Problem Set 5

1. Set $\Omega = (0,1)^2 \subset \mathbb{R}^2$, and let $k \in \mathbb{R}^2$ be given. Define the <u>k-pseudo-periodic</u> subspace of $H^1(\Omega)$ by

$$H^1_k(\Omega) = \left\{ u \in H^1(\Omega) \,:\, \exists\, \hat{u} \in H^1_{per}(\Omega) \text{ s.th. } u = \hat{u}e^{ikx} \right\}.$$

$[x = (x_1, x_2) \in \mathbb{R}^2]$ Define $\nabla_k = \nabla\big|_{H^1_k(\Omega)}$ by

$$\mathcal{D}(\nabla_k) = H^1_k(\Omega)$$

$$\nabla_k f = \nabla f \quad \forall\, f \in \mathcal{D}(\nabla_k),$$

where ∇ is the weak gradient operator with domain $H^1(\Omega)$.

(a) Characterize $H^1_k(\Omega)$ in terms of boundary conditions on $\partial\Omega$.

(b) Find the adjoint ∇_k^* — specify its domain and its action.

(c) Show that the map $u \mapsto ue^{ikx}$ defines a Hilbert-space isomorphism of $L^2(\Omega)$.

(d) Show that the map $u \mapsto ue^{ikx}$, restricted to $H^1_{per}(\Omega)$ is a linear homeomorphism $K : H^1_{per}(\Omega) \longrightarrow H^1_k(\Omega)$.

(e) Find ∇_k in terms of its action on $\hat{u}$, the "periodic part" of an element $u = \hat{u}e^{ikx}$ of $H^1_k(\Omega)$, that is, find $\tilde{\nabla}_k := K^{-1}\nabla_k K$ — specify its domain and its action.

(f) Find the adjoint of $\tilde{\nabla}_k$.