Math 7384 Solutions to problems

4. Because of the relation $E_s E_t = E_{\min\{s,t\}}$, we have,

for $s \le t$, $(E_t - E_s)^2 = E_t^2 - E_t E_s - E_s E_t + E_s^2 = E_t - E_s$.

Thus $\forall\ u \in \mathcal{H}$ and $s \le t$,

$$\|(E_t - E_s)u\|^2 = ((E_t - E_s)u, (E_t - E_s)u) = ((E_t - E_s)^2 u, u)$$

$$= ((E_t - E_s)u, u) = \sigma(t;u) - \sigma(s;u).$$

Since $\sigma(\cdot\,;u)$ is left-continuous, we have

$\lim\limits_{s \to t^-} \|(E_t - E_s)u\| = 0$, and therefore $E_s u \to E_t u$

as $s \to t^-$. Since this holds for all $u \in \mathcal{H}$,

E_s converges strongly to E_t as $s \to t^-$.

Now consider the canonical resolution of the identity in $L^2(\mathbb{R})$:

$E_t f = \chi_{(-\infty,t]} f$. Notice that $\forall\ s < t \in \mathbb{R}$, $\chi_{(s,t)} \in L^2(\mathbb{R})$

is nonzero in L^2 and that $(E_t - E_s)\chi_{(s,t)} = \chi_{(s,t)}$.

Thus $\|E_t - E_s\| \ge 1$ [in fact $\|E_t - E_s\| = 1$]. This means

that E_s does not tend to E_t in the operator norm

as $s \to t^-$.

5. $\forall\, u, v \in \mathcal{H}\ \forall\, z \in \mathbb{C}\ \forall\, t \in \mathbb{R}$,

$$(E_t R_z u, v) = (R_z u, E_t v) = \int \frac{1}{s-z}\, d_s(E_s f, E_t g)$$

$$= \int \frac{1}{s-z}\, d_s(E_t E_s u, v) = \int \frac{1}{s-z}\, d_s(E_s E_t u, v)$$

$$= (R_z E_t u, v). \quad \text{Thus } E_t R_z = R_z E_t.$$

6. We have derived the representation

$$((-i\partial - z)^{-1} u, v) = \int \frac{1}{k-z} \hat{f}(k) \hat{g}(k)\, dk \quad \forall\, u, v \in L^2(\mathbb{R}).$$

Thus $\forall\, u \in L^2(\mathbb{R})$, $(-i\partial - z)^{-1} u = \left(\frac{1}{k-z} \hat{f}(k)\right)^{\vee} = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{k-z}\right)^{\vee} * f$.

If $\operatorname{Im}(z) > 0$, then $|e^{-i(k-z)x}| = e^{-(\operatorname{Im} z)x}$ tends exponentially to zero as $x \to \infty$, and thus

$$\mathcal{F}\left(\sqrt{2\pi}\, i e^{izx} \chi_0^\infty(x)\right) = \int_0^\infty i e^{izx} e^{-ikx}\, dx = i \int_0^\infty e^{-i(k-z)x}\, dx = \frac{1}{k-z}.$$

If $\operatorname{Im} z < 0$, $e^{-i(k-z)x} \to 0$ exponentially as $x \to -\infty$, and thus $\mathcal{F}\left(-\sqrt{2\pi}\, i\, e^{izx} \chi_{-\infty}^0\right) = \int_{-\infty}^0 -i e^{-i(k-z)x}\, dx = \frac{1}{k-z}$.

$$\text{Thus } \left(\frac{1}{k-z}\right)^{\vee} = \begin{cases} i\sqrt{2\pi}\, e^{izx} \chi_0^\infty(x) & \text{if } \operatorname{Im} z > 0, \\ -i\sqrt{2\pi}\, e^{izx} \chi_{-\infty}^0(x) & \text{if } \operatorname{Im} z < 0, \end{cases}$$

and we obtain an expression for $(-i\partial - z)^{-1}$ as an explicit convolution:

$$(-i\partial - z)u = f$$

$$\Rightarrow \left((-i\partial - z)^{-1} f\right)(x) = u(x) = \frac{1}{\sqrt{2\pi}}\left(\left(\frac{1}{k-z}\right)^{\vee} * f\right)(x)$$

$$= \begin{cases} i\displaystyle\int_{-\infty}^{x} f(y)\, e^{iz(x-y)}\, dy & \text{if } \operatorname{Im} z > 0, \\ i\displaystyle\int_{x}^{\infty} f(y)\, e^{iz(x-y)}\, dy & \text{if } \operatorname{Im} z < 0. \end{cases}$$

Note: One can obtain $\mathcal{F}(\chi_0^{\infty} e^{izx}) = \frac{1}{\sqrt{2\pi}} \frac{1}{i(k-z)}$ $(\operatorname{Re} z > 0)$ also as follows:

$$\frac{1}{k-(\alpha+i\varepsilon)} = \frac{k-\alpha}{(k-\alpha)^2+\varepsilon^2} + i\frac{\varepsilon}{(k-\alpha)^2+\varepsilon^2} \xrightarrow{\varepsilon\to 0} (\text{P.V.})\frac{1}{k-\alpha} + i\pi\delta \quad \left(\text{distributional sense}\right)$$

$$\mathcal{F}(\operatorname{sgn} x) = \sqrt{\frac{2}{\pi}}\frac{1}{ik} \quad \text{and} \quad \mathcal{F}(1) = \sqrt{2\pi}\,\delta$$

$$\Rightarrow \mathcal{F}(\chi_0^{\infty}) = \mathcal{F}\left(\tfrac{1}{2}(\operatorname{sgn} x + 1)\right) = \frac{1}{i}\frac{1}{\sqrt{2\pi}}\left(\frac{1}{k} + i\pi\delta\right) = \frac{1}{i\sqrt{2\pi}}\lim_{\varepsilon\to 0}\frac{1}{k+i\varepsilon}$$

$$\Rightarrow \mathcal{F}(\chi_0^{\infty} e^{izx}) = \frac{1}{i\sqrt{2\pi}}\frac{1}{k+iz} = \frac{1}{\sqrt{2\pi}}\frac{1}{i(k-z)}.$$