Math 7384 Solutions to Problems

7. Let $\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}$ be in the domain of A^*. This means that the map $\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \longmapsto \left(A\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right)$ is bounded, and thus the Riesz theorem (b/c of the density of $D^2 \oplus D^1$ in $D^1 \oplus L^2$) provides a ~~func~~ pair $\begin{pmatrix} f_* \\ g_* \end{pmatrix}$ such that $\left(A\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right) = \left(\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_* \\ g_* \end{pmatrix} \right)$.

This equality, written out in integral form is

$$\int \omega^2 g_1(\omega)\overline{f_2(\omega)}\,d\omega - \int \omega^2 f_1(\omega)\overline{g_2(\omega)}\,d\omega$$

$$= \int \omega^2 f_1(\omega)\overline{f_*(\omega)} + \int g_1(\omega)\overline{g_*(\omega)}\,d\omega \quad \forall \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \in \mathcal{D}(A).$$

By taking $f_1 = 0$ ~~and~~ and letting g_1 range over all functions in C_c^∞ (for example), we obtain ~~$g_*(\omega) = (\omega^2 f_2(\omega))$~~ ~~so that $f_2 g_* \in D^2$. By taking $g_1 = 0$, we obtain $f_*(\omega) = -g_2(\omega) \in D^1$. Thus $\begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \in$~~ $\omega^2 f_2(\omega) = g_*(\omega) \in L^2$, and $-g_2(\omega) = f_*(\omega) \in D^1$. Thus, $\begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \in D^2 \oplus D^1 = \mathcal{D}(A)$, ~~so~~ so ~~that~~ that $\mathcal{D}(A^*) = \mathcal{D}(A)$. Moreover,

$$A^*\begin{pmatrix} f_2 \\ g_2 \end{pmatrix} = \begin{pmatrix} f_* \\ g_* \end{pmatrix} = \begin{pmatrix} -g_2 \\ \omega^2 f_2 \end{pmatrix} = -A\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}.$$

This means that $A^* = -A$.

8. Let the projection P be self-adjoint, and let $v \in \text{Null}\, P$ and $w \in \mathcal{H}$ be given. Then $(v, Pw) = (Pv, w) = (0, w) = 0$. Thus $v \perp Pw$ and thus $\text{Null}\, P \perp \text{Ran}\, P$.

Now suppose that $\text{Null}\, P \perp \text{Ran}\, P$, and let $v, w \in \mathcal{H}$ be given. Since $P(I-P)w = (P-P^2)w = 0$, $(I-P)w \in \text{Null}\, P$.

We have $0 = ((I-P)w, Pv) = (w, Pv) - (Pw, Pv)$.

Similarly, $0 = (Pw, (I-P)v) = (Pw, v) - (Pw, Pv)$

Thus, $(w, Pv) = (Pw, v)$.

Second part — easy enough.

9. Easy enough.

10. 'Tis simple to show that $\frac{d}{dx} W[u_1, \bar{u}_2] = (\tau \bar{u}_2')' u_1 - (\tau u_1')' \bar{u}_2$, and this same expression ~~obtained by~~ set equal to zero is obtained by multiplying $-\sigma(\tau u_1')' - k^2 u_1 = 0$ by $\bar{u}_2$ and subtracting from the same equation with $\bar{u}_2$ and u_1 switched. Thus W is constant.

Applying this result to a scattering field u with $u(x) = e^{ikx} + re^{-ikx}$ $(x < -L)$ and $u(x) = te^{ikx}$ $(x > L)$ gives $W[u, \bar{u}] = u\bar{u}' - \bar{u}u' = 2[-k + k|r|^2]$ $(x < -L)$ and $W[u, \bar{u}] = -2k|t|^2$ $(x > L)$. ~~Thus~~ Since W is constant, $W(-L) = W(L)$ and so $|r|^2 + |t|^2 = 1$.