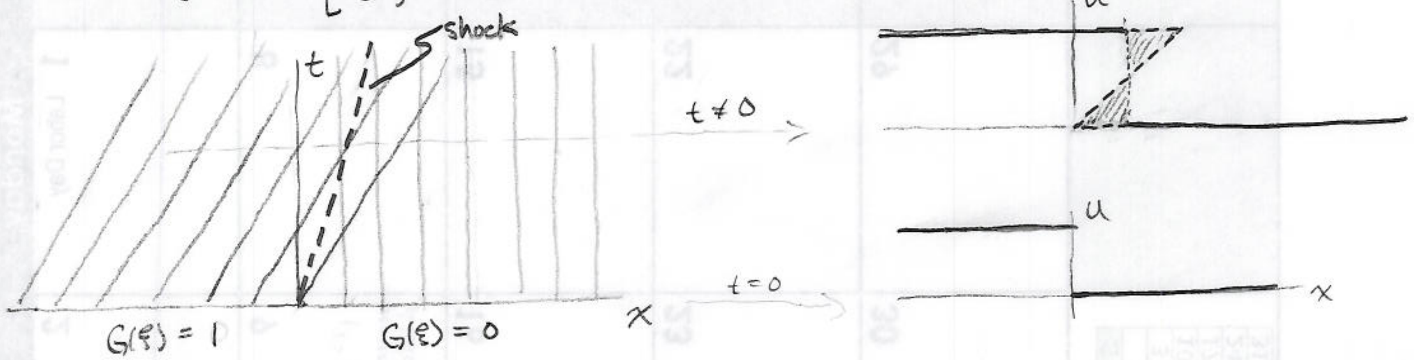
Two simple examples

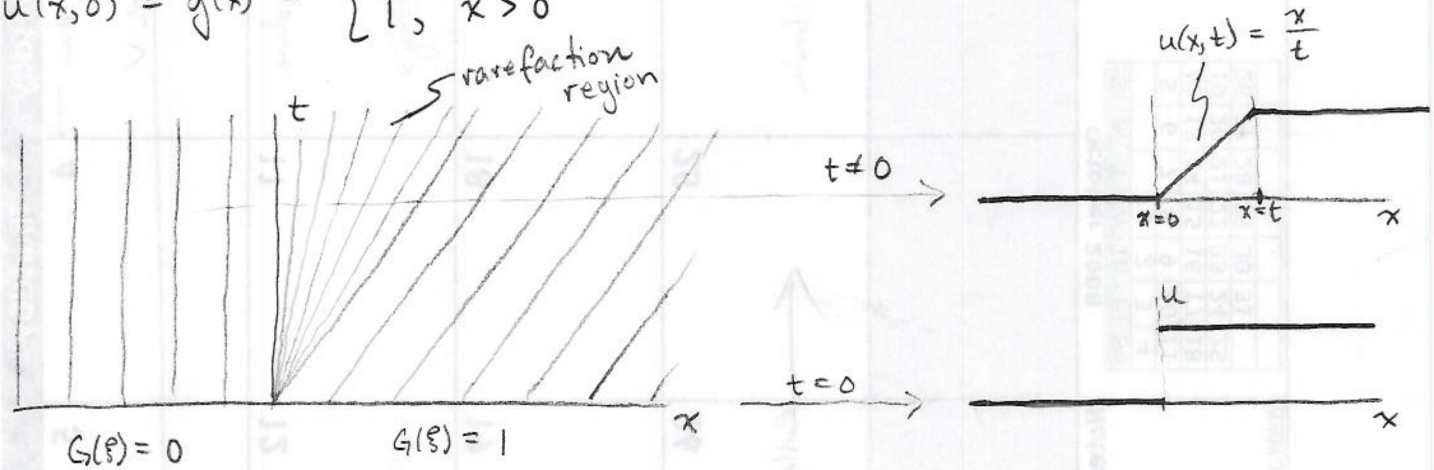
For the equation $u_t + uu_x = 0$, consider two initial functions

① $u(x,0) = g(x) = \begin{cases} 1, & x<0 \\ 0, & x>0 \end{cases}$.



The characteristic lines cross in the region $0 < x < t$, and the shock is at $x = \frac{1}{2}t$.

② $u(x,0) = g(x) = \begin{cases} 0, & x<0 \\ 1, & x>0 \end{cases}$.



In the region $0 < x < t$, the initial data defines no characteristic lines. By filling them in in a linear manner, we may define $u(x,t) = \frac{x}{t}$ in this region — this satisfies $u_t + uu_x = 0$.

This is an instance of <u>rarefaction</u> of the gas.

An ideal gas in a thin tube

See Section 7.2.3 of J. Billingham and A.C. King, *Wave Motion*.

In $\mathbb{R}^3$, the dynamical equations for an ideal gas without viscosity are (with ρ = density, $\bar{u}$ = velocity)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{u}) = 0$$

(conservation of mass, with flux density $F = \rho \bar{u}$)

$$\frac{\partial \bar{u}}{\partial t} + \underbrace{\bar{u} \cdot \nabla \bar{u}}_{\substack{\text{advection} \\ \text{of velocity}}} + \underbrace{k \gamma \rho^{\gamma - 2} \nabla \rho}_{\substack{\text{acceleration due} \\ \text{to forcing by gradient} \\ \text{of density (related to pressure)}}} = 0$$

In 1D (a thin tube), the equations are

$$\rho_t + (\rho u)_x = 0$$

$$u_t + \underbrace{u u_x}_{\substack{\text{notice that } u u_x \\ \text{arises from advection} \\ \text{of velocity } u}} + k \gamma \rho^{\gamma - 2} \rho_x = 0$$

This system of nonlinear first-order PDEs can be written in Riemann-invariant form using the functions u and the local sound speed

$$c = (\gamma k \rho^{\gamma - 1})^{1/2} \quad :$$

$$\text{(RI)} \qquad \begin{aligned} \frac{\partial}{\partial t}(u + \delta c) + (u + c) \frac{\partial}{\partial x}(u + \delta c) &= 0, \\ \frac{\partial}{\partial t}(u - \delta c) + (u - c) \frac{\partial}{\partial x}(u - \delta c) &= 0, \end{aligned}$$

in which $\delta = \frac{2}{\gamma - 1}$. The functions $u + \delta c$ and $u - \delta c$ are the Riemann invariants; they are invariant along the characteristic lines of speeds $u + c$ and $u - c$.

Let us solve the problem of a gas in a tube, extending from $x=0$ to ∞, that is initially at rest, $u(x,0)=0$, with uniform sound speed $c(x,0)=c_0$. At time $t=0$, a piston at the end is pushed into the gas at a constant speed of V.

The simple initial condition allows for a simplification of the system (RI) to one function of x and t: By setting

$$u = \delta(c-c_0)$$

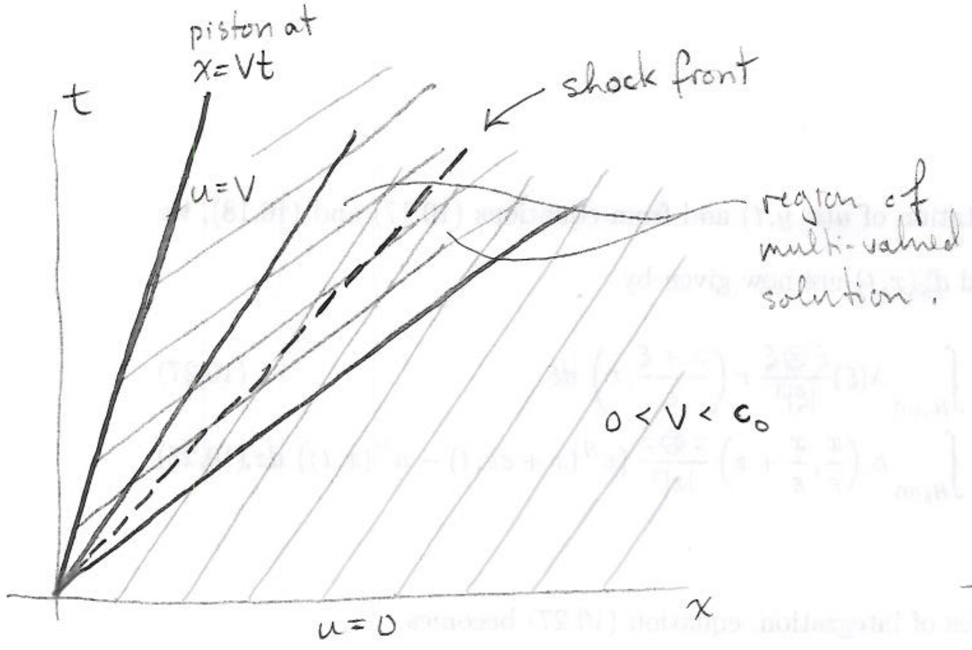
throughout the gas, the second equation and the initial condition are satisfied. The first equation reduces to

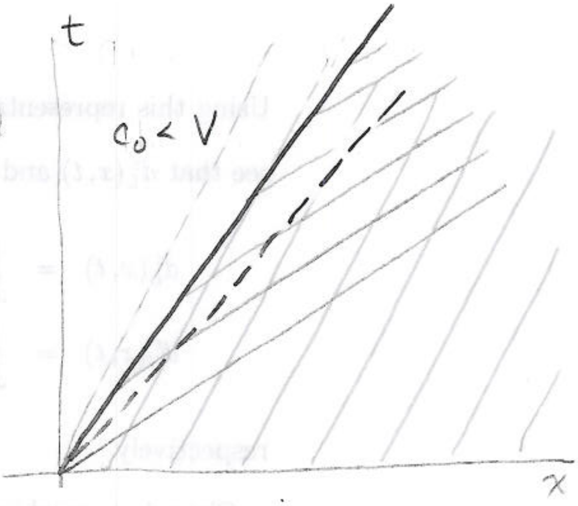
$$\frac{\partial u}{\partial t} + \left((1+\delta^{-1})u + c_0\right)\frac{\partial u}{\partial x} = 0$$

The characteristic lines for this equation are determined by the values of u on the lines $\{t=0\}$ and $\{x=Vt\}$, the latter being the position of the piston, at which $u=V$:

$$u(x,0) = 0$$
$$u(Vt,t) = V$$

If $V>0$, the characteristic lines emanating from the piston are faster than those emanating from $t=0$, and thus a shock occurs (see the figure).





The placement of the shock is determined by the rule of equal areas, as in the simple example ① on page 43.

The viscid Burgers equation: tempering shocks by diffusion.

$$\text{(B)} \qquad u_t + \underbrace{u u_x}_{\substack{\text{advection}\\ \text{of velocity}}} = \underbrace{\nu u_{xx}}_{\substack{\text{diffusion due}\\ \text{to viscocity}}} \quad ; \quad u(x,0) = F(x).$$

The transformation of dependent variable

$$u(x,t) = -2\nu(\log \varphi)_x$$

$$\varphi(x,t) = \exp\left(-\frac{1}{2\nu}\int_0^x u(y,t)\,dy\right)$$

converts (B) into the diffusion equation

$$\varphi_t = \nu \varphi_{xx},$$

$$\varphi(x,0) = \exp\left(-\frac{1}{2\nu}\int_0^x F(y)\,dy\right).$$