Toward a spectral representation of a self-adjoint operator T in a Hilbert space $\mathcal{H}$.

Fact If $z \notin \mathbb{R}$, then $\|(T-z)^{-1}\| \le \frac{1}{|\mathrm{Im}(z)|}$.

Proof For $v \in \mathcal{H}$, $\exists$ $u \in \mathcal{D}(T)$ s.th. $(T-z)u = v$. Put $z = x+iy$.
We have seen that $\|(T-z)u\|^2 = \|(T-x)u\|^2 + y^2\|u\|^2$.

$$\text{Thus } \frac{\|(T-z)^{-1}v\|}{\|v\|} = \frac{\|u\|}{\|(T-z)u\|} \le \left(\frac{\|u\|^2}{\|(T-x)u\|^2 + y^2\|u\|^2}\right)^{1/2}$$

$$\le \left(\frac{\|u\|^2}{y^2\|u\|^2}\right)^{1/2} = \frac{1}{|y|}. \quad \blacksquare$$

Let us denote the resolvent of T by $R_z = (T-z)^{-1}$.

The resolvent identity: For $z, z_1 \notin \mathbb{R}$,

$$R_{z_1} - R_z = (z_1 - z) R_{z_1} R_z$$

Proof: $(T-z_1)(R_{z_1} - R_z)(T-z) = (T-z) - (T-z_1) = z_1 - z$

$$\Longrightarrow \quad R_{z_1} - R_z = (T-z_1)^{-1}(z_1-z)(T-z)^{-1} = (z_1-z)R_{z_1}R_z \quad \blacksquare$$

By repeated use of the resolvent identity, we find that

$$\begin{aligned} R_{z_1} &= R_z + (z_1-z)R_{z_1}R_z \\ &= R_z + (z_1-z)R_z^2 + (z_1-z)^2 R_{z_1}R_z^2 \\ &= R_z + (z_1-z)R_z^2 + (z_1-z)^2 R_z^3 + (z_1-z)^3 R_{z_1}R_z^3 = \ldots \end{aligned}$$

R_z is analytic in H_+ and in $H_- = \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}$:

$$\frac{R_{z_1} - R_z}{z_1 - z} = R_z^2 + (z_1 - z) R_{z_1} R_z^2 ,$$

$$\| (z_1 - z) R_{z_1} R_z^2 \| \leq |z_1 - z| \, \|R_{z_1}\| \, \|R_z\|^2 \leq \frac{|z_1 - z|}{|\operatorname{Re} z_1| \, |\operatorname{Re} z|^2} ,$$

which tends to 0 as $z_1 \to z$.

So $\frac{d}{dz} R_z = R_z^2$ [Notice $\frac{d}{dz} \frac{1}{t-z} = \frac{1}{(t-z)^2}$ for scalars t.]

This implies that, $\forall \ u \in \mathcal{H}$, $(R_z u, u)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$, and

$(R_z u, u)$ analytic ✓

$$\frac{d}{dz}(R_z u, u) = (R_z^2 u, u) .$$

In addition if $R_z = v$, then
$(R_z u, u) = (v, (T - z) v) = (v, (T - x) v) + iy(v, v)$. Thus

$H_+ \to H_+$ ✓

$$\operatorname{Im} z > 0 \implies \operatorname{Im} (R_z u, u) > 0 .$$

Also, for $y \neq 0$,

decays @ ∞ ✓

$$|y| \, |(R_z u, u)| \leq |y| \, \|R_z u\| \, \|u\| \leq |y| \frac{\|u\|}{|y|} \|u\| = \|u\|^2$$

Thus there exists an increasing function $\sigma(t; u)$ of bounded variation such that

$$! \longrightarrow \quad (R_z u, u) = \int_{-\infty}^{\infty} \frac{1}{t - z} \, d\sigma(t; u) .$$

$\sigma(t;u)$ is the *spectral measure* for (R_z, u). We will take σ to be left continuous with $\sigma(-\infty;u)=0$ to make it *unique*.

Fact $R_z^* = R_{\bar z}$ $\forall$ $z \in \mathbb{C}\setminus\mathbb{R}$.

Proof $\forall$ $u,v \in \mathcal{H}$, $(R_z u, v) = (R_z u, (T-\bar z) R_{\bar z} v) = ((T-z)R_z u, R_{\bar z} v) = (u, R_{\bar z} v)$. ∎

Now suppose $y<0$. Then $\forall$ $u \in \mathcal{H}$,

$$(R_z u, u) = (u, R_{\bar z} u) = (R_{\bar z} u, u)^* = \left(\int_{-\infty}^{\infty} \frac{1}{t-\bar z}\, d\sigma(t;u) \right)^*$$

$$= \int_{-\infty}^{\infty} \frac{1}{t-z}\, d\sigma(t;u)$$

Thus the integral representation of $(R_z u, u)$ is valid for all $z \in \mathbb{C}\setminus\mathbb{R}$.

(*) *Fact* $\sigma(\infty;u) := \displaystyle\int_{-\infty}^{\infty} d\sigma(t;u) \le (u,u)$

Proof We have seen that $|y|\,|(R_{iy}u,u)| \le (u,u)$, or $\left| \int_{-\infty}^{\infty} \frac{y}{t-iy}\, d\sigma(t;u) \right| \le (u,u)$. The triangle inequality gives

$$\left| \int_{-A}^{A} \frac{y}{t-iy}\, d\sigma(t;u) \right| \le \left| \int_{-\infty}^{\infty} \frac{y}{t-iy}\, d\sigma(t;u) \right| + \left| \int_{-\infty}^{-A} \frac{y}{t-iy}\, d\sigma \right| + \left| \int_{A}^{\infty} \frac{y}{t-iy}\, d\sigma \right|$$

$$\overset{y\to\infty}{\Longrightarrow} \int_{-A}^{A} d\sigma(t;u) \le (u,u) + \left| \overset{\le 1\ \forall y}{\int_{-\infty}^{-A} d\sigma(t;u)} \right| + \left| \overset{\le 1\ \forall y}{\int_{A}^{\infty} d\sigma(t;u)} \right|$$

$$\overset{A\to\infty}{\Longrightarrow} \int_{-\infty}^{\infty} d\sigma(t;u) \le (u,u).$$

We now use the "polarization identity" for sesquilinear forms $[u,v]$ to obtain an integral representation for $(R_z u, v)$.

A sesquilinear form $[u,v]$ is linear in one coordinate (say u) and $[u,v] = [v,u]^*$. (This implies $[u,\alpha v] = \bar{\alpha}[u,v]$.)

Polarization: $\frac{1}{4}\sum_{k=0}^{3} i^k [u+i^k v, u+i^k v] =$

$$= \frac{1}{4}\sum_{k=0}^{3} i^k \left([u, i^k v] + [u,u] + [i^k v, u] + [i^k v, i^k u] \right)$$

$$= \frac{1}{4}\sum_{k=0}^{3} \left([u,v] + \underbrace{i^k [u,u] + i^{2k}[v,u] + i^k [v,u]}_{\text{Each vanishes over } \sum_k} \right)$$

$$= [u,v].$$

Since $(R_z u, v)$ is a sesquilinear form in $\mathcal{H}$, we put

$$\sigma(t; u,v) = \frac{1}{4}\sum_{i=0}^{3} i^k \sigma(t; u+i^k v) \qquad \text{(bdd var. on } \mathbb{R})$$

to obtain

$$\longrightarrow \qquad (R_z u, v) = \int_{-\infty}^{\infty} \frac{1}{t-z}\, \sigma(t; u,v) .$$

Observe that $\int_{-\infty}^{\infty} \frac{1}{t-z}\, d\sigma(t; v,u) = (R_z v, u) = (v, R_{\bar{z}} u)$

$$= \overline{(R_{\bar{z}} u, v)} = \left(\int_{-\infty}^{\infty} \frac{1}{t-\bar{z}}\, d\sigma(t; u,v) \right)^* = \int_{-\infty}^{\infty} \frac{1}{t-z}\, \overline{d\sigma(t; u,v)} .$$

Because of the uniqueness of $\sigma(t;u,v)$ (given left continuity and $\sigma(-\infty;u,v)=0$), we find that

$$\sigma(t;v,u) = \overline{\sigma(t;u,v)}.$$

'Tis straightforward to show linearity in the first argument of $\sigma(t;\cdot,\cdot)$ $\forall\, t$. So $\sigma(t;\cdot,\cdot)$ is a sesquilinear form for each $t \in \mathbb{R}$. Moreover, it is bounded because of $(*)$ p. 25: $\sigma(t;u,u) \leq (u,u)$. This means there exist bounded operators $E_t : \mathcal{H} \to \mathcal{H}$ s.th. $\sigma(t;u,v) = (E_t u, v)$ $\forall\, u,v \in \mathcal{H}$.

Here are the details: $\forall\, \gamma \in \mathbb{C}$, $u,v \in \mathcal{H}$,

$$
\begin{aligned}
0 &\leq \sigma(t; u+\gamma v, u+\gamma v) \\
&= \sigma(t;u,u) + \gamma\sigma(t;v,u) + \bar{\gamma}\sigma(t;u,v) + |\gamma|^2\sigma(t;v,v)
\end{aligned}
$$

For $\gamma = \alpha$ $(\alpha \in \mathbb{R})$ and for $\gamma = i\beta$ $(\beta \in \mathbb{R})$, we obtain

$$\sigma(t;u,u) + 2\alpha\,\mathrm{Re}\,\sigma(t;u,v) + \alpha^2\sigma(t;v,v) \geq 0$$
$$\sigma(t;u,u) + 2\beta\,\mathrm{Im}\,\sigma(t;u,v) + \beta^2\sigma(t;v,v) \geq 0$$

Since this holds $\forall\, \alpha \in \mathbb{R}$, $(\mathrm{Re}\,\sigma(t;u,v))^2 - \sigma(t;u,u)\sigma(t;v,v) \leq 0$ which implies

$$|\mathrm{Re}\,\sigma(t;u,v)| \leq \sigma(t;u,u)^{1/2}\sigma(t;v,v)^{1/2} \leq \|u\|\,\|v\| \quad ,$$

with an analogous inequality for $\mathrm{Im}\,\sigma(t;u,v)$.

Thus $|\sigma(t;u,v)| \le \sqrt{2}\,\|u\|\,\|v\|$, so that $\sigma(t;u,\cdot):\mathcal{H}\to\mathbb{C}$ is a bounded conjugate-linear functional on $\mathcal{H}$ whose norm is $\le \sqrt{2}\,\|u\|$. Thus the map

$$\mathcal{H} \to \mathcal{H}^* :: u \mapsto \sigma(t;u,\cdot)$$

is bounded by $\sqrt{2}$ and linear. The Riesz theorem provides a bounded linear operator $E_t : \mathcal{H}\to\mathcal{H}$ such that

The Riesz thm asserts a Hilbert space isomorphism $\varphi : \mathcal{H}^* \to \mathcal{H}$ w/ $w^*(v) = (\varphi(w), v)$ $\forall\, w^* \in \mathcal{H}^*$ & $v \in \mathcal{H}$.

$$\sigma(t;u,v) = (E_t u, v) \qquad \forall\ u,v \in \mathcal{H}.$$

[We have $\|E_t\| \le \sqrt{2}$, but in fact E_t is a projection, as we will see, so $\|E_t\| = 1$ unless $E_t = 0$.].

Since $\sigma(\infty;u,v) := \lim_{t\to\infty} \sigma(t;u,v)$ is bounded by $\sqrt{2}\,\|u\|\,\|v\|$, there is a bounded linear operator $E_\infty : \mathcal{H}\to\mathcal{H}$ s.th.

$$\sigma(\infty;u,v) = (E_\infty u, v) \qquad \forall\ u,v \in \mathcal{H}.$$

We obtain the following integral representation of the resolvent of T:

$$\longrightarrow \qquad (R_z u, v) = \int_{-\infty}^{\infty} \frac{1}{t-z}\, d(E_t u, v).$$

Fact $\forall\, s,t \in \mathbb{R}$, E_t is an orthogonal projection and

$$E_s E_t = E_{\min\{s,t\}}, \quad \text{or} \quad E_s \le E_t .$$

This shows that $E_s E_t = E_t E_s$.

Also, $E_\infty = I$ (the identity operator on $\mathcal{H}$).

Proof To see that E_t is self-adjoint, $\forall\, u,v \in \mathcal{H}$ write

$$(E_t u, v) = \sigma(t; u,v) = \overline{\sigma(t; v,u)} = \overline{(E_t v, u)} = (u, E_t v).$$

Next, recall that $\frac{d}{dz} R_z = R_z^2$ and $R_z^* = R_{\bar z}$, so

$$\frac{d}{dz}(R_z u,v) = (R_z^2 u, v) = (R_z u, R_{\bar z} v) = \int_{-\infty}^{\infty} \frac{1}{t-z}\, d(E_t u, R_{\bar z} v)$$

and

$$\frac{d}{dz}(R_z u,v) = \frac{d}{dz}\int_{-\infty}^{\infty} \frac{1}{t-z}\, d(E_t u, v)$$

$$= \int_{-\infty}^{\infty} \frac{1}{(t-z)^2}\, d(E_t u, v) = \int_{-\infty}^{\infty} \frac{1}{t-z}\, d\int_{-\infty}^{t} \frac{1}{s-z}\, d(E_s u, v).$$

By the uniqueness of the left-continuous spectral measures that vanish at $-\infty$, we obtain

$$\int_{-\infty}^{t} \frac{1}{s-z}\, d(E_s u, v) = (E_t u, R_{\bar z} v) = (R_z E_t u, v)$$

$$= \int_{-\infty}^{\infty} \frac{1}{s-z}\, d(E_s E_t u, v).$$

Thus $\underline{(E_s u, v) = (E_s E_t u, v) \text{ for } s \le t}$ and

and $\underline{(E_t u, v) = (E_s E_t u, v) \text{ for } s \geq t}$. This is because of the left continuity of $(E_t u, v)$ and because $d(E_s E_t u, v) = 0$ for $s \geq t \Rightarrow (E_s E_t u, v) = \text{const}$ $= (E_t E_t u, v) = (E_t u, v)$ for $s \geq t$.

E_t is a projection because $E_t E_t = I$, and it is orthogonal because E_t is self adjoint. This is because $\mathrm{Ran}\, E_t \perp \mathrm{Null}\, E_t^* = \mathrm{Null}\, E_t$.

For $s \leq t$, $E_s \leq E_t$ (ie $(E_s u, u) \leq (E_t u, u)\ \forall\, u \in \mathcal{H}$) because
$(E_s u, u) = (E_t E_s E_t u, u) = (E_s E_t u, E_t u) \leq (E_t u, E_t u) = (E_t u, u)$.
$\uparrow$ b/c E_s is a projection.

To see that $E_\infty = I$, $\forall\ u, v \in \mathcal{H}$ write

$$(E_t E_\infty u, v) = (E_\infty u, E_t v) = \lim_{s \to \infty} (E_s u, E_t v) = \lim_{s \to \infty} (E_t E_s u, v)$$

$$= \lim_{s \to \infty} (E_t u, v) = (E_t u, v),$$

so $E_t E_\infty = E_t \quad \forall\, t \in \mathbb{R}$. Thus

$$(R_z E_\infty u, v) = \int_{-\infty}^{\infty} \frac{1}{t-z} (E_t E_\infty u, v) = \int_{-\infty}^{\infty} \frac{1}{t-z} (E_t u, v) = (R_z u, v)$$

so that $R_z E_\infty u = R_z u \ \forall\ u \in \mathcal{H}$, and since R_z is injective, $E_\infty u = u \ \forall\ u \in \mathcal{H}$.

As an exercise, prove that $E_s \to E_t$ strongly as $s \to t^-$, that is, E_t is strongly left continuous.
This really means that $E_s \to E_t$ "pointwise" as $s \to t^-$:

$$E_s u \to E_t u \quad \text{as } s \to t^- \quad \forall\ u \in \mathcal{H}.$$

Notice that strong (i.e. pointwise) convergence of operators $A_s \to A$ implies weak convergence ($(A_t u, v) \to (Au, v)$ as $s \to t$ $\forall\, u, v \in \mathcal{H}$) and is implied by operator-norm (i.e. uniform on the unit ball) convergence ($\|A_t - A\| \to 0$).

The properties we have shown about the family $\{E_t\}_{t\in\mathbb{R}}$ make it a <u>resolution of the identity</u> on $\mathcal{H}$:

$\forall\, s,t \in \mathbb{R}$:

(i) $E_s E_t = E_{\min\{s,t\}}$
(ii) E_t is self-adjoint
$\Big\}\Rightarrow$ E_t is an orthogonal proj. and $E_s \le E_t$

(iii) E_t is strongly (i.e. pointwise) left continuous

(iv) $E_{-\infty} := \lim_{t\to-\infty} E_t = 0$ strongly
$E_{\infty} := \lim_{t\to\infty} E_t = I$ strongly

As an exercise, prove that E_t <u>reduces</u> R_z, that is, E_t <u>projects</u> R_z onto $\operatorname{Ran}(E_t)$. More precisely,

$$E_t R_z = R_z E_t \quad \forall\ t \in \mathbb{R},$$

$\to$ We write $$R_z = \int_{-\infty}^{\infty} \frac{1}{t-z}\, dE_t .$$

Now that we have a resolution of the identity associated to the resolvent of T, we will show that T is also represented by an integral with respect to the same resolution in a natural way:

$$(Tu, v) = \int_{-\infty}^{\infty} t \, d(E_t u, v) .$$

To prove this, first define an operator T_1 in $\mathcal{H}$ by

defn. of standard "diagonal" operator associated to $\{E_t\}$

$$\mathcal{D}(T_1) = \Big\{ u \in \mathcal{H} : \int_{-\infty}^{\infty} t \, d(E_t u, v) \text{ converges as an improper Stieltjes integral and is bounded by } C\|v\| \text{ for some constant } C \Big\}$$

$\forall\, u \in \mathcal{D}(T_1)$, $T_1 u$ is the unique element of $\mathcal{H}$ s.th.

$$(T_1 u, v) = \int_{-\infty}^{\infty} t \, d(E_t u, v) \quad \forall \; v \in \mathcal{H} .$$

We write $T_1 u = \int_{-\infty}^{\infty} t \, dE_t u$, or simply $\underline{T_1 = \int_{-\infty}^{\infty} t \, dE_t}$.

We will prove that $T = T_1$.

It is also true (see Akhiezer and Glazman §63) that

$$\mathcal{D}(T) = \Big\{ u \in \mathcal{H} : \int_{-\infty}^{\infty} t^2 \, d(E_t u, u) < \infty \Big\} .$$

To prove that $T = T_1$, it suffices to prove that

$$\begin{cases} (T_1 - z) R_z = I \\ R_z (T_1 - z) = I_{\mathcal{D}(T_1)} . \end{cases}$$

The second shows that $\mathcal{D}(T_1) \subset \mathcal{D}(T)$ since $\mathcal{D}(T) = \mathrm{Ran}(R_z)$.
The first shows that $\mathcal{D}(T) \subset \mathcal{D}(T_1)$ and that $T_1 - z = T - z$, so that $T_1 = T$.

$(T_1 - z) R_z = I$

First we prove that $((T_1 - z) R_z u, v) = (u, v) \quad \forall\, u, v \in \mathcal{H}$.
This tacitly involves proving that $R_z u \in \mathcal{D}(T_1)$.

$$\int_{-A}^{A} (t-z)\, d(E_t R_z u, v) = \int_{-A}^{A} (t-z)\, d(R_z u, E_t v)$$

$$= \int_{-A}^{A} (t-z)\, d_t \int_{-\infty}^{\infty} \frac{1}{s-z}\, d_s(E_s u, E_t v) = \int_{-A}^{A} (t-z)\, d \int_{-\infty}^{t} \frac{1}{s-z}\, d(E_s u, v)$$

$$= \int_{-A}^{A} \frac{t-z}{t-z}\, d \int_{-\infty}^{t} d(E_s u, v) = \int_{-A}^{A} d(E_s u, v) \xrightarrow{A \to \infty} \int_{-\infty}^{\infty} d(E_s u, v)$$

$$= (E_\infty u, v) = (u, v)$$

Thus $\int_{-\infty}^{\infty} (t-z)\, d(E_t R_z u, v)$ exists and is equal to (u, v),

that is, $R_z u \in \mathcal{D}(T_1)$ and $((T_1 - z) R_z u, v) = (u, v)$

$\forall\, u, v \in \mathcal{H}$, so $(T_1 - z) R_z = I$. ✓

Next we show that $\forall\, u \in \mathcal{D}(T_1)$, $R_z(T-z)u = u$.

Assuming $u \in \mathcal{D}(T_1)$, we have

$$\int_{-\infty}^{\infty} (t-z)\, d(E_t u, v) = ((T_1 - z)u, v).$$

$R_z(T_1 - z) = I_{\mathcal{D}(T_1)}$

$$(R_z(T-z)u, v) = \int_{-\infty}^{\infty} \frac{1}{t-z}\, d(E_t (T_1 - z)u, v) = \int_{-\infty}^{\infty} \frac{1}{t-z}\, d((T_1 - z)u, E_t v)$$

$$= \int_{-\infty}^{\infty} \frac{1}{t-z}\, d_t \int_{-\infty}^{\infty} (s-z)\, d_s(E_s u, E_t v)$$

$$= \int_{-\infty}^{\infty} \frac{1}{t-z}\, d \int_{-\infty}^{t} (s-z)\, d(E_s u, v) = \int_{-\infty}^{\infty} \frac{t-z}{t-z}\, d \int_{-\infty}^{t} d(E_s u, v)$$

$$= \int_{-\infty}^{\infty} d(E_t u, v) = (E_\infty u, v) = (u, v).$$

Thus $R_z(T-z)u = u$. ✓

The characterization of $\mathcal{D}(T)$ by

$$\mathcal{D}(T) = \left\{ u \in \mathcal{H} : \int_{-\infty}^{\infty} t^2\, d(E_t u, u) < \infty \right\}$$

is left as reading (Akhiezer/Glazman §63) or an exercise.