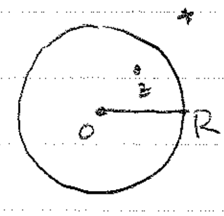
Toward the <u>Nevanlinna Theorem</u> for complex analytic functions from the upper-half plane to itself.

Let $f(z)$ be a complex-analytic function defined in an open set that contains the closed disk $\{z : |z| \le R\} =: \overline{D_R}$ of radius R about 0. For $z \in \mathbb{C}$ with $|z| < R$, the Cauchy integral formula gives



$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta ,$$

where $C_R = \{z : |z| = R\}$ is the circle of radius R.

<u>Define the reflection z^* of z about C_R</u> by

$$z^* = R^2 \bar{z}^{-1} .$$

Since $|z^*| > R$, $f(\zeta)/(\zeta - z^*)$ is analytic in a neighborhood of D_R, and we obtain

$$0 = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z^*}\, d\zeta .$$

Subtracting the two integrals yields

$$f(z) = \frac{1}{2\pi i} \int_{C_R} f(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z^*} \right) d\zeta .$$

Let us put $\zeta = Re^{it}$ and $z = \rho e^{i\phi}$. So $d\zeta = iRe^{it} dt$.

This parametrization gives, for $\rho < R$,

$$f(\rho e^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{it})\left(\frac{Re^{it}}{Re^{it}-\rho e^{i\phi}} - \frac{e^{it}}{e^{it}-R\rho^{-1}e^{i\phi}}\right)dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{it})\left(\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\phi-t)}\right)dt$$

Notice that the integral kernel in this expression is positive; it is known as the Poisson kernel:

$$K(R,t;\rho,\phi) = \frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\phi-t)}$$

Since this kernel is real-valued, we obtain integral representations for the real and imaginary parts of f. If $f(z) = u(z) + iv(z)$,

$$u(\rho e^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{it})\left(\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\phi-t)}\right)dt$$

$$v(\rho e^{it}) = \frac{1}{2\pi}\int_0^{2\pi} v(Re^{it})\left(\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\phi-t)}\right)dt$$

The Poisson kernel produces a harmonic function (u or v) in the disk in terms of its boundary values on the circle.

Now notice that the Poisson kernel is the real part of an analytic kernel:

$$(\dagger) \qquad \frac{\zeta + z}{\zeta - z} = \frac{Re^{it} + \rho e^{i\phi}}{Re^{it} - \rho e^{i\phi}} = \frac{R + \rho e^{i(\phi - t)}}{R - \rho e^{i(\phi - t)}}$$

$$= \frac{R^2 - \rho^2 + i\, 2R\rho \sin(\phi - t)}{R^2 + \rho^2 - 2R\rho \cos(\phi - t)}$$

$$= K(R, t; \rho, \phi) + i L(R, t; \rho, \phi)$$

[Note: as $\rho \to R$, K approaches the delta-function, or the identity as a convolution operator; and L as a convolution operator approaches the Hilbert transform on the circle.]

The function $g(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{it}) \frac{Re^{it} + z}{Re^{it} - z} dt$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(Re^{it})\, K(R, t; z)\, dt + i \frac{1}{2\pi} \int_0^{2\pi} u(Re^{it})\, L(R, t; z)\, dt$$

is analytic at all z in the open disk D_R and its real part is equal to $u(z)$.
Also notice that $g(0) = \int_0^{2\pi} u(Re^{it})\, dt = u(0)$, which is real.
Since an analytic function is determined up to an additive imaginary constant by its real part, we obtain

$$(*) \qquad \boxed{f(z) = i\beta + \frac{1}{2\pi} \int_0^{2\pi} u(Re^{it}) \frac{Re^{it} + z}{Re^{it} - z} dt, \quad v(0) = \beta.}$$

Source: Akhiezer & Glazman Ch. VI

Representation Theorems for analytic functions

Denote by D the open unit disk: $D = \{z : |z| < 1\}$; and by H_+ the open upper half plane: $H_+ = \{z : \operatorname{Im}(z) > 0\}$.

Theorem 1 A function $f : D \to \mathbb{C}$ is analytic and has a nonnegative real part if and only if there exists a real number β and an increasing function $\sigma : [0, 2\pi] \to \mathbb{R}$ such that, $\forall z \in D$,

* Note $\beta = \operatorname{Im} f(0)$

$$f(z) = i\beta + \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, d\sigma(t).$$

Theorem 2 A function $f : H_+ \to \mathbb{C}$ is analytic and has a nonnegative imaginary part if and only if there exist real numbers α and μ and an increasing function $\sigma : \mathbb{R} \to \mathbb{R}$ such that, $\forall z \in H_+$,

$$f(z) = \alpha + \mu z + \int_{-\infty}^{\infty} \frac{1 + tz}{t - z}\, d\sigma(t).$$

Theorem 3 A function $f : H_+ \to \mathbb{C}$ is analytic, has a nonnegative imaginary part, and satisfies

$$\limsup_{y \to \infty} |y f(iy)| < \infty$$

if and only if $\exists$ an increasing function $\sigma : \mathbb{R} \to \mathbb{R}$ of BV s.th. $\forall z \in H_+$,

$$f(z) = \int_{-\infty}^{\infty} \frac{1}{t - z}\, d\sigma(t).$$

<u>Proof of Theorem 1</u> Assuming the given representation of f, and setting $\Phi(t,z) = (e^{it}+z)/(e^{it}-z)$, we have

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \int_0^{2\pi} \frac{\Phi(t,z+\Delta z)-\Phi(t,z)}{\Delta z}\, d\sigma(t)$$

$$\longrightarrow \int_0^{2\pi} \frac{\partial \Phi}{\partial z}(t,z)\, d\sigma(t) \quad \text{as } \Delta z \to 0.$$

The convergence is valid by the following reasoning:
The difference quotient of Φ is continuous in t and Δz for $(t,\Delta z) \in [0,2\pi] \times \{\Delta z : |\Delta z| \le \varepsilon\}$ for some $\varepsilon > 0$, and thus Φ is uniformly continuous on this compact set.
Thus one obtains uniform convergence of the difference quotients to $\partial\Phi/\partial z\,(t,z)$ as $\Delta z \to 0$, and the convergence of the integrals follows. Thus f is analytic.
Recall from (†)p.9 that $\operatorname{Re} \Phi(t,z) > 0$. Given that σ is increasing, we find that $\operatorname{Re} f(z) \ge 0$.

Now assume $f : D \to \mathbb{C}$ is analytic and that $\operatorname{Re} f(z) \ge 0\ \forall z \in D$. By the representation (*)p.9, we have for $|z| < R < 1$,

$$(\circ) \qquad f(z) = \int_0^{2\pi} \frac{Re^{it}+z}{Re^{it}-z}\, d\sigma_R(t) + i \operatorname{Im} f(0)$$

in which $\sigma_R(t) = \frac{1}{2\pi}\int_0^t \operatorname{Re} f(Re^{is})\, ds \quad \forall\, t \in [0,2\pi]$.

Note: The Helly convergence theorem can be generalized by replacing f with a uniformly convergent sequence $f_n \to f \Rightarrow \int_a^b f_n \, d\sigma_n \to \int_a^b f \, d\sigma$

Since $\operatorname{Re} f(z) \geq 0$, σ_R is increasing on $[0, 2\pi]$, and the Helly selection theorem provides a sequence $\{R_j\}_{j=1}^{\infty}$ with $R_j \to 1$ as $j \to \infty$ and an increasing function $\sigma : [0, 2\pi) \to \mathbb{R}$ such that $\forall\, t \in [0, 2\sigma]$,

$$\lim_{j \to \infty} \sigma_{R_j}(t) = \sigma(t).$$

From the Helly convergence theorem and $(*)$ p.11, we obtain

$$f(z) = i \operatorname{Im} f(0) + \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, d\sigma(t). \quad \blacksquare$$

<u>Proof of Theorem 2</u> The given representation has nonnegative imaginary part whenever $\operatorname{Im} z \geq 0$ because

$$\operatorname{Im} \frac{1 + tz}{t - z} = \frac{1 + t^2}{|t - z|^2} \operatorname{Im} z \geq 0.$$

Analyticity will be shown later.

Let $f : H_+ \to \mathbb{C}$ be analytic with $\operatorname{Im} f(z) \geq 0$ $\forall\, z \in H_+$. Define a function $g : D \to \mathbb{C}$ by

$$g(\zeta) := -i f\left(i \frac{1 + \zeta}{1 - \zeta} \right) \quad \forall\, \zeta \in D.$$

This is well defined because the map $\zeta \mapsto i \frac{1+\zeta}{1-\zeta} = z$ takes D onto H_+. Also, g is analytic and $\forall\, \zeta \in D$, $\operatorname{Re} g(\zeta) = \operatorname{Im} f(z) \geq 0$ $(z = i(1+\zeta)/(1-\zeta))$. By Theorem 1, there is an increasing function $\rho : [0, 2\pi] \to \mathbb{R}$ and a real number β such that $\forall\, \zeta \in D$,

(*) $$g(\varsigma) = i\beta + \int_0^{2\pi} \frac{e^{is}+\varsigma}{e^{is}-\varsigma}\, d\rho(s)$$

$$= i\beta + \frac{1+\varsigma}{1-\varsigma}\mu + \int_{0+0}^{2\pi-0} \frac{e^{is}+\varsigma}{e^{is}-\varsigma}\, d\rho(s),$$

in which $\int_{0-0}^{2\pi-0} \cdot\, d\rho(s)$ means $\int_{(0,2\pi)} d\mu_\rho$ (Lebesgue-Stieltjes integral)

and $\mu = \left(\rho(2\pi) - \rho(2\pi-0) + \rho(0+0) - \rho(0)\right)$.

For each $z \in H_+$, $\exists\, \varsigma \in D$ s.th. $z = i\frac{1+\varsigma}{1-\varsigma}$, namely $\varsigma = \frac{z-i}{z+i}$,

so $f(z) = ig(\varsigma) = -\beta + \mu z + \int_{0+0}^{2\pi-0} \frac{z\cot\frac{s}{2} - 1}{\cot\frac{s}{2} + z}\, d\rho(s)$.

Putting $\alpha = -\beta$, $t = -\cot\frac{s}{2}$ $(t \in \mathbb{R})$, and $\sigma(t) = \rho(2\operatorname{arccot}(-t))$, this becomes

$$f(z) = \alpha + \mu z + \int_{-\infty}^{\infty} \frac{1+zt}{t-z}\, d\sigma(t) .$$

Finally, given any representation of this form, the transformation $\varsigma = \frac{z-i}{z+i}$ taking H_+ to D produces a function $g(\varsigma) = f(z)$ admitting the representation (*) p.13. Since g is analytic, so is f. ∎

<u>Proof of Theorem 3</u> Given σ as in the theorem, the function f defined by the integral has $\operatorname{Im} f(z) \geq 0$ $\forall\, z \in H_+$ since the integrand has positive imaginary part. The analyticity can be established by showing uniform (in t) convergence of the difference quotients of the integrand with respect to z; this is left to the reader.

Now suppose that $f: H_+ \to \mathbb{C}$ is analytic, $\operatorname{Im} f \geq 0$, and $\limsup_{y\to\infty} |y f(iy)| < \infty$. By Theorem 2, there exists an increasing function $\rho: \mathbb{R} \to \mathbb{R}$ and real numbers α and μ such that $\forall\, z \in H_+$,

$$f(z) = \alpha + \mu z + \int_{-\infty}^{\infty} \frac{1+tz}{t-z}\, d\rho(t).$$

By the assumption on $yf(iy)$, we obtain, for some M,

$$\left| \alpha y + i\mu y^2 + \int_{-\infty}^{\infty} \frac{y(1+ity)}{t-iy}\, d\rho(t) \right| \leq M \quad \forall\, y \geq 0,$$

and taking real and imaginary parts yields

(1) $$y\left| \alpha + \int_{-\infty}^{\infty} \frac{(1-y^2)t}{t^2+y^2}\, d\rho(t) \right| \leq M$$

(2) $$y^2\left| \mu + \int_{-\infty}^{\infty} \frac{1+t^2}{t^2+y^2}\, d\rho(t) \right| \leq M$$

$\forall\, y > 0$

Inequality (2) yields $\mu = 0$ and (1) yields

$$\alpha = \lim_{y\to\infty} \int_{-\infty}^{\infty} \frac{(y^2-1)t}{t^2+y^2}\, d\rho(t) = \int_{-\infty}^{\infty} t\, d\rho(t).$$

Thus

$$f(z) = \int_{-\infty}^{\infty} t\, d\rho(t) + \int_{-\infty}^{\infty} \frac{1+tz}{t-z}\, d\rho(t) = \int_{-\infty}^{\infty} \frac{1+t^2}{t-z}\, d\rho(t).$$

Inequality (2) implies $\int_{-\infty}^{\infty} y^2 \frac{1+t^2}{t^2+y^2}\, d\rho(t) \leq M \quad \forall\, y \geq 0$,

so that $$\int_{-\infty}^{\infty} (1+t^2)\, d\rho(t) \leq M \quad \left(\text{take } b \to \infty \text{ in } \int_{-b}^{b}\right).$$

Because of this, the increasing function $\sigma : \mathbb{R} \to \mathbb{R}$ defined by

$$\sigma(t) = \int_{-\infty}^{t} (1+s^2)\, d\rho(s)$$

is of bounded variation, and $d\mu_\sigma(t) = (1+t^2)\, d\mu_\rho(t)$.
Finally,

$$f(z) = \int_{-\infty}^{\infty} \frac{1+t^2}{t-z}\, d\rho(t) = \int_{-\infty}^{\infty} \frac{1}{t-z}\, d\sigma(t). \qquad \blacksquare$$

Math 7384 : Problems

1. Prove that the integrator functions σ in Theorem 1 and in Theorem 2 of the class notes are of bounded variation on $[0, 2\pi]$.

2. Find α, μ and $\sigma(t)$ so that the integral formula of Theorem 2, p.10, produces the following functions defined on the upper half plane:

(a) $f(z) = \frac{-1}{z}$

(b) $f(z) = \frac{4z}{1-z^2}$

(c) $f(z) = i$

3. Find an integral representation akin to those of Theorems 1, 2, 3 that characterizes exactly those analytic functions defined outside the closed unit disk whose real part is nonnegative.
$|z| > 1$ Prove that your representation holds if and only if the function has these properties (analytic and $\mathrm{Re} f \geq 0$).