Math 7384 Problem Set 5

12. Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(A) \subset \mathcal{H}$. For all $u \in \mathcal{D}(A) \setminus \{0\}$, define the Rayleigh quotient

$$\mathcal{R}(u) = \frac{(Au, u)}{(u, u)}.$$

Suppose that, for some real number λ_*, the part of the spectrum of A below λ_*, that is, $\sigma(A) \cap (-\infty, \lambda_*)$ consists of a finite number of eigenvalues, not necessarily distinct,

$$\lambda_1 \le \lambda_2 \le \dots \le \lambda_n,$$

with associated orthonormal eigenfunctions $\{\varphi_k\}_{k=1}^{n}$.

Let $V_k = \operatorname{span}\{\varphi_i\}_{i=1}^{k} \quad \forall\ k = 1, \dots, n$.

Prove that, $\forall\ k = 1, \dots, n$,

$$\lambda_k = \max_{\substack{u \in V_k \\ u \neq 0}} \mathcal{R}(u) = \min_{\substack{u \perp V_{k-1} \\ u \in \mathcal{D}(A) \setminus \{0\}}} \mathcal{R}(u) = \min_{\substack{V \subset \mathcal{D}(A) \\ \dim V = k}} \max_{\substack{u \in V \\ u \neq 0}} \mathcal{R}(u).$$

13. Let the sequence $\{\mu_k\}_{k=1}^{\infty}$ be an increasing listing of the numbers in the set $\left\{\sum_{i=1}^{n} m_i^2 \,:\, m_i \in \mathbb{Z} \ \forall\, i = 1,\dots,n\right\}$.

Let Ω be a bounded open set in $\mathbb{R}^n$. Prove that there exist positive numbers c_1 and c_2 such that the Dirichlet eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ of Ω satisfy

$$c_1 \mu_k \leq \lambda_k \leq c_2 \mu_k .$$

14. Prove that the map

$$(Tf_\pm)^\wedge_k = -i\,\xi_k(\omega)\,\hat{f}_{\pm,k}\,,$$

with $\xi_k(\omega)$ defined as in the class notes, defines a bounded linear operator $T : H^{1/2}(\Gamma_\pm) \to H^{-1/2}(\Gamma_\pm) = [H^{1/2}(\Gamma_\pm)]^*$ through the action

$$(Tf_\pm)v_\pm = \sum_{k=1}^{\infty} -i\,\xi_k(\omega)\,\hat{f}_{\pm,k}\,\hat{v}_{\pm,k} \qquad (v_\pm \in H^{1/2}(\Gamma_\pm))$$

15. Prove the statement on p. 81: For $u \in H^1_{(0)}(\Omega)$, $u \neq 0$,

$$\hat{a}(u,v) - \omega^2 b(u,v) = 0 \quad \forall\, v \in H^1_{(0)}(\Omega)$$

$$\Longleftrightarrow \begin{cases} \hat{a}_r(u,v) - \omega^2 b(u,v) = 0 \quad \forall\, v \in H^1_{(0)}(\Omega), \\ (u|_{\Gamma_\pm})^\wedge_k = 0 \quad \forall\, k \text{ s.th. } \xi_k(\omega) \in \mathbb{R}. \end{cases}$$