Math 7384 Problem Set 3

7. Define an operator A in the Hilbert space $D^1 \oplus L^2(\mathbb{R})$, where $D^1 = \{ f \in L^2(\mathbb{R}) : \omega f(\omega) \in L^2 \}$ and the inner product is $\left(\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right)_{D^1 \oplus L^2}$

$$= \int \omega^2 f_1(\omega) \overline{f_2(\omega)}\, d\omega + \int g_1(\omega) \overline{g_2(\omega)}\, d\omega$$

by

$$\begin{cases} \mathcal{D}(A) = D^2 \oplus D^1, \quad D^2 = \{ f \in L^2 : \omega^2 f(\omega) \in L^2 \} \\ A \begin{pmatrix} f \\ g \end{pmatrix}(\omega) = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{pmatrix} f(\omega) \\ g(\omega) \end{pmatrix}. \end{cases}$$

Prove that A is anti-self-adjoint.

8. Prove that a projection $P : \mathcal{H} \to \mathcal{H}$ in a Hilbert space ($P^2 = P$) is orthogonal if and only if it is self-adjoint. [P is orthogonal if $\mathrm{Null}(P) \perp \mathrm{Ran}(P)$.]

Prove that, if P is a projection, then so is $I - P$, and that P is orthogonal if and only if $I - P$ is.

9. Prove that the matrix $\begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$ is anti-self-adjoint with respect to the inner product

$$\left(\begin{bmatrix} \omega^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right)$$

and that the matrices

$$P^+ = \frac{1}{2} \begin{bmatrix} 1 & -(i\omega)^{-1} \\ -i\omega & 1 \end{bmatrix} \quad \text{and} \quad P_\omega^- = \frac{1}{2} \begin{bmatrix} 1 & (i\omega)^{-1} \\ i\omega & 1 \end{bmatrix}$$

are <u>complementary orthogonal projections</u> onto the eigenspaces of $\begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$.

10. Let u_1 and u_2 be solutions of the equation

$$-\sigma(\tau w')' - k^2 w = 0$$

on $\mathbb{R}$, where σ and τ are as in the notes, p. 48, and $k \in \mathbb{R}$. Prove that the function

$$W[u_1, u_2](x) := \tau(u_1 \bar{u}_2' - \bar{u}_2 u_1').$$

For the scattering fields $\sqrt{2\pi}\, w(x;k)$ defined in the notes, prove that, in the representation on p. 49,

$$|r|^2 + |t|^2 = 1 \qquad \text{(conservation of energy flux)}.$$