Problem Set 1

1. Advection equations with nonconstant advection rates in the absence of external sources are naturally written as conservation laws:

$$\begin{cases} u_t + \nabla_x \cdot (\bar{c}(x) u) = 0 & x \in \mathbb{R}^n \\ & t > 0 \\ u(x,0) = f(x) \end{cases}$$

in which f is the initial concentration of stuff.
This equation can be rewritten as

(1) $$u_t + \bar{c}(x) \cdot \nabla_x u = -(\nabla \cdot \bar{c}(x)) u$$

Find the explicit solution of the 1D problem

$$\begin{cases} u_t + (x^2 u)_x = 0 \\ u(x,0) = f(x) \end{cases}.$$

Use the method of characteristic curves by finding the curves $x_y(t)$ with $x_y(0) = y^{\in \mathbb{R}}$ such that the left-hand-side of the PDE in the form (1) is equal to $\frac{du}{dt}$ along the curve. Then solve for u as a function of y and t, and ultimately for u as a function of x and t. [It is useful to check your result!]

Discuss the domain of validity of the solution.

Problem Set 1, cont.

2. Find the dispersion relation $\omega = W(k)$ that relates frequency to wavenumber of solutions $e^{i(kx-\omega t)}$ to the linearized KdV equation

$$u_t + c\,u_x + \varepsilon u_{xxx} = 0$$

and determine the phase velocity and group velocity of wave packets as a function of the wavenumber k.

3. Prove that the solution of the 1D heat equation

$$u_t = \Delta u$$
$$u(x,0) = f(x)$$

is even in x for all t whenever f is even and that the solution is odd in x for all t whenever f is odd.

Find the fundamental solution of the heat equation on the half-line $[0,\infty)$ with the stipulation $u(0,t)=0$ $\forall\, t>0$, that is, find the function

$$\Phi_D(x,y,t)\,,\quad x,y \geq 0,\ t>0$$

such that the solution of the initial-boundary-value problem

$$u_t = \Delta u \qquad x>0,\ t>0$$
$$u(x,0) = f(x) \qquad x \geq 0$$
$$u(0,t) = 0 \qquad t>0$$

is equal to

$$u(x,t) = \int_0^\infty \Phi(x,y,t)\, f(y)\, dy$$