Spectral theory for the wave equation in 1D.

The 1D wave equation for $u(x,t)$:

$$u_{tt} = c^2 u_{xx}$$

Notice that this equation admits oscillatory traveling waves of the form

$$e^{ik(x-ct)} \quad \text{(forward)}$$
$$e^{ik(x+ct)} \quad \text{(backward)}$$

We would like to show that the general complex solution is of the form

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\hat{a}_+(k) e^{ik(x-ct)} + \hat{a}_-(k) e^{ik(x+ct)} \right) dk,$$

in which
$$\begin{cases} \hat{a}_+(k) + \hat{a}_-(k) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} u(x,0)\, e^{-ikx}\, dx \\ \hat{a}_+(k) - \hat{a}_-(k) = \dfrac{i}{ck} \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \frac{\partial}{\partial t}(x,0)\, e^{-ikx}\, dx \end{cases}$$

Convert the wave equation into a first-order system with the definition $v = u_t$:

$$u_{tt} = c^2 u_{xx} \iff \begin{cases} u_t = v \\ v_t = c^2 u_{xx} \end{cases}, \quad \text{or}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ c^2\partial^2 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

The operator A defined by

$$\begin{cases} \mathcal{D}(A) = H^2(\mathbb{R}) \oplus H^1(\mathbb{R}) \\ A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ c^2\partial^2 & 0 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \end{cases}$$

is anti-self-adjoint in the Hilbert space

$H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$ with inner product

$$\left(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\right)_{H^1 \oplus L^2} := \left(\begin{bmatrix} -c^2\partial^2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\right)_{L^2 \oplus L^2}$$

$$= c^2(\partial u_1, \partial u_2)_{L^2} + (v_1, v_2)_{L^2}.$$

exercise [Prove this.] Let $\{E_k\}_{k\in\mathbb{R}}$ be a resolution of the identity in L^2 for the operator $-i\partial$:

$$u = \int_{-\infty}^{\infty} dE_k u \quad \forall \ u \in L^2$$

$$-i\partial u = \int_{-\infty}^{\infty} k \, dE_k u \quad \forall \ u \in H^1 = \mathcal{D}(-i\partial).$$

The domain of $-\partial^2 u$ as a self-adjoint operator is $H^2(\mathbb{R})$, and its spectral representation is

$$-\partial^2 u = \int_{-\infty}^{\infty} k^2 \, dE_k u \quad \left(= \int_{-\infty}^{\infty} \lambda \, dE_{\sqrt{\lambda}} u \right)$$

Thus A has the spectral representation

$$(*) \qquad A\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c^2\partial^2 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ c^2\partial^2 u \end{bmatrix}$$

$$= \int_{-\infty}^{\infty} \begin{bmatrix} dE_k v \\ -c^2k^2 dE_k u \end{bmatrix} = \int_{-\infty}^{\infty} \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} dE_{\frac{\omega}{c}} \begin{pmatrix} u \\ v \end{pmatrix}$$

The matrix $\begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$ has eigenvalues $\pm i\omega$ and

exercise → it is anti-self-adjoint with respect to the inner product

$$\Big(\underbrace{\begin{bmatrix} \omega^2 & 0 \\ 0 & 1 \end{bmatrix}}_{\Lambda} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \Big)$$

(natural inner product on $\mathbb{C}^2$ for spec. rep. of wave equation)

Let us write the <u>spectral resolution</u>

of the identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in $\mathbb{C}^2$ corresponding to $\begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$ as an anti-self-adjoint matrix.

for $-i\omega$ for $i\omega$

This means that we find projections P_ω^+ and P_ω^- such that

$$P_\omega^+ + P_\omega^- = I$$

$$-i\omega P_\omega^+ + i\omega P_\omega^- = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$$

with $P_\omega^{\pm 2} = P_\omega^{\pm}$ and both $P_\omega^{\pm}$ <u>self-adjoint</u>

$\Lambda = \begin{bmatrix} \omega^2 & 0 \\ 0 & 1 \end{bmatrix}$ with respect to the inner product $\left(\Lambda \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}\right)$.

One calculates that

$$P_\omega^+ = \frac{1}{2}\begin{bmatrix} 1 & -(i\omega)^{-1} \\ -i\omega & 1 \end{bmatrix}, \quad P_\omega^- = \frac{1}{2}\begin{bmatrix} 1 & (i\omega)^{-1} \\ i\omega & 1 \end{bmatrix}.$$

We now use this together with $(*)$ to obtain a representation of $A\begin{bmatrix} u \\ v \end{bmatrix}$ as a <u>Fourier integral</u>.

Recall that

$$(E_k u)(x) = \int_{-\infty}^{k} \hat{u}(\xi) e^{i\xi x} \, d\xi$$

for all $u \in L^2$.

$$A\begin{bmatrix}u\\v\end{bmatrix} = \int \begin{bmatrix}0 & 1\\ -\omega^2 & 0\end{bmatrix} dE_{\frac{\omega}{c}}\begin{bmatrix}u\\v\end{bmatrix} = \int \left(-i\omega P_\omega^+ + i\omega P_\omega^-\right) dE_{\frac{\omega}{c}}\begin{pmatrix}u\\v\end{pmatrix}$$

$$= \int \left(-ick\, P_{ck}^+ + ick\, P_{ck}^-\right) dE_k\begin{pmatrix}u\\v\end{pmatrix} = \int -ick\left(P_{ck}^+ - P_{ck}^-\right) dE_k\begin{pmatrix}u\\v\end{pmatrix}$$

$$= \frac{1}{\sqrt{2\pi}}\int -ick\left(P_{ck}^+ - P_{ck}^-\right) d\int_{-\infty}^{k}\begin{pmatrix}\hat u(\xi)\\ \hat v(\xi)\end{pmatrix} e^{i\xi x}\, d\xi$$

$$= \frac{1}{\sqrt{2\pi}}\int -ick\left(P_{ck}^+ - P_{ck}^-\right)\begin{pmatrix}\hat u(k)\\ \hat v(k)\end{pmatrix} e^{ikx}\, dk$$

$$(*) \qquad = \frac{1}{\sqrt{2\pi}}\int -ick\left(\hat a_+(k)\begin{pmatrix}1\\-ick\end{pmatrix} - \hat a_-(k)\begin{pmatrix}1\\ick\end{pmatrix}\right) e^{ikx}\, dk$$

(†) where $\begin{pmatrix}\hat u(k)\\ \hat v(k)\end{pmatrix} = a_+(k)\begin{pmatrix}1\\-ick\end{pmatrix} + a_-(k)\begin{pmatrix}1\\ick\end{pmatrix}$

is the decomposition of $\begin{pmatrix}\hat u\\ \hat v\end{pmatrix}$ w/r.to the eigenspaces of A:

Now, let us return to the wave equation

Need Stone's thm. etc, to justify the following.

$$\frac{d}{dt}\begin{pmatrix}u\\v\end{pmatrix} = A\begin{bmatrix}u\\v\end{bmatrix}, \quad \text{or}$$

$$\frac{\partial}{\partial t}\begin{pmatrix}u(x,t)\\ v(x,t)\end{pmatrix} = \begin{bmatrix}0 & 1\\ c^2\partial_{xx} & 0\end{bmatrix}\begin{bmatrix}u(x,t)\\ v(x,t)\end{bmatrix} = \begin{bmatrix}v(x,t)\\ \frac{\partial^2}{\partial x^2}u(x,t)\end{bmatrix}.$$

Let us write $\hat{u} = \hat{u}(k,t)$ and set $\hat{u}(k,0) = \hat{u}(k)$.
etc. . . .

The decomposition (†) gives

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int \hat{u}(k,t)e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int \left(\hat{a}_+(k,t) + \hat{a}_-(k,t)\right)e^{ikx}\,dk$$

$$v(x,t) = \frac{1}{\sqrt{2\pi}}\int \hat{v}(k,t)e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int -i\omega\left(\hat{a}_+(k,t) - \hat{a}_-(k,t)\right)e^{ikx}\,dk$$

Using $\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{bmatrix} u \\ v \end{bmatrix}$ with the representation (*) for A

and differentiating the expressions above in t gives

$$\frac{\partial u}{\partial t} = \frac{1}{\sqrt{2\pi}}\int \frac{\partial}{\partial t}(\hat{a}_+ + \hat{a}_-)e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int -i\omega(\hat{a}_+ - \hat{a}_-)e^{ikx}\,dk$$

$$\frac{\partial v}{\partial t} = \frac{1}{\sqrt{2\pi}}\int -i\omega\frac{\partial}{\partial t}(\hat{a}_+ - \hat{a}_-)e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int (-i\omega)^2(\hat{a}_+ + \hat{a}_-)e^{ikx}\,dk$$

Equating Fourier coefficients yields

$$\left(\frac{\partial}{\partial t} + i\omega\right)\hat{a}_+ + \left(\frac{\partial}{\partial t} - i\omega\right)\hat{a}_- = 0$$

$$\left(\frac{\partial}{\partial t} + i\omega\right)\hat{a}_+ - \left(\frac{\partial}{\partial t} - i\omega\right)\hat{a}_- = 0$$

$$\Longrightarrow \quad \left(\frac{\partial}{\partial t} + i\omega\right)\hat{a}_+ = 0, \qquad \left(\frac{\partial}{\partial t} - i\omega\right)\hat{a}_- = 0$$

The solutions of these ODEs are

$$\begin{cases} \hat{a}_+(k,t) = \hat{a}_+(k)\, e^{-i\omega t} \\ \hat{a}_-(k,t) = \hat{a}_-(k)\, e^{i\omega t} \end{cases}$$

Thus we obtain the solution

$\omega = ck$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int \left(\hat{a}_+(k) e^{i(kx-\omega t)} + \hat{a}_-(k) e^{i(kx+\omega t)} \right) dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\underset{\text{(forward waves)}}{\hat{a}_+(k) e^{ik(x-ct)}} + \underset{\text{(backward waves)}}{\hat{a}_-(k) e^{ik(x+ct)}} \right) dk$$

The "physical field" is the <u>real part</u> of u!

$$\mathrm{Re}\, u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big(\hat{a}_+^r(k) \cos(k(x-ct)) - \hat{a}_+^i \sin(k(x-ct)) +$$
$$+ \hat{a}_-^r(k) \cos(k(x+ct)) - \hat{a}_-^i \sin(k(x+ct)) \Big)$$

$\int_0^\infty$!

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \Big((\hat{a}_+^r(k) + \hat{a}_+^r(-k)) \cos(k(x-ct)) - (\hat{a}_+^i(k) + \hat{a}_+^i(-k)) \sin(k(x-ct))$$
$$+ (\hat{a}_-^r(k) + \hat{a}_-^r(-k)) \cos(k(x+ct)) - (\hat{a}_-^i(k) + \hat{a}_-^i(-k)) \sin(k(x+ct)) \Big) dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \left(\hat{\alpha}_+(k) \cos(k(x-ct) + \delta_+) + \hat{\alpha}_-(k) \cos(k(x+ct) + \delta_-) \right) dk$$

Thus, if one is interested only in real solutions of the wave equation, an integral over $k>0$ suffices!

One often takes $\begin{cases} a_+(k) = 0 & \text{for } k<0 \\ a_-(k) = 0 & \text{for } k>0 \end{cases}$

So the solution (real) is

$$\mathrm{Re}\, u(x,t) = \mathrm{Re}\int_0^\infty \left(\hat{a}_+(k)e^{ik(x-ct)} + \hat{a}_-(k)e^{ik(x+ct)}\right)dk$$

$$= \mathrm{Re}\int_0^\infty \left(\check{a}(k)e^{-ikx} + \hat{b}(k)e^{ikx}\right)e^{-i\omega t}\,dk$$

$$(\omega = ck).$$