Math 7384 Problem Set 6

16. Consider a nonlinear spring-mass system attached to a line in which motion is governed by the nonlinear Schrödinger equation:

$$\begin{cases} iu_t + u_{xx} + \lambda |u|^2 u = 0 & \text{for } x \in \mathbb{R}\setminus\{0\} \\ i\dot{y} - \omega_0 y + \tau\left[u_x(0+,\cdot) - u_x(0-,\cdot)\right] + \mu |y|^2 y = 0 \end{cases}$$

$\dot{y} = \frac{dy}{dt}$

with $u(x,t)$ continuous and $y(t) = u(0,t)$, and $u_x(0\pm,t) = \lim_{x\to 0^\pm} u_x(x,t)$.

Find "trapped modes" of this system, that is, solutions that decay exponentially as $|x| \to \infty$. This can be done by shifting a solitary wave of the NLS equation to the right and to the left and "gluing" them together at $x=0$ to obtain solutions of the form

$$u(x,t) = Ae^{i\omega t}\cdot \begin{cases} \operatorname{sech}(\beta x - \eta), & x \le 0 \\ \operatorname{sech}(\beta x + \eta), & x \ge 0 \end{cases}.$$

Find the relations between ω, A, β, and η.

17. Consider an infinite chain of beads experiencing nearest-neighbor interactions:

$$m\ddot{y}_n = k(y_{n+1} - 2y_n + y_{n-1}), \quad n \in \mathbb{Z},$$

where $\ddot{y}_n = \frac{d^2y}{dt^2}(t)$ and m and k are positive numbers.

(a) Find all solutions of the form $y_n(t) = Ae^{i(kn-\omega t)}$ (A constant). Give the frequency band $I \subset \mathbb{R}$ for which such states exist and determine the "dispersion relation" between k and ω.

(b) By changing the mass of the 0^{th} bead to $M \neq m$,

$$\begin{cases} m\ddot{y}_n = k(y_{n+1} - 2y_n + y_{n-1}), & n \neq 0 \\ M\ddot{y}_0 = k(y_1 - 2y_0 + y_{-1}), \end{cases}$$

construct a trapped mode of the form

$$y_n(t) = u_n e^{-i\omega t}, \quad u_n \to 0 \text{ as } |n| \to \infty,$$

and show that ω is not in the propagation band I.

(c) For all $\omega \in I$, find scattering states of the form

$$y_n(t) = \begin{cases} (Je^{ikn} + Re^{-ikn})e^{-i\omega t}, & n \leq 0 \\ Te^{ikn}e^{-i\omega t}, & n \geq 0. \end{cases}$$