To find trapped modes, one first finds the eigenfrequencies and eigenfunctions of the "real" problem

$$(*) \qquad \hat{a}^{\omega}_{r}(u,v) - (\omega^2+1)\, b(u,v) = 0 \qquad \forall \; v \in H^1_{(0)}(\Omega)$$

and then checks the second condition on p. 81, i.e., whether the propagating harmonics vanish.

But $(*)$ is not linear in ω^2 because of the dependence of $\hat{a}^{\omega}_{r}$ through the Fourier multipliers $-i\xi_k$, which in T^{ω}_{r} are all nonnegative:

$$-i\xi_k = \sqrt{\lambda_k - \omega^2} \quad \text{for} \quad \lambda_k > \omega^2 .$$

In fact, T^{ω}_{r} can be expressed as

$$(T^{\omega}_{r} f)^{\wedge}_{k} = \operatorname{Re}\sqrt{\lambda_k - \omega^2}\, \hat{f}_k \quad , \quad k = 1, 2, \ldots .$$

The strategy to deal with this type of nonlinear eigenvalue problem is to fix a value of ω in $\hat{a}^{\omega}_{r}$ and consider the eigenvalues γ of the linear eigenvalue problem

$$\hat{a}^{\omega}_{r}(u,v) - \gamma\, b(u,v) = 0 \qquad \forall \; v \in H^1_{(0)}(\Omega)$$

We have seen that this problem admits an increasing (nondecreasing) sequence $\{\gamma^{\omega}_{j}\}_{j=1}^{\infty}$ of eigenvalues that tends to ∞ as $j \to \infty$.

We will prove that, for each $j \in \{1, 2, 3, \ldots\}$, γ_j^ω is a continuous decreasing function of $\omega \geq 0$. To do this, we use their min-max characterization:

$H = H^1_{(0)}(\Omega)$

$$\gamma_j^\omega = \min_{\substack{V_j < H \\ \dim(V_j) = j}} \max_{\substack{u \in V_j \\ u \neq 0}} \frac{\hat{a}_r^\omega(u,u)}{b(u,u)} .$$

Recall that

$$\hat{a}_r^\omega(u,v) := \int_\Omega \tau \nabla u \cdot \nabla \bar{v} + \int_\Omega \rho u \bar{v} + \int_\Gamma (T_r^\omega u)\bar{v}$$

For $\omega_2 \geq \omega_1 > 0$, observe that

$$\mathrm{Re}\sqrt{\lambda_k - \omega_2^2} \leq \mathrm{Re}\sqrt{\lambda_k - \omega_1^2} \quad , \quad k = 1, 2, 3, \ldots$$

so that, $\forall\, u \in H$,

$$0 \leq \hat{a}_r^{\omega_1}(u,u) - \hat{a}_r^{\omega_2}(u,u) = \int_\Gamma \left((T_r^{\omega_1} - T_r^{\omega_2}) u \right) \bar{u}$$

$$= \sum_{k=1}^{\infty} \mathrm{Re}\left(\sqrt{\lambda_k - \omega_1^2} - \sqrt{\lambda_k - \omega_2^2} \right) \left| (u|_\Gamma)^\wedge_k \right|^2$$

$$\leq \sum_{k=1}^{\infty} \sqrt{\omega_2^2 - \omega_1^2} \left| (u|_\Gamma)^\wedge_k \right|^2 = \sqrt{\omega_2^2 - \omega_1^2}\, \|u\|_{L^2(\Gamma)}$$

$$\leq \sqrt{\omega_2^2 - \omega_1^2}\, \|u\|_{H^{1/2}(\Gamma)} \leq C_1 \sqrt{\omega_2^2 - \omega_1^2}\, \|u\|_{H^1(\Omega)}$$

$$\leq C_2 \sqrt{\omega_2^2 - \omega_1^2}\, \hat{a}_r^{\omega_2}(u,u)$$

Rearrangement and division by $b(u,u)$ yields

$$\frac{a_r^{\omega_2}(u,u)}{b(u,u)} \leq \frac{a_r^{\omega_1}(u,u)}{b(u,u)} \leq \left(1 + C_2\sqrt{\omega_2^2 - \omega_1^2}\right)\frac{\omega_r^{\omega_2}(u,u)}{b(u,u)} .$$

For each j-dimensional subspace V_j of H, taking $\max\limits_{\substack{u \in V_j \\ u \neq 0}}$ of each part of this inequality preserves the inequalities; then taking the minimum over all such V_j again preserves inequality yielding

$$\gamma_j^{\omega_2} \leq \gamma_j^{\omega_1} \leq \left(1 + C_2\sqrt{\omega_2^2 - \omega_1^2}\right)\gamma_j^{\omega_2} ,$$

or $$0 \leq \gamma_j^{\omega_1} - \gamma_j^{\omega_2} \leq C_2\sqrt{\omega_2^2 - \omega_1^2}\,\gamma_j^{\omega_2} ,$$

which proves the continuity and the decreasing property of γ_j^{ω} with respect to ω.

Now, ω is an eigenvalue of the problem

$$\hat{a}_r^{\omega}(u,v) - (\omega^2+1)\,b(u,v) = 0 \qquad \forall \; v \in H$$

if and only if

$$\omega^2+1 = \gamma_j^{\omega} \qquad \text{for some } j \in \{1, 2, 3, \ldots\} .$$

Since ω^2+1 is increasing and γ_j^{ω} decreasing, there is an infinite sequence of numbers ω_j with $\omega_j \to \infty$ such that

$$\omega_j^2 = \gamma_j^\omega - 1$$

We have seen before that $\gamma_j^\omega > 1$ since $\gamma_j^\omega - 1$ satisfies

$$a_\Gamma^\omega(u,u) - (\gamma_j - 1)\, b(u,u) = 0,$$

where u is the eigenfunction and $a_\Gamma^\omega(u,v) = \int_\Omega \tau \nabla u \cdot \nabla \bar v + \int_\Gamma (T_\Gamma^\omega u)\bar v$.

The picture is this: