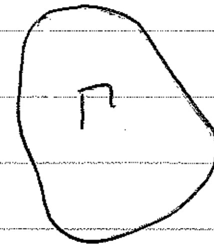
Let $\Gamma \subset \mathbb{R}^n$ be a bounded domain, and let



$$\{\lambda_k\}_{k=1}^{\infty},$$

$\lambda_k > 0$ be the Dirichlet eigenvalues for Γ, that is, the eigenvalues of $-\nabla\cdot\nabla$ with zero boundary condition on $\partial\Gamma$, as discussed above.

Let
$$\{\varphi_k\}_{k=1}^{\infty} \quad \int_{\Gamma} \varphi_k \cdot \overline{\varphi_j} = \delta_{ij}$$
be the associated eigenfunctions.

For a function $v \in L^2(\Gamma)$, let $\{\hat{v}_k\}$ be the Fourier transform with respect to $-\nabla\cdot\nabla_D =: \Delta_D$ (Dirichlet):

$$v(x) = \sum_{k=1}^{\infty} \hat{v}_k \varphi_k(x), \quad \|v\|_{L^2} = \sum_{k=1}^{\infty} |\hat{v}_k|^2$$

$$(-\Delta_D v)^{\wedge}_k = \lambda_k \hat{v}_k \; ; \; (-\Delta v)(x) = \sum_{k=1}^{\infty} \lambda_k \hat{v}_k \varphi_k(x)$$

The Laplacian $-\Delta_D$ has a positive square root:

$$(\sqrt{-\Delta}\, v)^{\wedge}_k = \sqrt{\lambda_k}\, \hat{v}_k \; ;$$

$$(\sqrt{-\Delta}\, v)(x) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \hat{v}_k \varphi_k(x)$$

In the usual Fourier transform in $\mathbb{R}^n$,

$$(-\Delta v)^\wedge(\xi) = (-\nabla\cdot(\nabla v))^\wedge(\xi) = -i\xi\cdot i\xi\,\hat{v}(\xi)$$
$$= |\xi|^2\hat{v}(\xi)$$

$$\Rightarrow (\sqrt{-\Delta}\,v)^\wedge(\xi) = |\xi|\,\hat{v}(\xi)$$

Thus $\|\nabla v\|_{L^2}^2 = \|\xi\hat{v}(\xi)\|_{L^2}^2 = \|\,|\xi|\,\hat{v}(\xi)\|_{L^2}^2$
$$= \|(\sqrt{-\Delta}\,\hat{v})(\xi)\|_{L^2}^2 = \|\sqrt{-\Delta}\,v\|_{L^2}^2 .$$

* So $\|\nabla v\|_{L^2}^2 = \|\sqrt{-\Delta}\,v\|_{L^2}^2 = \sum_{k=1}^{\infty}|\hat{v}_k|^2\lambda_k$

$$\|v\|_{H_0^1(\Gamma)}^2 = \sum_{k=1}^{\infty}(\lambda_k+1)|\hat{v}_k|^2$$

We have $v\in H_0^1(\Gamma)$ iff $\{\sqrt{\lambda_k+1}\,\hat{v}_k\}_{k=1}^{\infty}\in l_2$.

Spectral resolution:

If P_k is the projection onto the span of φ_k, we have
$$I = \sum_{k=1}^{\infty}P_k$$

For $v\in\mathcal{D}(-\Delta_D)$, $-\Delta_D v = \sum_{k=1}^{\infty}\lambda_k P_k v$

For $u\in H_1^0$, $(\nabla u,\nabla u) = \sum_{k=1}^{\infty}\lambda_k|\hat{u}_k|^2$

The max-min principle. Using the spectral representation of $(\nabla u, \nabla v) = \sum \lambda_k \hat{u}_k \bar{\hat{v}}_k$, one can deduce that

Exercise
$$\lambda_k = \inf_{\substack{V < H_0^1 \\ \dim V = k}} \sup_{\substack{u \in V \\ u \neq 0}} \frac{\int_\Gamma |\nabla u|^2}{\int_\Gamma |u|^2}$$

This principle allows one to obtain bounds on the eigenvalues λ_k. Arrange the double sequence $\left\{ \sum_{i=1}^n m_i^2 : m_i \in \mathbb{Z}, \ i=1,\dots,n \right\}$ in an increasing way: $\{\mu_k\}_{k=1}^\infty$.
Then one shows that $\exists$ $c_1, c_2 > 0$ s.th.

Exercise
$$c_1 \mu_k \leq \lambda_k \leq c_2 \mu_k .$$

To do this, one considers a box $[a_1, b_1] \times \dots \times [a_n, b_n] = B_1$ contained in Γ and a box $[c_1, d_1] \times \dots \times [c_n, d_n] = B_2$ that contains Γ. The Dirichlet eigenvalues of B_1 and B_2 can be found explicitly; they are
$$\left\{ \sum_{i=1}^n \left(\frac{m_i}{\pi (b_i - a_i)} \right)^2, \ m_1, \dots, m_n \in \mathbb{Z} \right\}$$

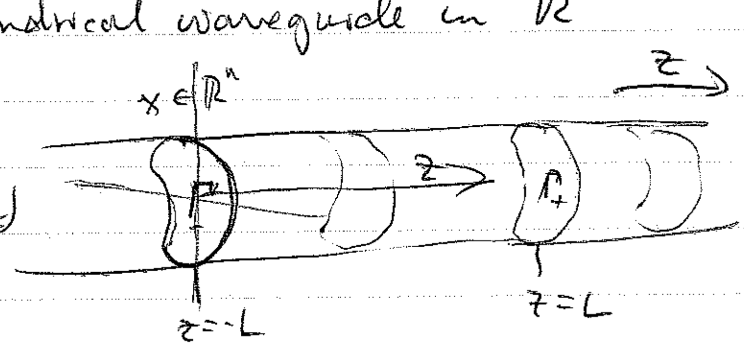
and

$$\left\{ \sum_{i=1}^n \left(\frac{m_i}{\pi (d_i - c_i)} \right)^2, \ m_1, \dots, m_n \in \mathbb{Z} \right\} .$$

Then one uses the min-max principle to obtain the appropriate comparison.

Let S denote a cylindrical waveguide in $\mathbb{R}^{n+1}$ with cross-section Γ:

$$S = \Gamma \times \mathbb{R} \ni (x,z)$$



The Fourier transform for S is the tensor product of the Fourier transforms for Γ and $\mathbb{R}$:

$$u(x,z) \longmapsto (\mathcal{F}u)_k(\xi) = \int_{\mathbb{R}}\int_{\Gamma} u(x,z)\,\overline{w_0(x,z;k,\xi)}\,dx\,dz,$$

where w_0 are the undistorted wave functions

$$w_0(x,z;k,\xi) = \varphi_k(x)\,e^{i\xi z}$$

The partial F. transform w.r.to x alone is

$$\hat{u}_k(z) = \int_{\Gamma} u(x,z)\,\overline{\varphi_k(x)}\,dx$$

The inverse F. transform gives the Fourier integral-sum

$$u(x,z) = \sum_{k=1}^{\infty}\int_{\mathbb{R}} (\mathcal{F}u)_k(\xi)\, w_0(x,z;k,\xi)\,d\xi$$

Notice that $w_0(x,z;k,\xi)$ satisfy

$$-\Delta w_0 = (\lambda_k + \xi^2)\, w_0$$

The wave functions $w_0(x,z;k,\xi)$ are the generalized, or extended eigenfunctions of $-\Delta_D$ in $\mathcal{S}$.

A major task of harmonic scattering ~~theory~~ is to construct the distorted wave functions $w(x,z;k,\xi)$ that arise when the ~~function~~ "incident field" $w_0(x,z;k,\xi)$ is scattered by a defect, or obstacle in the waveguide:
This obstacle is (for us) a local variation of τ and ρ satisfying

$$\left.\begin{aligned} 0 < \tau_- \le \tau(x,z) \le \tau_+ < \infty \\ 0 < \rho_- \le \rho(x,z) \le \rho_+ < \infty \end{aligned}\right\} \text{ for } -L < z < L$$

$$\tau(x,z) = \rho(x,z) = 1 \qquad \text{for } |z| \ge L.$$

Given a fixed frequency $\omega > 0$, there is a finite set of wave functions $w_0(x,z;k,\xi)$ that satisfy

$$-\Delta w_0 = \omega^2 w_0 \quad \text{in } \mathcal{S},$$

namely, those for which $\omega^2 = \lambda_k + \xi^2$, ie

$$\begin{cases} \lambda_k \le \omega^2 \\ \xi^2 = \omega^2 - \lambda_k \end{cases}$$

Define the "propagation exponents"

$$\xi_k(\omega) = \begin{cases} \sqrt{\omega^2 - \lambda_k} > 0 & \text{if } \lambda_k < \omega^2 \\ i\sqrt{\lambda_k - \omega^2} \in i\mathbb{R}_+ & \text{if } \lambda_k \geq \omega^2 \end{cases}$$

For a finite set of integers k, we have

* $\xi_k(\omega) > 0$; these are the <u>propagating harmonics</u>

because the associated wave functions

$$w_0 = \varphi_k(x)\, e^{\pm i \xi z}\, e^{-i\omega t}$$

are traveling waves (one to the right $(+i\xi)$ & one to the left.)

The rest of the k-values, ~~corresponding~~ with

* $-i\xi_k(\omega) > 0$ correspond to <u>evanescent harmonics</u>,

or exponential harmonics, because the ~~form~~ functions

$\varphi_k(x)\, e^{\pm i \xi z}$ are exponentially growing or decaying in z.

Now consider $w_0(x,z;k,\xi)$ to be an <u>incident field</u>, where $\xi = \xi_k(\omega)$ (and $\lambda_k < \omega^2$).
Define a scattered field $u^{sc}(x,z;k,\xi)$ such that

$$
\begin{cases}
\nabla \cdot \tau \nabla (w_0 + u^{sc}) + \rho (w_0 + u^{sc}) = 0 & \text{(weak sense)} \\
w_0 + u^{sc} = 0 \quad \text{on } \partial S & \\
\text{For } z > L, \; u^{sc}(x,z;k,\xi) = \sum_{\ell=1}^{\infty} \varphi_\ell(x)\, e^{i\xi_\ell(\omega) z}\, c_\ell^{+} & \\
\text{For } z < L, \; u^{sc}(x,z;k,\xi) = \sum_{\ell=1}^{\infty} \varphi_\ell(x)\, e^{-i\xi_\ell(\omega) z}\, c_\ell^{-} &
\end{cases}
$$

radiating, or <u>outgoing</u>

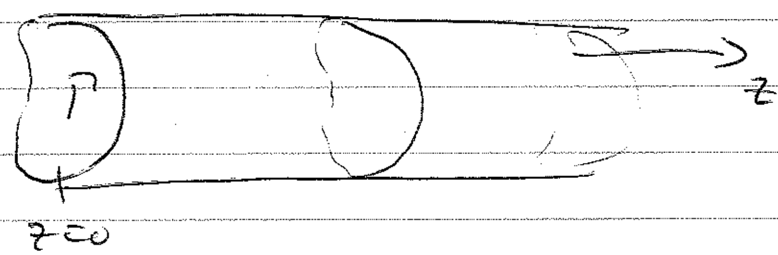
where we remember that $\omega^2 = \lambda_\ell + \xi_\ell^2(\omega)$.

The first two conditions declare that $w_0 + u^{sc}$ is the new field that satisfies the "distorted" waveguide, and that u^{sc} is the field one adds to w_0 to accommodate the distortion.

The final condition is a "radiating" or "outgoing" condition, that is physically meaningful and guarantees (as we will see) generically a <u>unique solution</u>.

<u>Boundary values</u>, or <u>traces</u>, of H_0^1 functions in a half-tube.

Let Γ be identified with $\Gamma \times \{0\} \subset S$, which is the cross-sectional part of the boundary of the half-waveguide $\Gamma \times (0,\infty) =: S_+$



Denote by $H^1_{(0)}(S_+)$ the subset of $H^1_0(S)$ of H^1 functions in ~~the~~ S_+ that ~~are~~ vanish on the boundary of the waveguide but not on the cross-section Γ:

$$H^1_{(0)}(S_+) = H^1_0(S) \cap H^1(S_+)$$

Function in $H^1_{(0)}(S_+)$ can be assigned boundary values, or traces on Γ in a natural way.

We first consider $u \in C^\infty(\bar{S}_+) \cap H^1_{(0)}(S_+)$, which has clearly ~~values~~ defined values on Γ: We have

$$(\lambda_k+1)^{1/2} |\hat{u}_k(0)|^2 = -2\,\mathrm{Re} \int_0^\infty \sqrt{\lambda_k+1}\, \overline{\hat{u}_k(z)}\, \frac{d\hat{u}_k}{dz}(z)\, dz$$

FTC & $2ab \le a^2+b^2$

$$\le \int_0^\infty (\lambda_k+1) |\hat{u}_k(z)|^2 dz + \int_0^\infty \left| \frac{d\hat{u}_k}{dz}(z) \right| dz$$

Summing over k, we obtain

$$\sum_{k=1}^{\infty}(\lambda_k+1)^{1/2}|\hat{u}_k(0)|^2 \leq \sum_{k=1}^{\infty}\int_0^{\infty}|\hat{u}_k(z)|^2\,dz + \sum_{k=1}^{\infty}\int \lambda_k|\hat{u}_k(z)|^2\,dz + \sum_{k=1}^{\infty}\int\left|\frac{d\hat{u}_k}{dz}(z)\right|^2 dz$$

$$= \|u\|^2_{L^2(S_+)} + \int_{S_+}\|\nabla_x u\|^2_{L^2(S_+)}(z)\,d\cancel{x}z + \int_{S_+}\left|\frac{\partial}{\partial z}u(x,z)\right|^2 dx\,dz$$

$$= \|u\|_{H^1(S_+)}$$

Since $C^\infty(\bar{S}_+)\cap H^1_{(0)}(S_+)$ is dense in $H^1_{(0)}(S_+)$, the map $C^\infty \cap H^1_{(0)}(S_+) \longrightarrow H^{1/2}(\Gamma) :: u \longmapsto u|_\Gamma$ can be extended to all of $H^1_{(0)}(S_+)$. Here, $H^{1/2}(\Gamma)$ is the Hilbert space of functions on Γ s.th.

$$H^{1/2}(\Gamma) = \left\{ f \in L^2(\Gamma) : \left[(\lambda_k+1)^{1/4}\hat{f}_k\right]^\vee \in L^2, \text{ or } \{(\lambda_k+1)^{1/4}\hat{f}_k\} \in l^2 \right\}$$

with norm $\|f\|^2_{H^{1/2}(\Gamma)} = \sum_{k=1}^{\infty}(\lambda_k+1)^{1/2}|\hat{f}_k|^2$.

We have just seen that the trace map

$$T : H^1_{(0)}(S_+) \longrightarrow H^{1/2}(\Gamma)$$

is bounded.

In fact, T possesses a <u>right inverse</u> $E : H^{1/2}(\Gamma) \to H^1_{(0)}(S_+)$:

$$TEv = v \quad \forall\ v \in H^{1/2}(\Gamma).$$

It is given, for $v \in C^\infty(\bar{\Gamma})$, by

$$(Ev)\hat{}_k(z) = \hat{u}_k e^{-(1+\lambda_k)^{1/2} z}$$

<u>Exercise</u> In fact, Ev minimizes the H^1 norm over all $u \in H^1_{(0)}(S_+)$ s.th. $Tu = v$. (Exercise)

<u>Boundary normal derivatives</u>

Set $\Gamma_- = \Gamma \times \{-L\}$, $\Gamma_+ = \Gamma \times \{L\}$,

$$\Omega = \Gamma \times (-L, L),$$

so that $\partial\Omega = \partial\Gamma \times \{-L, L\} \cup \Gamma_- \cup \Gamma_+$

(All parts of the boundary are oriented outward.)

Because of the outgoing condition, u^{sc} satisfies

$$z \geq L: \quad \frac{\partial}{\partial z} u^{sc} = \sum_{\ell=1}^{\infty} c_\ell^+ \, i\xi_\ell(\omega) e^{i\xi_\ell(\omega) z}$$

$$z \leq L: \quad \frac{\partial}{\partial z} u^{sc} = \sum_{\ell=1}^{\infty} c_\ell^- \, (-i\xi_\ell(\omega)) e^{-i\xi_\ell(\omega) z}$$

Since $|\xi_k(\omega)| = |\sqrt{\omega^2 - \lambda_k}| \sim \sqrt{\lambda_k}$ as $k \to \infty$,

the map

$$* \quad \begin{cases} T : H^{1/2}(\Gamma_\pm) \longrightarrow [H^{1/2}(\Gamma_\pm)]^* =: H^{-1/2}(\Gamma_\pm) \\ (\widehat{Tf_\pm})_k = -i\xi_k(\omega) \hat{f}_{\pm,k} \end{cases}$$

<u>Exercise</u> is <u>bounded</u> from $H^{1/2}$ to its dual, called $H^{-1/2}$ and

the expression $\{-i\xi_k(\omega) \hat{f}_{\pm,k}\}$ is understood as

$$(Tf_\pm)v_\pm = \sum_{k=1}^{\infty} -i\xi_k(\omega) \hat{f}_{\pm,k} \hat{v}_{\pm,k}$$

for $v_\pm \in H^{1/2}(\Gamma_\pm)$ and $f_\pm \in H^{1/2}(\Gamma_\pm)$.

The map T <u>characterizes</u> <u>the outgoing condition</u>

<u>Exercise</u> Since u outgoing $\Longleftrightarrow$ $\boxed{(Tu + \partial_n u)\big|_{\Gamma_\pm} = 0}$

for $u \in C^\infty(\bar{S}_+) \cap H^1_{(0)}(S_+)$

We now obtain the <u>weak formulation</u> of <u>the scattering problem</u>.

Formally, the field $u = w_0 + u^{sc}$ satisfies

$$\begin{cases} \nabla \cdot \tau \nabla u - \omega^2 \rho u = 0 & \text{in } \Omega \\ T u^{sc} + \partial_n u^{sc} = 0 & \text{on } \Gamma \end{cases}$$

Multiplying by $\bar{v}$ and integrating by parts gives

$$\int_\Omega \tau \nabla u \cdot \nabla \bar{v} - \omega^2 \int_\Omega \rho u \bar{v} = \int_{\Gamma_\pm} (\partial_n u) \bar{v}$$

$$= \int_{\Gamma_\pm} -(Tu)\bar{v} + \int_{\Gamma_\pm} (Tu + \partial_n u)\bar{v}$$

$$= \int_{\Gamma_\pm} -(Tu)\bar{v} + \int_{\Gamma_\pm} (T + \partial_n) w_0 \bar{v}$$

Now put

$$\hat{a}(u,v) = \int_\Omega \tau \nabla u \cdot \nabla \bar{v} + \int_\Omega \rho u \bar{v} + \int_{\Gamma_\pm} (Tu)\bar{v}$$

$$b(u,v) = \int_\Omega \rho u \bar{v}$$

Both of these are <u>bounded</u> forms in $H^1_{(\alpha)}(\Omega)$.

The scattering problem in its H^1-form is to seek $u \in H^1_{(0)}(\Omega)$ such that

$$\breve{a}(u,v) - (w^2+1)\,b(u,v) = f(v) \quad \forall\ v \in H^1_{(0)}(\Omega),$$

where $f(v) := \int_{\Gamma_\pm} (T+\partial_n)w_0\,\bar{v}$.

Exercise In fact, $f \in H^{1/2}(\Gamma_\pm)^* = H^{-1/2}(\Gamma_\pm)$.

Again, one can define an invertible operator

$$A : H^1_{(0)}(\Omega) \longrightarrow H^1_{(0)}(\Omega)$$

Exercise such that

$$\breve{a}(u,v) = (Au,v).$$

Also let $\tilde{f} \in H^{1\,*}_{(0)}$ be such that

$$(\tilde{f},v) = f(v), \quad \text{and again,} \quad (Bu,v) = b(u,v).$$

The scattering problem becomes

$$(Au,v) + (Bu,v) = (\tilde{f},v) \quad \forall\ v \in H^1_{(0)}(\Omega),$$

or $$\boxed{u + A^{-1}Bu = A^{-1}\tilde{f}}$$

Trapped modes

A *trapped mode* is a nonzero field $u \in H^1_{(0)}(\Omega)$ that satisfies

$$u + A^{-1}Bu = 0,$$

i.e., u is a field that exists in the guide in the absence of a source field f.

Define the *real part* of $\hat{a}$ by

$$\hat{a}_r(u,v) = \int_\Omega \tau \nabla u \cdot \nabla \bar{v} + \int_\Omega \rho u \bar{v} + \int_{\Gamma_\pm} (\mathrm{Re}\,T) u \bar{v},$$

where $((\mathrm{Re}\,T)\hat{f}_\pm)(g_\pm) = \sum_{-i\xi_k(\omega)\in\mathbb{R}} -i\xi_k(\omega)\hat{f}_{\pm,k}\overline{\hat{g}_{\pm,k}}$

for $f_\pm, g_\pm \in H^{-1/2}(\Gamma_\pm)$.

Then $u \neq 0$ is a trapped mode iff

Exercise

$$\boxed{\begin{aligned} &\hat{a}(u,v) - \omega^2 b(u,v) = 0 \quad \forall\, v \in H^1_{(0)} \\ &\iff \begin{cases} \hat{a}_r(u,v) - \omega^2 b(u,v) = 0 \quad \forall\, v \in H^1_{(0)} \\ (u|_{\Gamma_\pm})^{\wedge}_k = 0 \quad \forall\, k \text{ with } \xi_k(\omega) \in \mathbb{R} \end{cases} \end{aligned}}$$

- The second eqn. gives a new condition for each propagating harmonic