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Linear and Nonlinear Waves
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CHAPTER 4

Burgers' Equation

The simplest equation combining both nonlinear propagation effects and diffusive effects is Burgers' equation

$$c_t + cc_x = \nu c_{xx}. \tag{4.1}$$

In (2.28) we saw that this is an exact equation for waves described by

$$\rho_t + q_x = 0, \qquad q = Q(\rho) - \nu\rho_x, \tag{4.2}$$

in the case that $Q(\rho)$ is a quadratic function of ρ. In general, if the two effects are important in a problem, there is usually some way of extracting (4.1) either as a precise approximation or as a useful basis for rough estimates.

For a general $Q(\rho)$ in (4.2), for example, the equation may be written

$$c_t + cc_x = \nu c_{xx} - \nu c''(\rho)\rho_x^2, \tag{4.3}$$

where $c(\rho) = Q'(\rho)$ as usual. The ratio of $\nu c''(\rho)\rho_x^2$ to νc_{xx} is of the order of the amplitude of the disturbance, and we therefore expect that (4.1) is a good approximation for small amplitude. We are then assuming that omission of this particular small amplitude term does not produce accumulating errors (as $t \to \infty$, say) which eventually lead to nonuniform validity. We know, in contrast, that to linearize the left hand side by $c_t + c_0 c_x$, where c_0 is some constant unperturbed value, would be disastrous in this respect. But as a check, we may verify that in the shock structure solution (see Section 2.4), where the diffusion terms are greatest, the term $\nu c''(\rho)\rho_x^2$ remains of smaller order than νc_{xx} in the strength of the shock. This kind of argument can be made the basis of formal perturbation expansions in terms of appropriate precisely defined small parameters. On the other hand, the fact that the terms retained in (4.1) represent identifiable and important phenomena, whereas the term $\nu c''(\rho)\rho_x^2$ appears more as a mathematical nuisance, leads one to suggest (4.1) as a useful overall description even beyond the range of strict validity.

In a similar fashion, Burgers' equation is relevant in higher order systems such as (3.2)–(3.3), when nonlinear propagation is combined with diffusion. Of course it is limited to the stable range and to parts of the solution where the lower order waves are dominant. The appropriate form is easily recognized and again can usually be substantiated by more formal procedures. In the case of (3.2)–(3.3), we know from (3.6) that the effective diffusivity is $\nu^* = \nu - (v_0 - c_0)^2\tau$ and we would use (4.1) with this value. Indeed, (3.6) is the fully linearized Burgers' equation for this system.

Our general purpose now is to show that the exact solution of Burgers' equation endorses the ideas regarding shocks that were developed in Chapter 2. That is, we want to confirm that as $\nu \to 0$ (in appropriate dimensionless form) the solutions of (4.1) reduce to solutions of

$$c_t + cc_x = 0, \tag{4.4}$$

with discontinuous shocks which satisfy

$$U = \frac{1}{2}(c_1 + c_2), \qquad c_2 > U > c_1, \tag{4.5}$$

and the shocks are located at the positions determined in Section 2.8.

4.1 The Cole-Hopf Transformation

Independently, Cole (1951) and Hopf (1950) noted the remarkable result that (4.1) may be reduced to the linear heat equation by the nonlinear transformation

$$c = -2\nu\frac{\varphi_x}{\varphi}. \tag{4.6}$$

This is similar to Thomas' earlier transformation of the exchange equations described in Section 3.4. It is again convenient to do the transformation in two steps. First introduce

$$c = \psi_x,$$

so that (4.1) may be integrated to

$$\psi_t + \frac{1}{2}\psi_x^2 = \nu\psi_{xx}.$$

Then introduce

$$\psi = -2\nu\log\varphi$$

to obtain

$$\varphi_t = \nu \varphi_{xx}. \tag{4.7}$$

The nonlinear transformation just eliminates the nonlinear term. The general solution of the heat equation (4.7) is well known and can be handled by a variety of methods.

The basic problem considered in Chapter 2 is the initial value problem:

$$c = F(x) \qquad \text{at } t = 0.$$

This transforms through (4.6) into the initial value problem

$$\varphi = \Phi(x) = \exp\left\{ -\frac{1}{2\nu} \int_0^x F(\eta)\, d\eta \right\}, \qquad t = 0, \tag{4.8}$$

for the heat equation. The solution for φ is

$$\varphi = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \Phi(\eta) \exp\left\{ -\frac{(x-\eta)^2}{4\nu t} \right\} d\eta. \tag{4.9}$$

Therefore, from (4.6), the solution for c is

$$c(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{x-\eta}{t} e^{-G/2\nu} d\eta}{\displaystyle\int_{-\infty}^{\infty} e^{-G/2\nu} d\eta}, \tag{4.10}$$

where

$$G(\eta; x, t) = \int_0^{\eta} F(\eta')\, d\eta' + \frac{(x-\eta)^2}{2t}. \tag{4.11}$$

4.2 Behavior as $\nu \to 0$

The behavior of the exact solution (4.10) is now considered as $\nu \to 0$ while x, t and $F(x)$ are held fixed. [Strictly speaking this means we consider a family of solutions with $\nu = \epsilon \nu_0$ and take the limit as $\epsilon \to 0$, holding $\nu_0, x, t, F(x)$ fixed.] As $\nu \to 0$, the dominant contributions to the integrals in (4.10) come from the neighborhood of the stationary points of G. A stationary point is where

$$\frac{\partial G}{\partial \eta} = F(\eta) - \frac{x-\eta}{t} = 0. \tag{4.12}$$

Let $\eta = \xi(x,t)$ be such a point; that is, $\xi(x,t)$ is defined as a solution of

$$F(\xi) - \frac{(x-\xi)}{t} = 0. \tag{4.13}$$

The contribution from the neighborhood of a stationary point, $\eta = \xi$, in an integral

$$\int_{-\infty}^{\infty} g(\eta) e^{-G(\eta)/2\nu} d\eta,$$

is

$$g(\xi) \sqrt{\frac{4\pi\nu}{G''(\xi)}}\; e^{-G(\xi)/2\nu};$$

this is the standard formula of the method of steepest descents.

Suppose first that there is only one stationary point $\xi(x,t)$ which satisfies (4.13). Then

$$\int_{-\infty}^{\infty} \frac{x-\eta}{t} e^{-G/2\nu} d\eta \sim \frac{x-\xi}{t} \sqrt{\frac{4\pi\nu}{G''(\xi)}}\; e^{-G(\xi)/2\nu}, \tag{4.14}$$

$$\int_{-\infty}^{\infty} e^{-G/2\nu} d\eta \sim \sqrt{\frac{4\pi\nu}{G''(\xi)}}\; e^{-G(\xi)/2\nu}, \tag{4.15}$$

and in (4.10) we have

$$c \sim \frac{x-\xi}{t}, \tag{4.16}$$

where $\xi(x,t)$ is defined by (4.13). This asymptotic solution may be rewritten

$$c = F(\xi)$$
$$x = \xi + F(\xi)t. \tag{4.17}$$

It is exactly the solution of (4.4) which was discussed in (2.5) and (2.6); the stationary point $\xi(x,t)$ becomes the characteristic variable.

However, we saw that in some cases (4.17) gives a multivalued solution after a sufficient time, and discontinuities must be introduced. Yet the solution (4.10) for Burgers' equation is clearly single-valued and continuous for all t. The explanation is that when this stage is reached there are two stationary points that satisfy (4.13), and the foregoing analysis of the asymptotic behavior requires modification. If the two stationary points are denoted by ξ_1 and ξ_2 with $\xi_1 > \xi_2$, there will be contributions as shown in (4.14) and (4.15) from both ξ_1 and ξ_2. Therefore the dominant behavior will be included if we take

$$c \sim \frac{\frac{x-\xi_1}{t}\{G''(\xi_1)\}^{-1/2}e^{-G(\xi_1)/2\nu} + \frac{x-\xi_2}{t}\{G''(\xi_2)\}^{-1/2}e^{-G(\xi_2)/2\nu}}{\{G''(\xi_1)\}^{-1/2}e^{-G(\xi_1)/2\nu} + \{G''(\xi_2)\}^{-1/2}e^{-G(\xi_2)/2\nu}}. \quad (4.18)$$

When $G(\xi_1) \neq G(\xi_2)$, the accentuation by the small denominator ν in the exponents makes one or the other of the terms overwhelmingly large as $\nu \to 0$. If $G(\xi_1) < G(\xi_2)$, we have

$$c \sim \frac{x-\xi_1}{t};$$

if $G(\xi_1) > G(\xi_2)$,

$$c \sim \frac{x-\xi_2}{t}.$$

In each case (4.17) applies with either ξ_1 or ξ_2 for ξ. But the choice is now unambiguous. Both ξ_1 and ξ_2 are functions of (x,t); the criterion $G(\xi_1) \gtrless G(\xi_2)$ will determine the appropriate choice of ξ_1 or ξ_2 for given (x,t). The changeover from ξ_1 to ξ_2 will occur at those (x,t) for which

$$G(\xi_1) = G(\xi_2).$$

From (4.11), this is when

$$\int_0^{\xi_2} F(\eta')\,d\eta' + \frac{(x-\xi_2)^2}{2t} = \int_0^{\xi_1} F(\eta')\,d\eta' + \frac{(x-\xi_1)^2}{2t}. \quad (4.19)$$

Since ξ_1 and ξ_2 both satisfy (4.13), the condition may be written

$$\frac{1}{2}\{F(\xi_1)+F(\xi_2)\}(\xi_1-\xi_2) = \int_{\xi_2}^{\xi_1} F(\eta')\,d\eta'. \quad (4.20)$$

This is exactly the shock determination obtained in (2.45). The changeover in the choice of terms in (4.18) leads to the discontinuity in $c(x,t)$ in the limit $\nu \to 0$. All the details of Section 2.8 can be confirmed similarly. We conclude that solutions of Burgers' equation approach those described by (4.4) and (4.5) as $\nu \to 0$.

In reality ν is fixed, but it is relatively small and we expect that the limit solution for $\nu \to 0$ will often be a good approximation. For this argument, since ν is a dimensional quantity, we have to introduce a nondimensional measure of ν by comparing it with some other quantity of the same dimension. This is not hard to do. In the single hump problem, for example, where $F(x)$ is as shown in Fig. 2.9, we may introduce the parameter

$$A = \int_{-\infty}^{\infty}\{F(x) - c_0\}\,dx. \quad (4.21)$$

The dimensions of A and ν are both L^2/T, so that

$$R = \frac{A}{2\nu} \quad (4.22)$$

is a dimensionless number, and by "ν small" we mean $R \gg 1$. If the length of the hump is L, the number R measures the ratio of the nonlinear term $(c-c_0)c_x$ to the diffusion term νc_{xx}, *in those regions where the x scale for the derivatives is L*. (Inside shocks, for example, the x scale is of smaller order.) It will be convenient to refer to R as the Reynolds number, following the practice in viscous flow.

Even with the meaning of "small ν" decided, there are distinctions between the limit solution $\nu \to 0$ and the solution for fixed small ν. As we saw in (2.26), the shock thickness tends to infinity if the strength tends to zero. Therefore for fixed R, even if it is large, any solution that includes shock formation or a shock decaying as $t \to \infty$ will not always be well approximated by the discontinuity theory in these regions. As regards a shock formation region, the precise details are not usually important; one just wants a good estimate of *where* it forms, without details of the profile, and this is provided by the discontinuity theory. The effects of diffusion on decaying shocks as $t \to \infty$ is of more interest. We will explore these questions through typical examples in the following sections.

4.3 Shock Structure

The shock structure for (4.1) satisfies

$$-Uc_X + cc_X = \nu c_{XX}, \qquad X = x - Ut.$$

Hence

$$\frac{1}{2}c^2 - Uc + C = \nu c_X.$$

If $c \to c_1, c_2$ as $X \to \pm\infty$,

$$U = \frac{1}{2}(c_1 + c_2), \qquad C = \frac{1}{2}c_1c_2,$$