NLS **Cubic nonlinear Schrödinger equation.** Ref: Linear and Nonlinear Waves, G.B. Whitham Ch. 17.

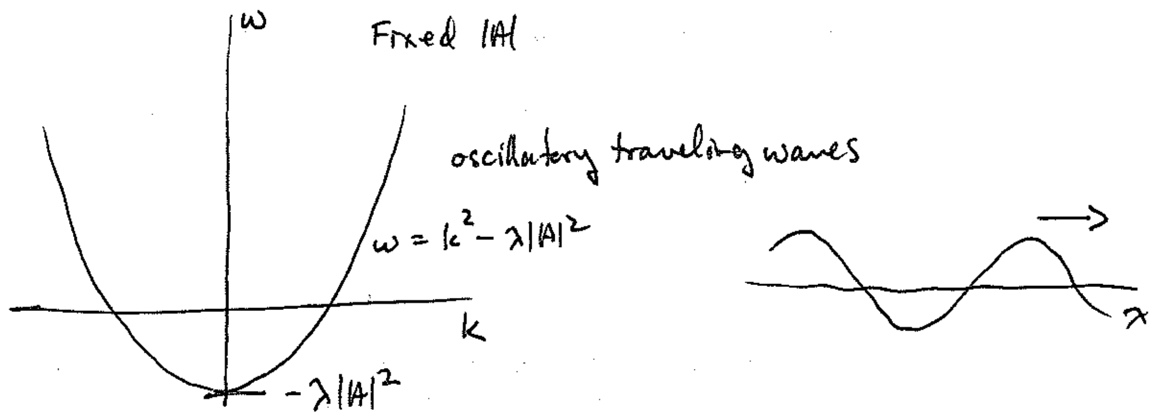
$$iu_t + u_{xx} + \lambda |u|^2 u = 0$$ (modulated beams in nonlinear optics)

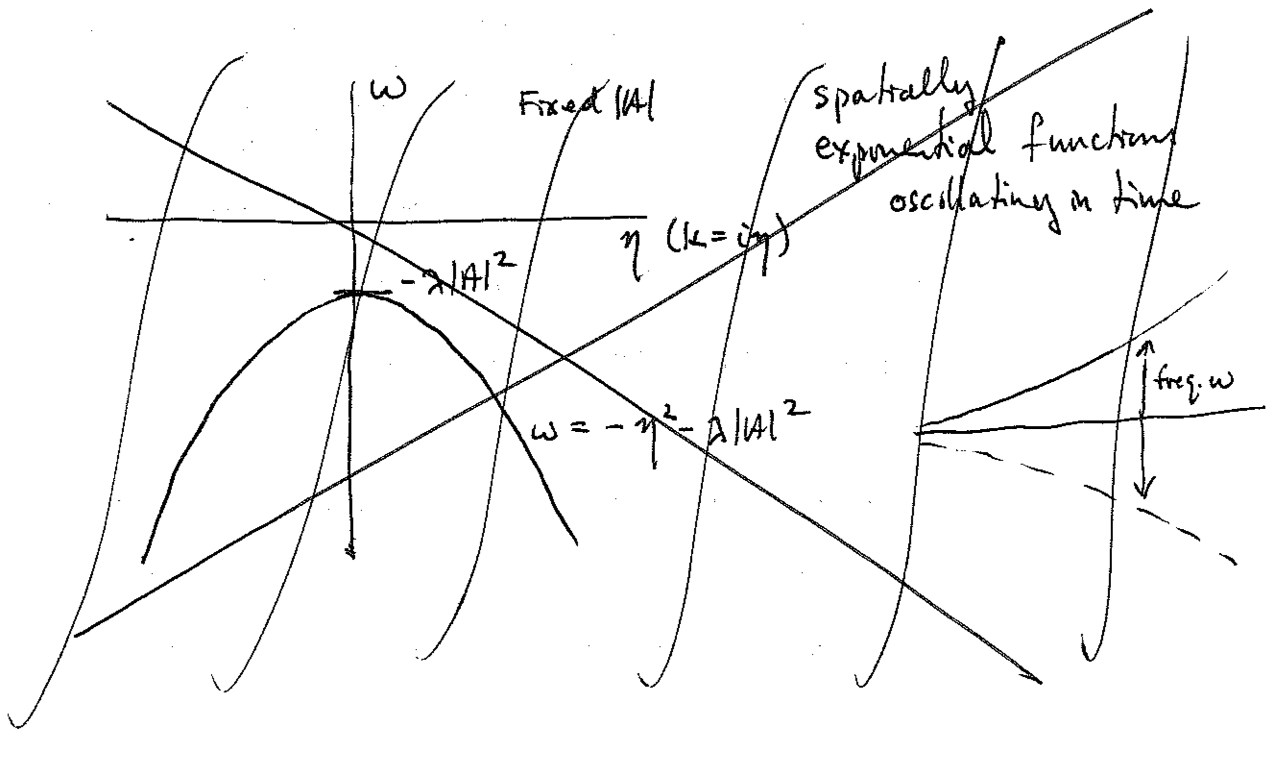
Traveling waves: $u = Ae^{i(kx-\omega t)}$

$\Rightarrow \quad \omega = k^2 - \lambda |A|^2$: amplitude-dep. dispersion relation.

Case $\lambda > 0$: $\exists$ two equilibrium solns oscillating spatially
- balance btw curvature and "reaction".

Dispersion reln:





Wave trains for $iu_t + u_{xx} + \lambda |u|^2 u = 0$

of the form $u = e^{i(kx - \omega t)} f(x - ct)$, f real-valued.

$$\Longrightarrow \quad f'' + i(2k - c) f' + (\omega - k^2) f + \lambda |f|^2 f = 0$$

Let us set $\begin{cases} 2k - c = 0 \\ \alpha := k^2 - \omega \end{cases}$

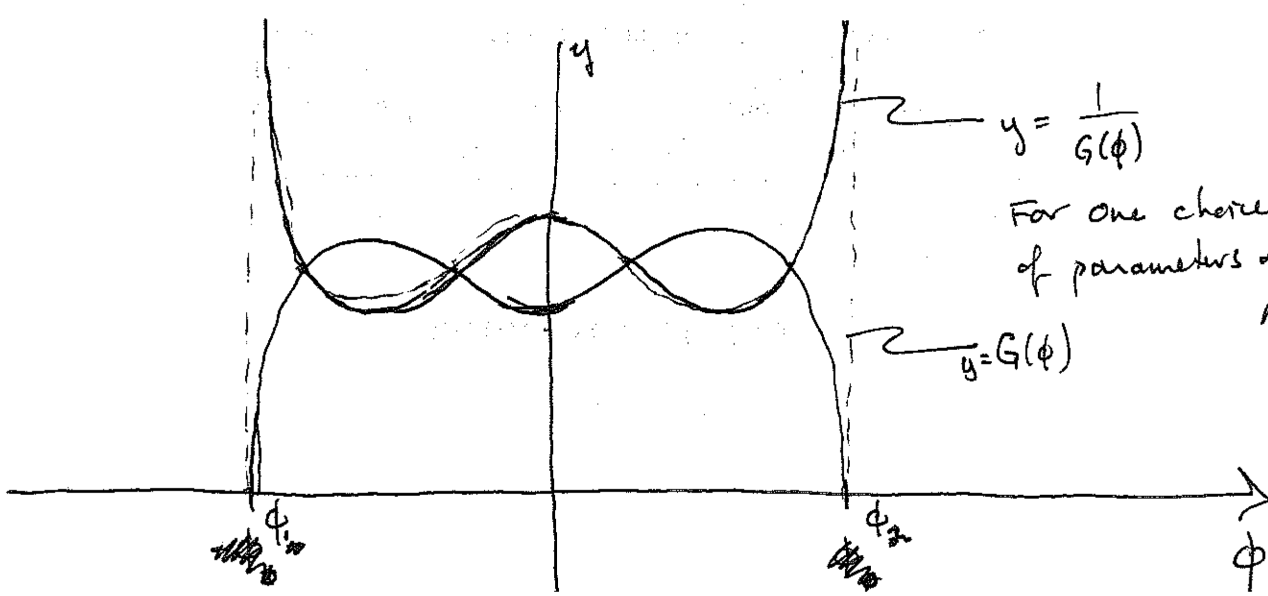
$$\Longrightarrow \quad f'' - \alpha f + \lambda f^3 = 0$$

$$\Longrightarrow \quad f' f'' = (\alpha f - \lambda f^3) f'$$

$$\Longrightarrow \quad (f')^2 = \alpha f^2 - \frac{\lambda}{2} f^4 + A \qquad (A \text{ arb. constant})$$

$$\Longrightarrow \quad \frac{df}{d\xi} = \pm \sqrt{\alpha f^2 - \frac{\lambda}{2} f^4 + A} =: G(f) \quad (\text{for } + \text{ sign})$$

$$\Longrightarrow \quad \frac{d\xi}{df} = \frac{\pm 1}{\sqrt{\alpha f^2 - \frac{\lambda}{2} f^4 + A}} = \frac{1}{G(f)} \quad (\text{w/ } + \text{ sign})$$



For one choice of parameters α, λ, A, $A > 0$.

Suppose that the polynomial $\alpha\phi^2 - \frac{\lambda}{2}\phi^4 + A$ has two simple roots ϕ_1 and ϕ_2 with $\phi_1 < \phi_2$ (so $A \neq 0$). Then the function $\frac{1}{G(\phi)}$ is integrable (its integral is finite) between ϕ_1 and ϕ_2 because of the square-root singularities at ϕ_1 and ϕ_2. This means that there is a periodic solution $f(\xi)$ with half-period $\frac{1}{2}L$

$$\frac{1}{2}L = \int_{\phi_1}^{\phi_2} \frac{1}{\sqrt{\alpha\phi^2 - \frac{\lambda}{2}\phi^4 + A}}\, d\phi .$$

The solution $f(\xi)$ is ~~defined~~ given on its increasing segment by the implicit formula

$$(+) \qquad \xi = \xi_0 + \int_{\phi_1}^{f(\xi)} \frac{1}{\sqrt{\alpha\phi^2 - \frac{\lambda}{2}\phi^4 + A}}\, d\phi ,$$

where $f(\xi_0) = \phi_1$. This gives

$$\xi_0 + \frac{1}{2}L = \xi_0 + \int_{\phi_1}^{\phi_2} \frac{1}{G(\phi)}\, d\phi, \text{ so that } f(\xi_0 + \tfrac{1}{2}L) = \phi_2$$

On its decreasing segments one has

$$(++) \qquad \xi = \xi_0 + \frac{1}{2}L - \int_{f(\xi)}^{\phi_2} \frac{1}{\sqrt{\alpha\phi^2 - \frac{\lambda}{2}\phi^4 + A}}\, d\phi ,$$

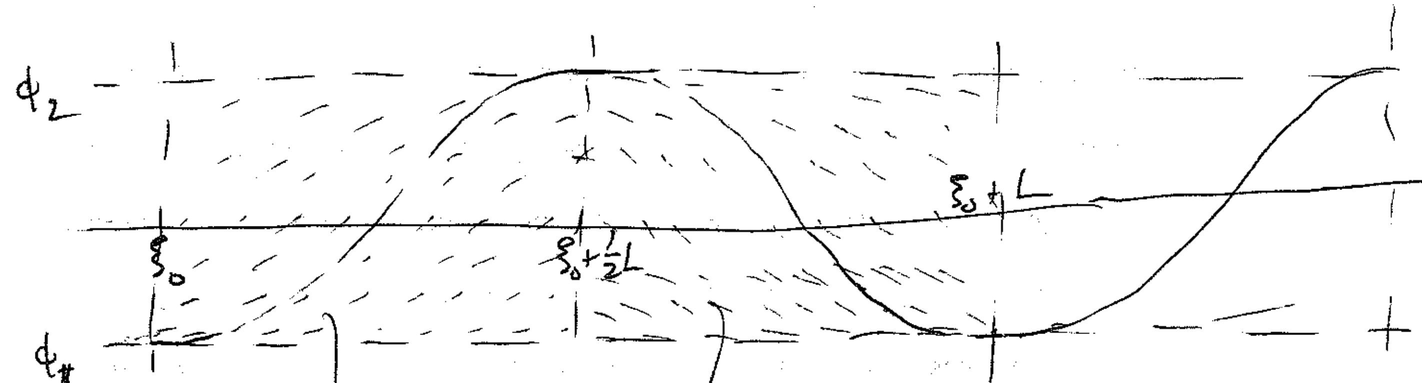
so $f(\xi_0 + L) = \phi_1 = f(\xi_0)$.

Equations (†) and (††) describe one period of a periodic solution of $f'' - \alpha f + \lambda f^3 = 0$.

The resulting ~~wave~~ wave-train solution of NLS is

$$u = e^{i(kx - \omega t)} f(x - ct).$$

The function f is an <u>elliptic function</u>



<u>Solitary waves</u>

If we set $A = 0$, the polynomial becomes $\alpha\phi^2 - \frac{\lambda}{2}\phi^4$, which has a double root at $\phi_1 = 0$. Because of this, the function $\dfrac{1}{\sqrt{\alpha\phi^2 - \frac{\lambda}{2}\phi^4}}$ is not finitely integrable on $[\phi_1, \phi_2]$, and the "period" L is ∞, that is, the solution is no longer periodic but instead decays exponentially as $|\xi| \to \infty$.

In this case, one checks that a solution (all other solutions are obtained by a shift in ξ) is

$$f(\xi) = \left(\frac{2\alpha}{\lambda}\right)^{\frac{1}{2}} \operatorname{sech}(\alpha^{1/2}\xi),$$

and we obtain solutions of NLS :

$$\longrightarrow \quad u(x,t) = e^{i(kx-\omega t)} \left(\frac{2\alpha}{\lambda}\right)^{1/2} \operatorname{sech}\left(\alpha^{1/2}(x-2kt)\right),$$

(recall $2k - c = 0$) with $\alpha = k^2 - \omega$.

This kind of a solution is called a <u>solitary wave</u>, and in some instances a <u>soliton</u>.