Math 7384 Solutions to problems

1. Since σ is increasing on the closed interval $[0, 2\pi]$, it is of bounded variation. (In fact $V_0^{2\pi}(\sigma) = \sigma(2\pi) - \sigma(0)$.)

[OR, put $z = 0$ in $f(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\sigma(t) + i\beta$

to obtain $f(0) = \int_0^{2\pi} d\sigma(t) + i\beta$. Thus

$\int_0^{2\pi} d\sigma(t) = \operatorname{Re} f(0)$, which proves that $V_0^{2\pi}(\sigma) = \operatorname{Re} f(0) < \infty$.]

In the expression $f(z) = \int_{-\infty}^{\infty} \frac{1+tz}{t-z}\, d\sigma(t) + \mu z + \alpha$,

with σ increasing, put $z = i$:

$$f(i) = \alpha + i\mu + \int_{-\infty}^{\infty} \frac{1+it}{t-i}\, d\sigma(t) = \alpha + i\mu + i\int_{-\infty}^{\infty} d\sigma(t).$$

Thus $\int_{\infty}^{-\infty} d\sigma(t) = \operatorname{Im} f(i) - \mu < \infty$,

which shows that σ is of bounded variation.

2.

(a) Set $\sigma(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}$, $(\alpha = 0, \mu = 0)$

Then $\displaystyle\int_{-\infty}^{\infty} \frac{1+tz}{t-z}\, d\sigma(t) = \left.\frac{1+tz}{t-z}\right|_{t=0} = -\frac{1}{z}$ ✓.

(b) Set $\sigma(t) = \begin{cases} 0, & t \le -1 \\ 1, & -1 < t \le 1 \\ 2, & 1 < t \end{cases}$ $(\alpha = 0, \mu = 0)$

Then $\displaystyle\int_{-\infty}^{\infty} \frac{1+tz}{t-z}\, d\sigma(t) = \left.\frac{1+tz}{t-z}\right|_{t=-1} + \left.\frac{1+tz}{t-z}\right|_{t=1}$

$$= \frac{1-z}{-1-z} + \frac{1+z}{1-z} = \frac{4z}{1-z^2}.$$

(c) $f(z) = i$; $g(\zeta) := -if\left(i\frac{1+\zeta}{1-\zeta}\right) = 1$

By the representation theorem for analytic fcns in a nbhd of $\bar{D}$, we have

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+\zeta}{e^{it}-\zeta}\, \mathrm{Re}(g(\zeta))\, dt = \int_0^{2\pi} \frac{e^{it}+\zeta}{e^{it}-\zeta}\, d\rho(t),$$

where $\rho(t) = \frac{t}{2\pi}$ on $[0, 2\pi]$. With

$\longrightarrow$ $\sigma(t) = \rho(2\operatorname{arccot}(-t)) = \frac{1}{\pi}\operatorname{arccot}(-t)$, we obtain

$$i = \int_{-\infty}^{\infty} \frac{1+tz}{t-z}\, d\sigma(t).$$

3. Let $f(z)$ be analytic in $\mathbb{C}\setminus\overline{D} = \{z \in \mathbb{C} : |z| > 1\}$ with $\operatorname{Re} f(z) \geq 0 \ \forall\ z \in \mathbb{C}\setminus\overline{D}$. Define $\forall\ \zeta \in D$,

$g(\zeta) = f(\frac{1}{\zeta})$, which is analytic in $D\setminus\{0\}$ and has nonnegative real part. g has a Laurent expansion about $\zeta = 0$:

$$g(\zeta) = \sum_{n=N}^{\infty} a_n \zeta^n$$

If $N = -\infty$, then g has an essential singularity at 0 and thus takes on all values in $\mathbb{C}\setminus\{z_0\}$ for some $z_0 \in \mathbb{C}$ in each neighborhood of 0, contradicting $\operatorname{Re} g \geq 0$. Thus $N \neq -\infty$, so g has a pole of order $N < \infty$ (assume $a_N \neq 0$)

$$g(\zeta) = \frac{1}{\zeta^N} h(\zeta), \text{ where } h \text{ is analytic at } 0$$

with $h(0) \neq 0$. Since h is continuous at 0 and $h(0) \neq 0$, $\operatorname{Re}\left(\frac{1}{\zeta^N} h(\zeta)\right)$ takes on both positive and negative values in each punctured neighborhood of 0, contradicting $\operatorname{Re} g \geq 0$. Thus $N = 0$ and g has a removable singularity at 0.

Theorem 1 now applies;

could include more details.

$$(+) \quad g(\zeta) = i\beta + \int_0^{2\pi} \frac{e^{it}+\zeta}{e^{it}-\zeta}\, d\sigma(t) \quad \text{for some}$$

increasing function σ on $[0, 2\pi]$.

$$\text{So} \quad f(z) = g\left(\tfrac{1}{z}\right) = i\beta + \int_0^{2\pi} \frac{e^{it}+z^{-1}}{e^{it}-z^{-1}}\, d\sigma(t)$$

$$= i\beta + \int_0^{2\pi} \frac{z+e^{-it}}{z-e^{-it}}\, d\sigma(t),$$

which is the representation we seek.

Now, assuming this representation of some function f defined in $\mathbb{C}\setminus\bar{D}$, the transformation $z \mapsto \frac{1}{\zeta}$ converts it into the form $(+)$ for $g(\zeta) = f\left(\frac{1}{\zeta}\right)$, which is, by Theorem, analytic with $\operatorname{Re} g(\zeta) \geq 0$. Thus $f(z) = g\left(\frac{1}{z}\right)$ is analytic with $\operatorname{Re} f(z) \geq 0$.