Some Sobolev Space Theory

The Sobolev space of k times weakly differentiable functions on Ω derivatives in $L^p(\Omega)$:

$$W^{k,p}(\Omega) = \left\{ u \in \mathcal{S}'(\Omega) : \partial^\alpha u \in L^p(\Omega) \ \forall \ |\alpha| \le k \right\},$$

with norm

$$\|u\|_{k,p} = \left(\sum_{|\alpha| \le k} \|\partial^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p}.$$

The L^2-Sobolev spaces are Hilbert spaces:

$$H^k(\Omega) = W^{k,2}(\Omega) = \left\{ u \in \mathcal{S}'(\Omega) : \partial^\alpha u \in L^2(\Omega) \ \forall \ |\alpha| \le k \right\},$$

with inner product

$$\langle u, v \rangle_{H^k} = \sum_{|\alpha| \le k} \int_\Omega \partial^\alpha u \, \partial^\alpha \bar{v} \, dx$$

and norm

$$\|u\|_{H^k} = \left(\sum_{|\alpha| \le k} \int |u|^2 dx \right)^{1/2}.$$

Some authors (as Folland) make a distinction in the definitions of H^k and $W^{k,2}$, but the definitions always coincide when $\partial\Omega$ is C^1 (see Thm 6.39 of Folland PDE).

If $\Omega = \mathbb{R}^n$, H^k can be characterized through the Fourier transform:

$$H^k(\mathbb{R}^n) = \left\{ u \in L^2 : (i\xi)^\alpha \hat{u} \in L^2 \quad \forall\ |\alpha| \le k \right\}.$$

By the Plancherel Theorem ($\mathcal{F} : L^2 \to L^2$ is unitary),

$$\|u\|_{H^k} = \Big(\sum_{|\alpha| \le k} \|(i\xi)^\alpha \hat{u}\|_{L^2}^2 \Big)^{1/2} = \Big[\int |\hat{u}(\xi)|^2 \sum_{|\alpha| \le k} |\xi^\alpha|^2 \Big]^{1/2}.$$

One can show that $\exists$ positive numbers C_1, C_2 such that

$$C_1 \sum_{|\alpha| \le k} |\xi^\alpha|^2 \le (1+|\xi|^2)^k \le C_2 \sum_{|\alpha| \le k} |\xi^\alpha|^2$$

and thereby conclude that that the H_k norm is <u>equivalent</u> <u>to</u> the norm

$$\|u\|_{H^k} \sim \Big[\int (1+|\xi|^2)^k |\hat{u}(\xi)|^2 \, d\xi \Big]^{1/2}$$

and that

$$u \in H^k(\mathbb{R}^n) \iff (1+|\xi|^2)^{k/2} \hat{u} \in L^2(\mathbb{R}^n).$$

This motivates the definition of fractional Sobolev spaces: For $s \in \mathbb{R}$,

$$H^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}' : (1+|\xi|^2)^{s/2} \hat{u} \in L^2 \right\},$$

with inner product

$$\langle u, v \rangle_{H^s} = \int (1+|\xi|^2)^s \hat{u} \bar{\hat{v}} \, dx$$

and norm

$$\|u\|_{H^s} = \Big[\int (1+|\xi|^2)^s |\hat{u}|^2 \, d\xi \Big]^{1/2}.$$

- <u>Fact</u> $H^s(\mathbb{R}^n)$ is a Hilbert space, that is, it is complete in the H^s-norm.

<u>Pf</u> Suppose that $\{u_i\}$ is a Cauchy sequence in H^s, that is, $\|u_i - u_j\|_{H^s} \to 0$ as $i,j \to \infty$.

By def of $\|\cdot\|_{H^s}$, this is equivalent to

$$\|(1+|\xi|^2)^{s/2}\hat{u}_i - (1+|\xi|^2)^{s/2}\hat{u}_j\|_{L^2} \to 0 \text{ as } i,j \to \infty.$$

Since L^2 is complete, $\exists\ v \in L^2$ s.th.

$$\|(1+|\xi|^2)^{s/2}\hat{u}_i - \hat{v}\|_{L^2} \to 0 \text{ as } i \to \infty.$$

Define u by $\hat{u}(\xi) = \hat{v}(\xi)(1+|\xi|^2)^{-s/2}$.

Then $u \in H^s$ and

$$\|u_i - u\|_{H^s} = \|\hat{u}_i(1+|\xi|^2)^{s/2} - \hat{u}(1+|\xi|^2)^{s/2}\|_{L^2}$$

$$= \|\hat{u}_i(1+|\xi|^2)^{s/2} - \hat{v}\|_{L^2} \to 0 \text{ as } i \to \infty.$$

Thus $\{u_i\}$ converges to $u \in H^s$.

There is a natural duality between H^s and H^{-s}:

For $u \in H^s$ and $v \in H^{-s}$, we have $\hat{u}(1+|\xi|^2)^{s/2} \in L^2$ and $\hat{v}(1+|\xi|^2)^{-s/2} \in L^2$, so $\hat{u}\bar{\hat{v}} \in L^1$, and we define

$$\langle u | v \rangle = \int \hat{u}(\xi)\bar{\hat{v}}(\xi)\, d\xi$$

which is bounded in both variables:

$$\langle u | v \rangle = \int \hat{u}(\xi)(1+|\xi|^2)^{s/2}\, \bar{\hat{v}}(\xi)(1+|\xi|^2)^{-s/2} d\xi \leq C \|u\|_{H^s} \|v\|_{H^{-s}}$$ [by Cauchy-Schwarz ineq.

In fact, for each $w \in H^s$, we obtain $v \in H^{-s}$ through

$$\hat{v}(\xi) = \hat{w}(\xi)(1+|\xi|^2)^s,$$

and obtain $\langle u, w \rangle_{H^s} = \langle u | v \rangle$,

which shows that H^{-s} is isomorphic to the dual of H^s.

Set $\mathbb{R}^{n+} = \{(x', z) \in \mathbb{R}^n : z > 0\}$. $[x' = (x_1, \ldots, x_{n-1}),\ z = x_n]$

The partial Fourier transform of $u \in \mathcal{S}(\overline{\mathbb{R}^{n+}})$:

$$\hat{u}(\xi, z) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int u(x', z) e^{-i\xi\cdot x'} dx', \quad \xi \in \mathbb{R}^{n-1}.$$

- $(\partial_{x'}^{\alpha_1} \partial_z^{\alpha_2} u)^\wedge(\xi, z) = |i\xi|^{\alpha_1} \partial_z^{\alpha_2} \hat{u}(\xi, z) \in L^2(\mathbb{R}^{n+}) \quad \forall\ |\alpha_1| + \alpha_2 \leq k$

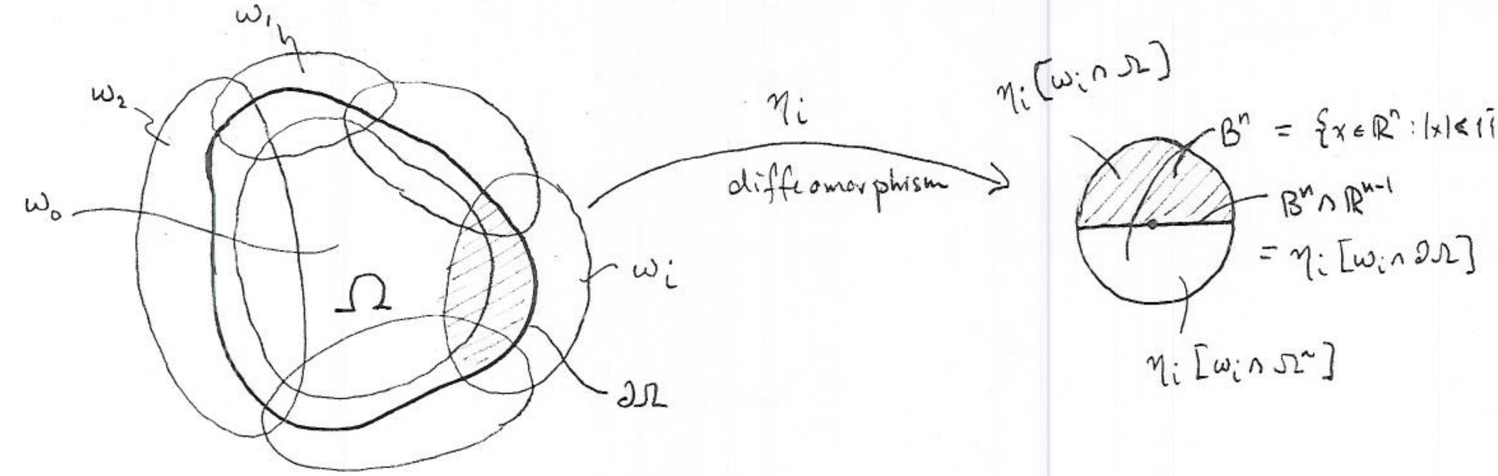
$\iff u(x', z) \in H^k(\mathbb{R}^{n+})$

$\iff (1+|\xi|^2)^{\frac{\ell}{2}} \partial_z^{k-\ell} \hat{u}(\xi, z) \in L^2 \quad \forall\ \ell$ s.th. $0 \leq \ell \leq k$,

- $\|u\|_{H^k(\mathbb{R}^{n+})} \underset{\uparrow}{\sim} \left(\sum_{\ell=0}^{k} \|(1+|\xi|^2)^{\frac{\ell}{2}} \partial_z^{k-\ell} \hat{u}(\xi, z)\|^2_{L^2(\mathbb{R}^{n+})} \right)^{1/2}$

equiv. as norm

Atlases and partitions of unity (POU) for bounded domains Ω with C^1 boundary $\partial\Omega$.



$\{\omega_i\}_{i=1}^n$ is an open cover of Ω such that $\bar{\omega}_0 \subset \Omega$ and for $i = 1, \ldots, n$, there is a diffeomorphism $\eta_i : \omega_i \to B^n$ such that $\eta_i[\omega_i \cap \Omega] = B^n \cap \mathbb{R}^{n+}$, $\eta_i[\omega_i \cap \partial\Omega] = B^n \cap \mathbb{R}^{n-1}$, and $\eta_i[\omega_i \cap \Omega^\sim] = B^n \cap \mathbb{R}^{n-}$.

$\{\lambda_i\}_{i=0}^n$ is a partition of unity, that is, a set of functions, s.th.

$$\begin{cases} \lambda_i \in C_c^\infty(\omega_i), \\ \sum_{i=0}^n \lambda_i(x) = 1 \quad \text{for all } x \in \Omega, \\ 0 \le \lambda_i(x) \le 1, \end{cases}$$

- Fact : If $u \in \mathcal{D}'(\Omega)$, the fllg are equivalent: [Ω bounded]

  * $u \in H^k(\Omega)$
  * $(u\phi) \in H^k(\Omega)$ for all $\phi \in C_c^\infty(\Omega)$
  * $(u\phi)^\wedge(1+|\xi|^2)^{k/2} \in L^2 \quad \forall \; \phi \in C_c^\infty(\Omega)$, where ϕ is extended by zero to all of $\mathbb{R}^n$.

The first two are equivalent by the <u>product rule</u> for differentiation of $u\phi$, where $u \in \mathcal{D}'(\Omega)$ and $\phi \in C_c^\infty(\Omega)$. This is straightforward to prove using the definition of ∂^α in $\mathcal{D}'(\Omega)$ and the product rule for products of smooth functions.

- It now makes sense to define

$$H^s(\Omega) = \left\{ u \in \mathcal{D}'(\Omega) : (u\phi)^\wedge(1+|\xi|^2)^{s/2} \in L^2 \quad \forall \; \phi \in C_c^\infty(\Omega) \right\},$$

in which $u\phi$ is extended by zero to $\mathbb{R}^n$.

- <u>Distributions on $\partial\Omega$</u>, assuming $\partial\Omega$ is C^∞ and Ω is bounded.

test functions

- $C^\infty(\partial\Omega)$ = the space if infinitely differentiable functions on $\partial\Omega$: By defn., $\phi \in C^\infty(\partial\Omega)$ iff $\lambda_i \phi \circ \eta_i^{-1}(x)$, for $x \in B^n \cap \mathbb{R}^{n-1}$, is in $C^\infty(\mathbb{R}^{n-1}) \; \forall \; i=1,\ldots,n$.

  This definition is independent of the choice of atlas and POU.

  Convergence of ϕ_j to ϕ in $C^\infty(\partial\Omega)$ is defined through the <u>uniform</u> convergence of all derivatives $\partial^\alpha(\lambda_i \phi_j \circ \eta_i^{-1})$, $x \in B^n \cap \mathbb{R}^{n-1}$, to $\partial^\alpha(\lambda_i \phi \circ \eta_i^{-1})$.

  This is independent of the altlas and POU and can be defined without use of the λ_i.

distributions

- $\mathcal{D}'(\partial\Omega)$ is the space of sequentially continuous linear functions from $C^\infty(\partial\Omega)$ to $\mathbb{C}$.

- Defn. of $H^s(\partial\Omega)$.

For $u \in \mathcal{D}'(\partial\Omega)$, $u = \sum_{i=1}^{n} \lambda_i u$, $\operatorname{supp}(\lambda_i u) \subset \omega_i \cap \partial\Omega$

$\lambda_i u$ can be "pushed forward" by η_i to a tempered distribution $\eta_{i*}\lambda_i$ on $\mathbb{R}^{n-1}$ as follows: For $\phi \in S(\mathbb{R}^{n-1})$, set

$$\langle \eta_{i*}(\lambda_i u), \phi \rangle = \langle u, \lambda_i \phi \circ \eta_i \rangle .$$

[If u is a function, then $\eta_{i*}(\lambda_i u) = \lambda_i u \circ \eta_i^{-1}$, extended by zero to $\mathbb{R}^{n-1}$.]

Defn. $H^s(\partial\Omega) = \left\{ u \in \mathcal{D}'(\partial\Omega) : \eta_{i*}(\lambda_i u) \in H^s(\mathbb{R}^{n-1}) \ \forall \ i = 1, \dots, n \right\}$.

Exercise: Prove that H^s is complete in $\|\cdot\|_{H^s}$ and that $H^{-s} \cong (H^s)^*$.

- Fact $C_c^\infty(\mathbb{R}^n)$ is dense in $H^k(\mathbb{R}^n)$.

Pf Since $\{ f \in H^k(\mathbb{R}^n) : \operatorname{supp} f$ is bounded$\}$ is dense in H^k (L. dom. conv. thm.), we will prove that $C_c^\infty(\mathbb{R}^n)$ is dense in this subspace of H^k.

For $u \in H^k$, with $\operatorname{supp}(u)$ bounded, and with μ_ε defined as before $\left[\int \mu_\varepsilon = 1, \ \mu_\varepsilon(x) = \frac{1}{\varepsilon^n} \mu_1(x/\varepsilon) \right]$, we have, for $|\alpha| \le k$,

$$\partial^\alpha (u * \mu_\varepsilon) = \partial^\alpha \int u(y) \mu_\varepsilon(x-y)\,dy = \int u(y) \partial^\alpha \mu_\varepsilon(x-y)\,dy$$

$$= \int \partial^\alpha u(y) \mu_\varepsilon(x-y)\,dy \quad \text{by defn. of } \partial^\alpha \text{ distributionally.}$$

We have shown that the latter convolution, $(\partial^\alpha u) * \mu_\varepsilon$, converges in L^2 to $\partial^\alpha u$ as $\varepsilon \to 0$.

Since this holds for each $|\alpha| \le k$, we find that

$$\| u * \mu_\varepsilon - u \|_{H^k} \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

We have also seen that $u * \mu_\varepsilon \in C^\infty$, and since $\mu_\varepsilon \in C_c^\infty$, $u * \mu_\varepsilon \in C_c^\infty$ also.

- <u>Fact</u>. $C^\infty(\bar\Omega)$ is dense in $H^k(\Omega)$ [Ω a bounded domain w/ <u>C^1</u> boundary in $\mathbb{R}^n$]

<u>Sketch of proof</u> [See Evans §5.3, for example]

* For $u \in H^k(\Omega)$, $u * \mu_\varepsilon$ is defined in $\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$

and $u * \mu_\varepsilon \to u$ in $H^k(V)$ $\forall$ open V with $\bar V \subset \Omega$.

* Define $\Sigma_i = \Omega_{1/i+2} \setminus \Omega_{1/i}$ (an open "sliver"), $\Sigma_0 = \Omega_{1/2}$,

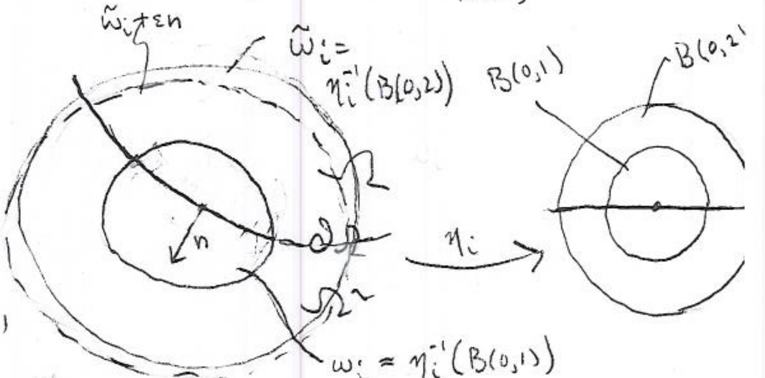
and let $\{\psi_i\}$ be a POU subordinate to $\{\Sigma_i\}$.

We have $\bigcup_{i=0}^\infty \Sigma_i = \Omega$ and $u = \sum_{i=0}^\infty \psi_i u$.

Define $u_\delta = \sum_{i=0}^\infty (\psi_i u) * \mu_{\varepsilon_i} \in C^\infty(\Omega)$ s.th. $\|(\psi_i u) * \mu_{\varepsilon_i} - \psi_i u\|_{H^k(\Omega)} \le \frac{\delta}{2^{i+1}}$,

so that $\|u_\delta - u\| \le \delta$

* Let $\{w_i, \eta_i\}_{i=1\dots n}$ be an atlas of Ω such that each η_i can be extended to an open $\tilde\omega_i$ w/ $\eta_i[\tilde\omega_i] = B(0,2)$,



and $(\tilde\omega_i \cap \Omega) + \varepsilon n \supset \omega_i \cap \Omega$ for ε suff small and $i \ne 0$.

Define $u_i^\varepsilon = u_\delta(\cdot - \varepsilon n)\big|_{\tilde\omega_i \cap \Omega + \varepsilon n}$ for $i \ne 0$ and $u_0^\varepsilon = u_\delta$

Then $u_i^\varepsilon \to u_\delta$ as $\varepsilon \to 0$ in $H^k(\omega_i \cap \Omega)$. [Convergence of shifts of $\partial^\alpha u|_{\omega_i \cap \Omega}$ in L^2, for $|\alpha| \le k$]

Let ε be s.th. $\|u_i^\varepsilon - u_\delta\|_{H^k(\omega_i \cap \Omega)} < \delta$, $i = 1, \dots, n$.

Set $w_i = \lambda_i u_i^\varepsilon$ and $w = \sum_{i=0}^n \lambda_i u_i^\varepsilon \in C^\infty(\bar\Omega)$.

Also, $u = \sum_{i=0}^n \lambda_i u$ because $\{\lambda_i\}_{i=0}^n$ is a POU.

Then $\|\partial^\alpha w - \partial^\alpha u\|_{L^2} \le \sum_{i=0}^n \|\partial^\alpha w_i - \partial^\alpha(\lambda_i u)\|_{L^2} \le \sum_{i=0}^n \|\partial^\alpha(\lambda_i(u_i^\varepsilon - u))\|_{L^2}$

$\le C_1 \sum_{i=0}^n \|u_i^\varepsilon - u\|_{H^k(\omega_i \cap \Omega)} < C_2 \delta$.

- Fact $C_c^\infty(\overline{\mathbb{R}^{n+}})$ is dense in $H^k(\mathbb{R}^{n+})$

Note that $C_c^\infty(\overline{\mathbb{R}^{n+}})$ includes restrictions to $\overline{\mathbb{R}^{n+}}$ of functions in $C_c^\infty(\mathbb{R}^n)$.
so functions in $C_c^\infty(\overline{\mathbb{R}^{n+}})$ have boundary values on $\mathbb{R}^{n-1}$ that are generically nonzero.

The proof of this fact is similar to (and easier than) that for Ω.

Traces, or boundary values, of functions in $H^k(\overline{\mathbb{R}^{n+}})$ on $\partial\mathbb{R}^{n+} = \mathbb{R}^{n-1}$.

- Fact If $u \in C_c^\infty(\overline{\mathbb{R}^{n+}}) \subset H^k(\mathbb{R}^{n+})$, $k \geq 1$,

then $\| u|_{\mathbb{R}^{n-1}} \|_{H^{k-\frac{1}{2}}(\mathbb{R}^{n-1})} \leq \| u \|_{H^k(\mathbb{R}^{n+})}$.

Proof Let $\hat{u}(\xi, z)$ be the partial fourier transform of $u(x', z)$.

Fundamental thm of calculus,

Then
$$(1+|\xi|^2)^{k-\frac{1}{2}} |\hat{u}(0,\xi)|^2 = -2\,\mathrm{Re}\int_0^\infty (1+|\xi|^2)^{k-\frac{1}{2}}\, \bar{\hat{u}}(\xi, z)\, \frac{\partial \hat{u}}{\partial z}(\xi, z)\, dz,$$

so
$$\|u\|^2_{H^{k-\frac{1}{2}}(\mathbb{R}^{n-1})} \leq \|(1+|\xi|^2)^{\frac{k}{2}} \hat{u}\|^2_{L^2(\mathbb{R}^{n+})} + \|(1+|\xi|^2)^{\frac{k-1}{2}} \frac{\partial \hat{u}}{\partial z}\|^2_{L^2(\mathbb{R}^+)}$$

use $2|ab| \leq |a|^2 + |b|^2$.

$$\leq \|u\|^2_{H^k(\mathbb{R}^{n+})}$$

see bottom of p. 3.

Now, by the BLT theorem and density of $C_c^\infty(\overline{\mathbb{R}^{n+}})$ in $H^k(\mathbb{R}^{n+})$,
the map $C_c^\infty(\overline{\mathbb{R}^{n+}}) \subset H^k(\overline{\mathbb{R}^{n+}}) \longrightarrow H^{k-\frac{1}{2}}(\mathbb{R}^{n-1})$
can be extended to a bounded linear trace operator

$$T_0 : H^k(\mathbb{R}^{n+}) \to H^{k-\frac{1}{2}}(\mathbb{R}^{n-1}).$$

- T_0 is onto:

<u>Fact</u> For $u \in H^{k-\frac{1}{2}}(\mathbb{R}^{n-1})$, $k \geq 1$, let $\tilde{u}(x', z)$ be defined on $\mathbb{R}^{n+}$ through its partial Fourier transform

$$\hat{\tilde{u}}(\xi, z) = \hat{u}(\xi) e^{-(1+|\xi|^2)^{1/2} z}$$

Then $T_0 \tilde{u} = u$ and

$$\|\tilde{u}\|_{H^k(\mathbb{R}^{n+})} = \sqrt{(k+1)/2}\, \|u\|_{H^{k-1/2}(\mathbb{R}^{n-1})} .$$

<u>Proof</u> For each l, $0 \leq l \leq k$,

$$\int_{\mathbb{R}^{n+}} (1+|\xi|^2)^{k-l} \left| \frac{\partial^l}{\partial z^l} \hat{\tilde{u}}(\xi, z) \right|^2 dz\, d\xi = \int_{\mathbb{R}^{n+}} (1+|\xi|^2)^k \left| \hat{\tilde{u}}(\xi, z) \right|^2 dz\, d\xi$$

$$= \int_{\mathbb{R}^{n-1}} \int_0^\infty (1+|\xi|^2)^k |\hat{u}(\xi)|^2 e^{-2(1+|\xi|^2)^{1/2} z} dz\, d\xi$$

$$= \frac{1}{2} \int_{\mathbb{R}^{n-1}} (1+|\xi|^2)^{k-\frac{1}{2}} |\hat{u}(\xi)|^2 d\xi = \frac{1}{2} \|u\|^2_{H^{k-1/2}(\mathbb{R}^{n-1})}$$

$$\text{So } \|\tilde{u}\|^2_{H^k(\mathbb{R}^{n+})} = \sum_{l=0}^{k} (1+|\xi|^2)^{k-l} \left| \frac{\partial^l}{\partial z^l} \hat{\tilde{u}}(\xi, z) \right|^2 dz\, d\xi = \frac{k+1}{2} \|u\|^2_{H^{k-1/2}(\mathbb{R}^{n-1})} .$$

This map, call it $R : H^{k-1/2}(\mathbb{R}^{n-1}) \to H^k(\mathbb{R}^{n+})$ is a bounded right inverse of T_0:

$$T_0 R = I \text{ on } H^{k-1/2}(\mathbb{R}^{n-1}) .$$

The trace and its right inverse on $H^k(\Omega)$, $k \geq 1$

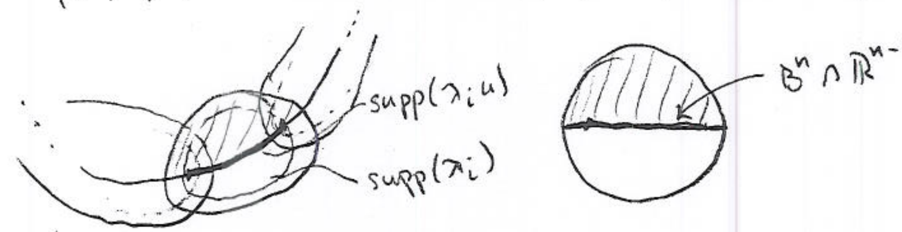
For a bounded domain Ω with C^∞ boundary $\partial\Omega$, let an atlas $\{(\omega_i, \eta_i)\}_{i=0}^n$ be given along with a POU $\{\lambda_i\}_{i=0}^n$ subordinate to $\{\omega_i\}$.

By using the product rule, it is straightforward to show that the norm $\|u\|_{H^k(\Omega)}$ is equivalent to $\sum_{i=0}^n \|\lambda_i u \circ \eta_i^{-1}\|_{H^k(B^n)}$, $k \in \mathbb{N}$.

For the fractional Sobolev spaces on $\partial\Omega$, $H^s(\partial\Omega)$, we can define the norm by

$$\|u\|_{H^s(\partial\Omega)} = \left[\sum_{i=1}^n \|\lambda_i u \circ \eta_i^{-1}\|^2_{H^s(\mathbb{R}^{n-1})}\right]^{1/2}, \quad \text{(dep. on } \omega_i, \eta_i, \lambda_i\text{)}$$

where, for $s < 0$, the distribution $\lambda_i u \circ \eta_i^{-1}$ is defined by $(\lambda_i u \circ \eta_i^{-1}, \phi) = (\lambda_i u, \phi \circ \eta_i)$ for all $\phi \in C_c^\infty(B^n \cap \mathbb{R}^{n-1})$.



Of course, one needs to prove that this is independent of the choice of atlas and POU and that $H^s(\partial\Omega)$ is complete.

- <u>Fact</u>: There is a surjective bounded trace operator, for $k \geq 1$, $k \in \mathbb{N}$,

  $T_0 : H^k(\Omega) \to H^{k-1/2}(\partial\Omega)$ whose restriction to $C^\infty(\bar{\Omega})$ is

  the boundary-value operator $T_0(u) = u|_{\partial\Omega}$.

<u>Pf</u> Using the definition on p. 10 of the norm in $H^s(\partial\Omega)$, for $u \in C^\infty(\bar{\Omega})$,

$$\|u\|_{H^{k-1/2}(\partial\Omega)} = \sum_{i=1}^{n} \|\lambda_i u \circ \eta_i^{-1}\|_{H^{k-\frac{1}{2}}(\mathbb{R}^{n-1})} \leq C_1 \sum_{i=1}^{n} \|\lambda_i u \circ \eta_i^{-1}\|_{H^k(\mathbb{R}^{n+})} \quad (\text{by } T_0 \text{ in } \mathbb{R}^{n+})$$

$$\leq C_2 \|u\|_{H^k(\Omega)} \quad (\text{by the product rule})$$

Since $C^\infty(\bar{\Omega})$ is dense in $H^k(\Omega)$ and $H^{k-\frac{1}{2}}(\partial\Omega)$ is complete,
this map is extendable to a bounded operator $T_0 : H^k(\Omega) \to H^{k-1/2}(\partial\Omega)$.

Next, for $u \in H^{k-\frac{1}{2}}(\partial\Omega)$, define $\tilde{u} \in H^k(\Omega)$ by

$$\tilde{u} = \sum_{i=1}^{n} \lambda_i R(\lambda_i u \circ \eta_i^{-1}) \circ \eta_i .$$

$$\|\tilde{u}\|_{H^k(\Omega)} \leq \sum_{i=1}^{n} \|\lambda_i R(\lambda_i u \circ \eta_i^{-1}) \circ \eta_i\|_{H^k(\Omega)}$$

$$\leq C_1 \sum_{i=1}^{n} \|R(\lambda_i u \circ \eta_i^{-1})\|_{H^k(\mathbb{R}^{n+})} \quad (\text{using the product rule})$$

$$\leq C_2 \sum_{i=1}^{n} \|\lambda_i u \circ \eta_i^{-1}\|_{H^{k-1/2}(\mathbb{R}^{n-1})} = C_2 \|u\|_{H^{k-1/2}(\partial\Omega)}$$

By defn. of $R(\lambda_i u \circ \eta_i^{-1})$, $T_0(R(\lambda_i u \circ \eta_i^{-1})) = \lambda_i u \circ \eta_i^{-1}$,
so by defn. of $\tilde{u}$, we find that $T_0(\tilde{u}) = u$.

If u is regular enough in Ω, then it also has a well defined normal derivative. The precise statement we will need is

Fact If $k \in \mathbb{N}$, $k \geq 2$, then there is a bounded linear operator

$T_1 : H^k(\Omega) \to H^{k-\frac{3}{2}}(\partial\Omega)$ such that, for $u \in C^\infty(\bar{\Omega})$,

$T_1 u = \partial_n u$ on $\partial\Omega$.

Proof 1: If $u \in H^k(\mathbb{R}^{n+})$, then $-\frac{\partial}{\partial z} u(x,z) \in H^{k-1}(\mathbb{R}^{n+})$.

If $u \in C_c^\infty(\overline{\mathbb{R}^{n+}})$ also, then $T_0\left(-\frac{\partial}{\partial z} u(x,z)\right)$ is the normal derivative of u on $\mathbb{R}^{n-1}$ and is bounded in $H^{k-\frac{3}{2}}$ by $\|\frac{\partial}{\partial z} u\|_{H^{k-1}}$, which is bounded by $\|u\|_{H^k}$.

2: If $u \in H^k(\Omega)$, we need to consider each open set w_i from our atlas.

Let us assume that the λ_i were chosen appropriately so that the vector field n can be extended smoothly into $w_i \cap \operatorname{supp}(\lambda_i)$.

Then $\nabla(\lambda_i u) \cdot n \in H^{k-1}(w_i)$ is bounded by $\|\lambda_i u\|_{H^k(w_i)}$.

The trace map T_0, as in the case of $\mathbb{R}^{n+}$, applied to $\nabla(\lambda_i u) \cdot n$ is bounded into $H^{k-3/2}(\partial\Omega)$. For $u \in C^\infty(\bar{\Omega})$,

$$T_1(u) := \sum_{i=1}^{n} T_0(\lambda_i u) \cdot n = \sum_{i=1}^{n} \nabla(\lambda_i u) \cdot n \Big|_{\partial\Omega} = \nabla u \cdot n \Big|_{\partial\Omega},$$

so T_1 is an extension of the normal derivative on $\partial\Omega$, and one can check that T_1 is bounded.