{f(k)}{\phi'(k)}\right)e^{i\phi(k)t}\,dk = O\left(\frac{1}{t}\right).$$

$$(t\to\infty)$$

[We assume $\frac{f(k)}{\phi'(k)} \to 0$ as $k\to\infty$ and that $\frac{d}{dk}\left(\frac{f(k)}{\phi'(k)}\right) \in L^1$.]

We see that the integrals that do not contain contributions from points of stationary phase decay as $\frac{1}{t}$ as $t\to\infty$.

Now we examine the integrals $\int_{-d}^{-c}$ and $\int_{c}^{d}$; they are handled similarly. We use $\int_c^d$ to illustrate.



We make a change of coordinate near k_* in which ϕ becomes quadratic:

$$\phi(k) = \phi(k_*) + \frac{\phi''(k_*)}{2}(k-k_*)^2 + \mathcal{O}\left((k-k_*)^3\right)$$

$$= \phi(k_*) + \frac{\phi''(k_*)}{2}(k-k_*)^2\left[1 + \mathcal{O}(k-k_*)\right]$$

Put $\xi = (k-k_*)\left[1+\mathcal{O}(k-k_*)\right]^{1/2} = (k-k_*)\left[1+\mathcal{O}(k-k_*)\right]$, $c \le k \le d$

and let $k = K(\xi)$ be the inverse map, $\alpha \le \xi \le \beta$.

Then $\phi(K(\xi)) = \phi(k_*) + \frac{\phi''(k_*)}{2}\xi^2$, and $K(0) = k_*$, $K'(0) = 1$.

Put $\gamma := \frac{\phi''(k_*)}{2}$. Now we transform the integral:

$$\int_c^d e^{i\phi(k)t} f(k)\,dk = \int_\alpha^\beta e^{i\phi(K(\xi))t} f(K(\xi))K'(\xi)\,d\xi = e^{i\phi(k_*)t}\int_{-\infty}^{\infty} e^{i\gamma\xi^2 t} g(\xi)\,d\xi,$$

(where $g(\xi) = \chi_{[\alpha,\beta]}(\xi) f(K(\xi))K'(\xi)$, so $g(0) = f(k_*)$)

$$= e^{i\phi(k_*)t} \lim_{\delta \to 0} \int_{-\infty}^{\infty} e^{i\gamma \xi^2 t - \delta \xi^2} g(\xi)\, d\xi \qquad \text{(Dominated convergence thm)}$$

$$= e^{i\phi(k_*)t} \lim_{\delta \to 0} \int_{-\infty}^{\infty} \left(\frac{1}{2(\delta - i\gamma t)}\right)^{1/2} e^{\frac{y^2}{4(i\gamma t - \delta)}} \hat{g}(y)\, dy \qquad \left[\int u(\xi) v(\xi) d\xi = \int \hat{u}(y) \hat{v}(-y)\, dy \right.$$

and

$$\frac{1}{\sqrt{2\pi}} \int e^{-w\xi^2 - iy\xi}\, d\xi = \left.\left(\frac{1}{2w}\right)^{1/2} e^{-\frac{y^2}{4w}} \right]$$

$\mathrm{Re}(w) > 0$, $-\pi/4 < \mathrm{Arg}(w^{1/2}) < \pi/4$

$$= e^{i\phi(k_*)t} \left(\frac{1}{-2i\gamma t}\right)^{1/2} \int_{-\infty}^{\infty} e^{\frac{-iy^2}{4\gamma t}} \hat{g}(y)\, dy$$

$$= e^{i\phi(k_*)t} \left(\frac{1}{-2i\gamma t}\right)^{1/2} \left[\int_{-\infty}^{\infty} \hat{g}(y)\, dy + \int_{-\infty}^{\infty} O(y^2/t)\, dy \right]$$

$$= e^{i\phi(k_*)t} \left(\frac{1}{-2i\gamma t}\right)^{1/2} \left[\sqrt{2\pi}\, g(0) + O\!\left(\frac{1}{t}\right) \right].$$

$$= \sqrt{\frac{2\pi}{|\phi''(k_*)|\, t}}\; e^{i\frac{\pi}{4}\operatorname{sgn}(\phi''(k_*))}\, e^{i\phi(k_*)t} \left(f(k_*) + O(1/t) \right) \qquad (t \to \infty)$$

With this result, we obtain large-time asymptotics for u along $x = x_0 + mt$ by using $f(k) = \frac{1}{\sqrt{2\pi}} \hat{u}_0(k) e^{ikx_0}$ and taking the sum of contributions from both points of stationary phase, $k_\pm = \pm \sqrt{\frac{a-m}{3b}}$.

$$u(x_0+mt,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}_0(k)e^{ikx_0}\, e^{i(km-W(k))t}\, dk$$

$$= \sqrt{\frac{1}{|\phi''(k_-)|t}}\; e^{i\frac{\pi}{4}\operatorname{sgn}(\phi''(k_-))}\; e^{i\phi(k_-)t}\; \hat{u}_0(k_-)e^{ik_-x_0} \;+$$

$$+\sqrt{\frac{1}{|\phi''(k_+)|t}}\; e^{i\frac{\pi}{4}\operatorname{sgn}(\phi''(k_+))}\; e^{i\phi(k_+)t}\; \hat{u}_0(k_+)e^{ik_+x_0}$$

$$+\;\mathcal{O}\left(\frac{1}{t}\right) \qquad (t\to\infty).$$

If $u_0(x)$ is <u>real</u>, then $\hat{u}_0(-k) = \overline{\hat{u}_0(k)}$.

Also, $k_+ = -k_-$, $\phi(k_+) = -\phi(k_-)$, and $\phi''(k_+) = -\phi''(k_-)$, so

(*) $$\boxed{u(x_0+mt,t) = 2\,\mathrm{Re}\left[\sqrt{\frac{1}{|\phi''(k_+)|t}}\; e^{i\frac{\pi}{4}\operatorname{sgn}(\phi''(k_+))}\; e^{i\phi(k_+)t}\; \hat{u}_0(k_+)e^{ik_+x_0}\right] + \mathcal{O}\left(\frac{1}{t}\right)}$$

If $u_0(x)$ is real and symmetric $(u_0(-x) = u_0(x))$, then $\hat{u}_0(k)$ is real and symmetric, so

(+) $$\boxed{u(x_0+mt,t) = \frac{2}{\sqrt{|\phi''(k_+)|t}}\,\hat{u}_0(k)\cos\left(\phi(k_+)t + k_+x_0 + \pi/4\right) + \mathcal{O}(1/t).}$$

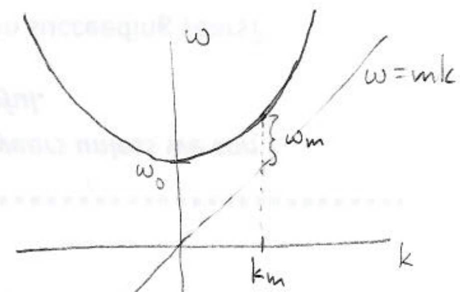
Here, k_+, $\phi(k_+)$, and $\phi''(k_+)$ can all be calculated easily from $\phi(k) = (m-a)k + bk^3$.

(*) is valid for $m < a$.

Quadratic dispersion : stationary phase analysis in the simplest case.

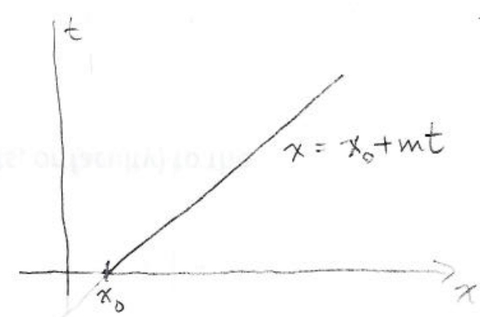
First, a few facts on the Fourier transform, in 1D valid whenever the integrals make sense:

- $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx$; $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{i\xi x}\, d\xi \;=:\; \check{\hat{f}}(x)$
- $f(x) = e^{-wx^2} \implies \hat{f}(\xi) = \left(\frac{1}{2w}\right)^{1/2} e^{-\frac{\xi^2}{4w}}$, $\operatorname{Re} w > 0$, $-\pi/4 < \arg(w^{1/2}) < \pi/4$
- $\int f(x) g(x)\, dx = \int \hat{f}(\xi)\, \hat{g}(-\xi)\, d\xi$



dispersion relation $\omega = W(k) = \omega_0 + \gamma k^2$

$$u(x,t) = \int_{-\infty}^{\infty} c(k)\, e^{i(kx - W(k)t)}\, dk, \text{ where } c(k) = \frac{1}{\sqrt{2\pi}}\hat{u}(x,0)$$



[Idea : Travel at speed m and follow what happens to the wave. If $\omega = \omega_0 + mk$, $x = x_0 + mt$,

then $e^{i(kx-\omega t)} = e^{i(kx_0 - \omega_0 t)}\, \underbrace{\cancel{e^{i(kmt - mkt)}}}_{1}$,]

In u, put $x = x_0 + mt$. The phase in the integrand is

$$\text{phase} = k(x_0+mt) - (\omega_0 + \gamma k^2)t = kx_0 - t(\omega_0 - mk + \gamma k^2) = kx_0 - t(\omega_m + \gamma(k-k_m)^2),$$

$$\begin{cases} \omega_m = \omega_0 - \frac{m^2}{4\gamma} \\ k_m = \frac{m}{2\gamma} \end{cases}$$

$$u(x_0+mt, t) = e^{-i\omega_m t}\int c(k)\, e^{ikx_0}\, e^{-it\gamma(k-k_m)^2}\, dk$$

[Put $\xi = k - k_m$, $c(k)e^{ikx_0} = a(\xi)$] $= e^{-i\omega_m t}\int a(\xi)\, e^{-it\gamma\xi^2}\, d\xi$

$\boxed{\text{Assume } a \in L^1}$

$$= e^{-i\omega_m t}\lim_{\varepsilon\to 0}\int a(\xi)\, e^{-it\gamma\xi^2 - \varepsilon\xi^2}\, d\xi = e^{-i\omega_m t}\lim_{\varepsilon\to 0}\left[2(it\gamma+\varepsilon)\right]^{-1/2}\int \check{a}(y)\, e^{-\frac{y^2}{4(it\gamma+\varepsilon)}}\, dy$$

$$= e^{-i\omega_m t}\left[2it\gamma\right]^{-1/2}\underbrace{\int \check{a}(y)\, e^{-\frac{iy^2}{4\gamma t}}\, dy}_{I}$$

$\left[(2it\gamma)^{1/2} = (2t|\gamma|)^{1/2}\exp\left(i\frac{\pi}{4}\operatorname{sgn}\gamma\right)\right]$

$$I = \underbrace{\int \check{a}(y)\,dy}_{a(0)} + \underbrace{\int \check{a}(y)\underbrace{\left(e^{\frac{-iy^2}{4\gamma t}} - 1\right)}_{\phi(y^2/t)}dy}_{E}$$

$$E = \int_{-\infty}^{-t^{1/4}} + \int_{-t^{1/4}}^{t^{1/4}} + \int_{t^{1/4}}^{\infty} \check{a}(y)\,\phi(y^2/t)\,dy$$

$$E_1 \qquad E_2 \qquad E_3$$

To estimate E_2, $\phi(y^2/t) = O(y^2/t)$, and, for $|y| < t^{1/4}$, $\phi(\frac{y^2}{t}) = O(\frac{1}{t^{1/2}})$

Thus, $|E_2| < \int_{-t^{1/4}}^{t^{1/4}} |\check{a}(y)|\, O(1/t^{1/2})\,dy \leq O(1/t^{1/2}) \int_{-\infty}^{\infty} |\check{a}(y)|\,dy = O(1/t^{1/2})$

because $\hat{a} \in L^1$.

To estimate E_3 (E_1 similar)

$$E_3 = \int_{t^{1/4}}^{\infty} \check{a}(y)\, e^{-\frac{iy^2}{4\gamma t}}\,dy$$

$\left[\text{Assume } \check{a}(y) < \frac{\text{const}}{|y^3|} \text{ for } y \text{ suff large}\right]$

$$|E_3| < \int_{t^{1/4}}^{\infty} |\check{a}(y)|\,dy < \int_{t^{1/4}}^{\infty} \frac{\text{const}}{y^3}\,dy \quad (\text{for } t \text{ suff. large})$$

$$= \left[\frac{\text{const}}{y^2}\right]_{t^{1/4}}^{\infty} = \frac{\text{const}}{t^{1/2}}$$

Thus, $I = a(0) + O(\frac{1}{t^{1/2}}) = c(k_m)e^{ik_m x_0} + O(1/t^{1/2})$

$$\boxed{u(x_0 + mt, t) = e^{-i\omega_m t}\,\frac{\exp(-i\frac{\pi}{4}\operatorname{sgn}\gamma)}{[2it\gamma]^{1/2}}\left[c(k_m)e^{ik_m x_0} + O(1/t^{1/2})\right]}$$

$\frac{1}{t^{1/2}}$ here.