Problem Set 2

1. Solve the 1D Schrödinger equation $iu_t = \gamma u_{xx}$, $\gamma > 0$, with initial data $u(x,0) = e^{-ax^2}$, $a > 0$, explicitly, in the spatial domain. The solution in Fourier form is

$$u(x,t) = \int_{-\infty}^{\infty} c(k)\, e^{i(kx - W(k)t)}\, dk,$$

where $c(k) = \frac{1}{\sqrt{2\pi}} u(\cdot,0)^\wedge$ is the Fourier transform of $u(x,0)$ and $\omega = W(k)$ is the dispersion relation for the Schrödinger equation. Use the fact that, if $f(x) = e^{-wx^2}$, then

$$\check{f}(\xi) = \hat{f}(\xi) = \left(\frac{1}{2w}\right)^{1/2} e^{-\frac{\xi^2}{4w}} \qquad (\operatorname{Re} w > 0,\ -\pi/4 < \arg(w^{1/2}) < \pi/4).$$

Then evaluate $u(x,t)$ along $x = x_0 + mt$ and show that your explicit solution possesses the asymptotics derived by the method of stationary phase.

2. [From L. C. Evans, Partial Differential Equations, GSM Vol. 19, p. 234]

Consider the viscous conservation law

$$(*) \qquad u_t + F(u)_x - a u_{xx} = 0 \quad \text{in } \mathbb{R} \times (0,\infty),$$

where $a > 0$ and F is uniformly convex.

(i) Show that u solves $(*)$ if $u(x,t) = v(x - \sigma t)$ and v is defined implicitly by

$$s = \int_c^{v(s)} \frac{a}{F(z) - \sigma z + b}\, dz \qquad (s \in \mathbb{R}),$$

where b and c are constants.

2. (cont.)

(ii) Demonstrate that one can find a travelling wave satisfying

$$\lim_{s \to -\infty} v(s) = u_l \quad \text{and} \quad \lim_{s \to \infty} v(s) = u_r$$

for $u_l > u_r$, if and only if

$$\sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r} .$$

(iii) Let u^ε denote the above travelling solution of $(*)$ for $a = \varepsilon$, with $u^\varepsilon(0,0) = \frac{u_l + u_r}{2}$. Compute $\lim_{\varepsilon \to 0} u^\varepsilon$ and explain your answer.

3. [From J. Billingham and A.C. King, Wave Motion, Camb. U. Press 2000 p. 267]

A piston confines an ideal gas within a semi-infinite tube of uniform cross-section. When $t = 0$, the gas is at rest and has sound speed c_0. For $t \geq 0$,

(a) The piston moves with a constant velocity $-V$ with $V > 0$. Show that the solution takes the form of an expansion fan and determine the solution

(b) The piston moves with velocity $A\omega \sin \omega t$, where A and ω are positive constants. Show that a shock wave first forms when $t = t_s = \dfrac{2c_0}{A\omega^2(\gamma+1)}$.