Math 7384 Problem Set 4

11. We have seen that the operator $A = \begin{bmatrix} 0 & 1 \\ c^2\partial^2 & 0 \end{bmatrix}$ is anti-self-adjoint in $H^1 \oplus L^2$ w/ $\mathcal{D}(A) = H^2 + H^1$, and thus $\frac{i}{c}A$ is self-adjoint.

a. If $\left(\frac{i}{c}A - z\right)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \in H^1 \oplus L^2$, prove that u satisfies the Helmholtz equation

$$u'' + z^2 u = -\left(zf + \frac{i}{c}g\right),$$

and find a similar equation for v.

b. Find an <u>explicit</u> expression for the standard resolution of the identity in $H^1 \oplus L^2$ associated with $\frac{i}{c}A$, that is, $\{E_t\}_{t\in\mathbb{R}}$ s.th.

$$\frac{i}{c}A\begin{pmatrix} u \\ v \end{pmatrix} = \int_{-\infty}^{\infty} k \, dE_k \begin{pmatrix} u \\ v \end{pmatrix}$$

I want a formula that gives $E_k\begin{pmatrix} u \\ v \end{pmatrix}(x) = \ldots$.

c. If $\begin{pmatrix} f \\ g \end{pmatrix} \in H^1 \oplus L^2$ and $z \in \mathbb{C}\setminus\mathbb{R}$ are given, find $u \in H^2$ such that, for some $v \in H^1$,

$$\left(\frac{i}{c}A - z\right)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

Do this as follows: (1) Find a spectral representation of $\left(\frac{i}{c}A - z\right)^{-1}$;
(2) from this, obtain the Fourier transform of u;
(3) obtain $u(x)$ in the form

$$u_z(x) = \int_{-\infty}^{\infty} G_z(x,y)\, m(y)\, dy .$$

Find explicity the integral kernel $G_z(x,y)$ as well as how the function m depends on f and g.

d. If $z = \lambda + ih$ $(\lambda \in \mathbb{R}, h \in \mathbb{R})$, find both

$$G_\lambda^{(1)}(x,y) := \lim_{h\to 0^+} G_z(x,y) ,$$

$$G_\lambda^{(2)}(x,y) := \lim_{h\to 0^-} G_z(x,y) .$$

If $u_\lambda^{(1)}(x) = \lim_{h\to 0^+} u_z(x)$ and $u_\lambda^{(2)}(x) = \lim_{h\to 0^-} u_z(x)$,

prove that

$$u_\lambda^{(1)}(x) = \int_{-\infty}^{\infty} G_\lambda^{(1)}(x,y)\, m(y)\, dy$$

$$u_\lambda^{(2)}(x) = \int_{-\infty}^{\infty} G_\lambda^{(2)}(x,y)\, m(y)\, dy$$

and that $\exists$ constants $C_\pm$, $B_\pm$ such that

$$u_\lambda^{(1)}(x) = C_+ e^{i\lambda x} \quad \text{for } x > L$$
$$u_\lambda^{(1)}(x) = C_- e^{-i\lambda x} \quad \text{for } x < -L$$

$$u_\lambda^{(2)}(x) = B_+ e^{-i\lambda x} \quad \text{for } x > L$$
$$u_\lambda^{(2)}(x) = B_- e^{i\lambda x} \quad \text{for } x < -L$$