CHAPTER 1

COMPLEX NUMBERS

In this chapter, we survey the algebraic and geometric structure of the complex number system. We assume various corresponding properties of real numbers to be known.

1. SUMS AND PRODUCTS

Complex numbers can be defined as ordered pairs (x, y) of real numbers that are to be interpreted as points in the *complex plane*, with rectangular coordinates x and y, just as real numbers x are thought of as points on the real line. When real numbers x are displayed as points $(x, 0)$ on the *real axis*, it is clear that the set of complex numbers includes the real numbers as a subset. Complex numbers of the form $(0, y)$ correspond to points on the y axis and are called *pure imaginary numbers*. The y axis is, then, referred to as the *imaginary axis*.

It is customary to denote a complex number (x, y) by z, so that

(1) $$z = (x, y).$$

The real numbers x and y are, moreover, known as the *real and imaginary parts* of z, respectively; and we write

(2) $$\operatorname{Re} z = x, \qquad \operatorname{Im} z = y.$$

Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal whenever they have the same real parts and the same imaginary parts. Thus $z_1 = z_2$ if and only if z_1 and z_2 correspond to the same point in the complex, or z, plane.

The *sum* $z_1 + z_2$ and the *product* z_1z_2 of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are defined as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \tag{3}$$

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, y_1x_2 + x_1y_2). \tag{4}$$

Note that the operations defined by equations (3) and (4) become the usual operations of addition and multiplication when restricted to the real numbers:

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0),$$
$$(x_1, 0)(x_2, 0) = (x_1x_2, 0).$$

The complex number system is, therefore, a natural extension of the real number system.

Any complex number $z = (x, y)$ can be written $z = (x, 0) + (0, y)$, and it is easy to see that $(0, 1)(y, 0) = (0, y)$. Hence

$$z = (x, 0) + (0, 1)(y, 0);$$

and, if we think of a real number as either x or $(x, 0)$ and *let i denote the pure imaginary number* $(0, 1)$, it is clear that*

$$z = x + iy. \tag{5}$$

Also, with the convention $z^2 = zz$, $z^3 = zz^2$, etc., we find that

$$i^2 = (0, 1)(0, 1) = (-1, 0),$$

or

$$i^2 = -1. \tag{6}$$

In view of expression (5), definitions (3) and (4) become

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2), \tag{7}$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2). \tag{8}$$

Observe that the right-hand sides of these equations can be obtained by formally manipulating the terms on the left as if they involved only real numbers and by replacing i^2 by -1 when it occurs.

2. ALGEBRAIC PROPERTIES

Various properties of addition and multiplication of complex numbers are the same as for real numbers. We list here the more basic of these algebraic properties and verify some of them. Most of the others are verified in the exercises.

*In electrical engineering, the letter j is used instead of i.

The commutative laws

$$z_1 + z_2 = z_2 + z_1, \qquad z_1z_2 = z_2z_1 \tag{1}$$

and the associative laws

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad (z_1z_2)z_3 = z_1(z_2z_3) \tag{2}$$

follow easily from the definitions in Sec. 1 of addition and multiplication of complex numbers and the fact that real numbers obey these laws. For example, if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = z_2 + z_1.$$

Verification of the rest of the above laws, as well as the distributive law

$$z(z_1 + z_2) = zz_1 + zz_2, \tag{3}$$

is similar.

According to the commutative law for multiplication, $iy = yi$. Hence one can write $z = x + yi$ instead of $z = x + iy$. Also, because of the associative laws, a sum $z_1 + z_2 + z_3$ or a product $z_1z_2z_3$ is well defined without parentheses, as is the case with real numbers.

The additive identity $0 = (0, 0)$ and the multiplicative identity $1 = (1, 0)$ for real numbers carry over to the entire complex number system. That is,

$$z + 0 = z \quad \text{and} \quad z \cdot 1 = z \tag{4}$$

for every complex number z. Furthermore, 0 and 1 are the only complex numbers with such properties (see Exercise 11).

There is associated with each complex number $z = (x, y)$ an additive inverse

$$-z = (-x, -y), \tag{5}$$

satisfying the equation $z + (-z) = 0$. Moreover, there is only one additive inverse for any given z, since the equation $(x, y) + (u, v) = (0, 0)$ implies that $u = -x$ and $v = -y$. Expression (5) can also be written $-z = -x - iy$ without ambiguity since (Exercise 10) $-(iy) = (-i)y = i(-y)$. Additive inverses are used to define subtraction:

$$z_1 - z_2 = z_1 + (-z_2). \tag{6}$$

So if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then

$$z_1 - z_2 = (x_1 - x_2, y_1 - y_2) = (x_1 - x_2) + i(y_1 - y_2). \tag{7}$$

For any *nonzero* complex number $z = (x, y)$, there is a number z^{-1} such that $zz^{-1} = 1$. This multiplicative inverse is less obvious than the additive one. To find it, we seek real numbers u and v, expressed in terms of x and y, such that

$$(x, y)(u, v) = (1, 0).$$

According to equation (4), Sec. 1, which defines the product of two complex numbers, u and v must satisfy the pair

$$xu - yv = 1, \qquad yu + xv = 0$$

of linear simultaneous equations; and simple computation yields the unique solution

$$u = \frac{x}{x^2+y^2}, \qquad v = \frac{-y}{x^2+y^2}.$$

So *the* multiplicative inverse of $z = (x, y)$ is

$$z^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) \qquad (z \neq 0). \tag{8}$$

The inverse z^{-1} is not defined when $z = 0$. In fact, $z = 0$ means that $x^2 + y^2 = 0$; and this is not permitted in expression (8).

The existence of multiplicative inverses enables us to show that *if a product z_1z_2 is zero, then so is at least one of the factors z_1 and z_2.* For suppose that $z_1z_2 = 0$ and $z_1 \neq 0$. The inverse z_1^{-1} exists; and, according to the definition of multiplication, any complex number times zero is zero. Hence

$$z_2 = 1 \cdot z_2 = (z_1^{-1}z_1)z_2 = z_1^{-1}(z_1z_2) = z_1^{-1} \cdot 0 = 0. \tag{9}$$

That is, if $z_1z_2 = 0$, either $z_1 = 0$ or $z_2 = 0$; or possibly both z_1 and z_2 equal zero. Another way to state this result is that *if two complex numbers z_1 and z_2 are nonzero, then so is their product z_1z_2.*

Division by a nonzero complex number is defined as follows:

$$\frac{z_1}{z_2} = z_1z_2^{-1} \qquad (z_2 \neq 0). \tag{10}$$

If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, equations (10) and (8) tell us that

$$\frac{z_1}{z_2} = \left(\frac{x_1x_2 + y_1y_2}{x_2^2+y_2^2}, \frac{y_1x_2 - x_1y_2}{x_2^2+y_2^2}\right) = \left(\frac{x_1x_2 + y_1y_2}{x_2^2+y_2^2}\right) + i\left(\frac{y_1x_2 - x_1y_2}{x_2^2+y_2^2}\right) \qquad (z_2 \neq 0). \tag{11}$$

Although expression (11) is not easy to remember, it can be obtained by writing [see Exercise 14(*b*)]

$$\frac{z_1}{z_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}, \tag{12}$$

multiplying out the products in the numerator and denominator on the right, and then using the property

$$\frac{z_1+z_2}{z_3} = (z_1+z_2)z_3^{-1} = z_1z_3^{-1} + z_2z_3^{-1} = \frac{z_1}{z_3} + \frac{z_2}{z_3} \qquad (z_3 \neq 0). \tag{13}$$

The motivation for starting with equation (12) appears in the next section.

Finally, we mention some expected identities involving quotients that follow from the relation

$$\frac{1}{z_2} = z_2^{-1} \qquad (z_2 \neq 0), \tag{14}$$

which is equation (10) when $z_1 = 1$. It enables us, for example, to write that equation in the form

$$\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right) \qquad (z_2 \neq 0). \tag{15}$$

Also, by observing that (see Exercise 6)

$$(z_1z_2)(z_1^{-1}z_2^{-1}) = (z_1z_1^{-1})(z_2z_2^{-1}) = 1 \qquad (z_1 \neq 0, z_2 \neq 0),$$

and hence that $(z_1z_2)^{-1} = z_1^{-1}z_2^{-1}$, one can use equation (14) to show that

$$\frac{1}{z_1z_2} = (z_1z_2)^{-1} = z_1^{-1}z_2^{-1} = \left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right) \qquad (z_1 \neq 0, z_2 \neq 0). \tag{16}$$

Another useful identity, to be derived in the exercises, is

$$\frac{z_1z_2}{z_3z_4} = \left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) \qquad (z_3 \neq 0, z_4 \neq 0). \tag{17}$$

EXAMPLE. Computations such as the following are now justified:

$$\left(\frac{1}{2-3i}\right)\left(\frac{1}{1+i}\right) = \frac{1}{(2-3i)(1+i)} = \frac{1}{5-i} \cdot \frac{5+i}{5+i} = \frac{5+i}{(5-i)(5+i)}$$
$$= \frac{5+i}{26} = \frac{5}{26} + \frac{i}{26} = \frac{5}{26} + \frac{1}{26}i.$$

EXERCISES

1. Verify that
   (*a*) $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i$; (*b*) $(2, -3)(-2, 1) = (-1, 8)$;
   (*c*) $(3, 1)(3, -1)\left(\frac{1}{5}, \frac{1}{10}\right) = (2, 1)$; (*d*) $\frac{1+2i}{3-4i} + \frac{2-i}{5i} = -\frac{2}{5}$;
   (*e*) $\frac{5}{(1-i)(2-i)(3-i)} = \frac{i}{2}$; (*f*) $(1-i)^4 = -4$.
2. Show that $(1+z)^2 = 1 + 2z + z^2$.
3. Verify that each of the two numbers $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$.
4. Show that
   (*a*) $\text{Im}(iz) = \text{Re}\, z$; (*b*) $\text{Re}(iz) = -\text{Im}\, z$; (*c*) $\frac{1}{1/z} = z \quad (z \neq 0)$;
   (*d*) $(-1)z = -z$.
5. Prove that multiplication is commutative, as stated in the second of equations (1), Sec. 2.
6. Use the associative and commutative laws for multiplication to show that
   $$(z_1z_2)(z_3z_4) = (z_1z_3)(z_2z_4).$$

7. Prove that if $z_1 z_2 z_3 = 0$, then at least one of the three factors is zero.
   Suggestion: Write $(z_1 z_2)z_3 = 0$ and use a similar result (Sec. 2) involving two factors.
8. Verify
   (*a*) the associative law for addition, stated in the first of equations (2), Sec. 2;
   (*b*) the distributive law (3), Sec. 2.
9. Use the associative law for addition and the distributive law to show that

$$z(z_1 + z_2 + z_3) = zz_1 + zz_2 + zz_3.$$

10. By writing $i = (0, 1)$ and $y = (y, 0)$, show that $-(iy) = (-i)y = i(-y)$.
11. (*a*) Write $(x, y) + (u, v) = (x, y)$ and point out how it follows that the complex number $0 = (0, 0)$ is unique as an additive identity.
   (*b*) Likewise, write $(x, y)(u, v) = (x, y)$ and show that the number $1 = (1, 0)$ is a unique multiplicative identity.
12. Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0)$$

   and then solving a pair of simultaneous equations in x and y.
   Suggestion: Use the fact that no real number x satisfies the given equation to show that $y \neq 0$.

   Ans. $z = \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$.
13. Derive expression (11), Sec. 2, for the quotient z_1/z_2 by the method described just after it.
14. (*a*) With the aid of relations (15) and (16) in Sec. 2, derive identity (17) there.
   (*b*) Use the identity derived in part (*a*) to establish the cancellation law

$$\frac{z_1 z}{z_2 z} = \frac{z_1}{z_2} \qquad (z_2 \neq 0, z \neq 0).$$

3. MODULI AND CONJUGATES

It is natural to associate any nonzero complex number $z = x + iy$ with the directed line segment, or vector, from the origin to the point (x, y) that represents z (Sec. 1) in the complex plane. In fact, we often refer to z as the point z or the vector z. In Fig. 1 the numbers $z = x + iy$ and $-2 + i$ are displayed graphically as both points and radius vectors.

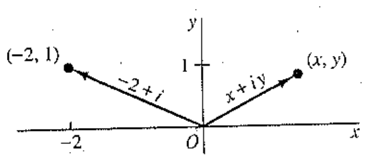

FIGURE 1

According to the definition of the sum of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the number $z_1 + z_2$ corresponds to the point $(x_1 + x_2, y_1 + y_2)$. It

also corresponds to a vector with those coordinates as its components. Hence $z_1 + z_2$ may be obtained vectorially as shown in Fig. 2. The difference $z_1 - z_2 = z_1 + (-z_2)$ corresponds to the sum of the vectors for z_1 and $-z_2$ (Fig. 3). Note that, by translating the radius vector $z_1 - z_2$ in Fig. 3, one can interpret $z_1 - z_2$ as the directed line segment from the point (x_2, y_2) to the point (x_1, y_1).

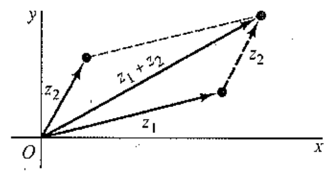

FIGURE 2

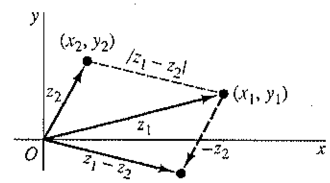

FIGURE 3

Although the product of two complex numbers z_1 and z_2 is itself a complex number represented by a vector, that vector lies in the same plane as the vectors for z_1 and z_2. Evidently, then, this product is neither the scalar nor the vector product used in ordinary vector analysis.

The vector interpretation of complex numbers is especially helpful in extending the concept of absolute values of real numbers to the complex plane. The *modulus*, or absolute value, of a complex number $z = x + iy$ is defined as the nonnegative real number $\sqrt{x^2 + y^2}$ and is denoted by $|z|$; that is,

$$|z| = \sqrt{x^2 + y^2}. \tag{1}$$

Geometrically, the number $|z|$ is the distance between the point (x, y) and the origin, or the length of the vector representing z. It reduces to the usual absolute value in the real number system when $y = 0$. Note that, while *the inequality* $z_1 < z_2$ *is meaningless unless both* z_1 *and* z_2 *are real*, the statement $|z_1| < |z_2|$ means that the point z_1 is closer to the origin than the point z_2 is.

EXAMPLE 1. Since $|-3 + 2i| = \sqrt{13}$ and $|1 + 4i| = \sqrt{17}$, the point $-3 + 2i$ is closer to the origin than $1 + 4i$ is.

The distance between two points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is $|z_1 - z_2|$. This is clear from Fig. 3, since $|z_1 - z_2|$ is the length of the vector representing $z_1 - z_2$. Alternatively, it follows from definition (1) and the expression

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

that

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

The complex numbers z corresponding to the points lying on the circle with center z_0 and radius R thus satisfy the equation $|z - z_0| = R$, and conversely. We refer to this set of points simply as the circle $|z - z_0| = R$.

EXAMPLE 2. The equation $|z-1+3i| = 2$ represents the circle whose center is $z_0 = (1, -3)$ and whose radius is $R = 2$.

It also follows from definition (1) that the real numbers $|z|$, $\text{Re}\, z = x$, and $\text{Im}\, z = y$ are related by the equation

(2) $$|z|^2 = (\text{Re}\, z)^2 + (\text{Im}\, z)^2.$$

Thus

(3) $$\text{Re}\, z \le |\text{Re}\, z| \le |z| \quad \text{and} \quad \text{Im}\, z \le |\text{Im}\, z| \le |z|.$$

The *complex conjugate*, or simply the conjugate, of a complex number $z = x + iy$ is defined as the complex number $x - iy$ and is denoted by $\overline{z}$; that is,

(4) $$\overline{z} = x - iy.$$

The number $\overline{z}$ is represented by the point $(x, -y)$, which is the reflection in the real axis of the point (x, y) representing z (Fig. 4). Note that $\overline{\overline{z}} = z$ and $|\overline{z}| = |z|$ for all z.

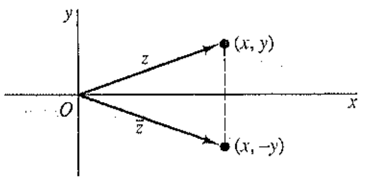

FIGURE 4

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2).$$

So the conjugate of the sum is the sum of the conjugates:

(5) $$\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2.$$

In like manner, it is easy to show that

(6) $$\overline{z_1 - z_2} = \overline{z}_1 - \overline{z}_2,$$

(7) $$\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2,$$

(8) $$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2} \qquad (z_2 \ne 0).$$

The sum $z + \overline{z}$ of a complex number $z = x + iy$ and its conjugate $\overline{z} = x - iy$ is the real number $2x$, and the difference $z - \overline{z}$ is the pure imaginary number $2iy$. Hence

(9) $$\text{Re}\, z = \frac{z + \overline{z}}{2}, \qquad \text{Im}\, z = \frac{z - \overline{z}}{2i}.$$

An important identity relating the conjugate of a complex number $z = x + iy$ to its modulus is

(10) $$z\overline{z} = |z|^2,$$

where each side is equal to $x^2 + y^2$. It suggests the method for determining a quotient z_1/z_2 that begins with expression (12), Sec. 2. That method is, of course, based on multiplying both the numerator and the denominator of z_1/z_2 by $\overline{z}_2$, so that the denominator becomes the real number $|z_2|^2$.

EXAMPLE 3. As an illustration,

$$\frac{-1+3i}{2-i} = \frac{(-1+3i)(2+i)}{(2-i)(2+i)} = \frac{-5+5i}{|2-i|^2} = \frac{-5+5i}{5} = -1 + i.$$

Also, see the example at the end of Sec. 2.

With the aid of identity (10), one can easily obtain various other properties of moduli from properties of conjugates noted above. We mention that

(11) $$|z_1 z_2| = |z_1||z_2|,$$

(12) $$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad (z_2 \ne 0).$$

Property (11) can be established by writing

$$|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1 \overline{z}_1)(z_2 \overline{z}_2) = |z_1|^2 |z_2|^2 = (|z_1||z_2|)^2$$

and recalling that a modulus is never negative. Property (12) can be verified in a similar way.

4. TRIANGLE INEQUALITY

Properties of moduli and conjugates in Sec. 3 enable us to give an algebraic derivation of the *triangle inequality,* which provides an upper bound for the modulus of the sum of two complex numbers z_1 and z_2:

(1) $$|z_1 + z_2| \le |z_1| + |z_2|.$$

This important inequality is geometrically evident in Fig. 2 of Sec. 3. Indeed, it is merely a statement that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides. We note from Fig. 2 that inequality (1) is actually an equality when the points z_1, z_2, and 0 are collinear.

We start the algebraic derivation by writing

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2)$$

and multiplying out the far right-hand side. This shows that

$$|z_1 + z_2|^2 = z_1\overline{z}_1 + (z_1\overline{z}_2 + \overline{z_1\overline{z}_2}) + z_2\overline{z}_2.$$

But

$$z_1\overline{z}_2 + \overline{z_1\overline{z}_2} = 2\,\text{Re}(z_1\overline{z}_2) \le 2|z_1\overline{z}_2| = 2|z_1||z_2|;$$

and so

$$|z_1 + z_2|^2 \le |z_1|^2 + 2|z_1||z_2| + |z_2|^2,$$

or

$$|z_1 + z_2|^2 \le (|z_1| + |z_2|)^2.$$

Since moduli are nonnegative, inequality (1) now follows.

An immediate consequence of the triangle inequality is the fact that

$$(2) \qquad |z_1 + z_2| \ge ||z_1| - |z_2||.$$

To derive inequality (2), we write

$$|z_1| = |(z_1 + z_2) + (-z_2)| \le |z_1 + z_2| + |-z_2|,$$

which means that

$$(3) \qquad |z_1 + z_2| \ge |z_1| - |z_2|.$$

This is inequality (2) when $|z_1| \ge |z_2|$. If $|z_1| < |z_2|$, we need only interchange z_1 and z_2 in inequality (3) to get

$$|z_1 + z_2| \ge -(|z_1| - |z_2|),$$

which is the desired result. Inequality (2) tells us, of course, that the length of one side of a triangle is greater than or equal to the difference of the lengths of the other two sides.

Useful alternative forms of inequalities (1) and (2) are obtained when z_2 is replaced by $-z_2$:

$$(4) \qquad |z_1 - z_2| \le |z_1| + |z_2|,$$

$$(5) \qquad |z_1 - z_2| \ge ||z_1| - |z_2||.$$

EXAMPLE 1. If a point z lies on the unit circle $|z| = 1$ about the origin, then

$$|z^3 - 2| \le |z|^3 + 2 = 3$$

and

$$|z^3 - 2| \ge ||z|^3 - 2| = 1.$$

The triangle inequality can be generalized by means of mathematical induction to sums involving any finite number of terms:

$$(6) \qquad |z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n| \qquad (n = 2, 3, \ldots).$$

To give details of the induction proof here, we note that when $n = 2$, inequality (6) is just inequality (1). Furthermore, if inequality (6) is assumed to be valid when $n = m$, it must also hold when $n = m + 1$ since, by the triangle inequality,

$$\begin{aligned} |(z_1 + z_2 + \cdots + z_m) + z_{m+1}| &\le |z_1 + z_2 + \cdots + z_m| + |z_{m+1}| \\ &\le (|z_1| + |z_2| + \cdots + |z_m|) + |z_{m+1}|. \end{aligned}$$

EXAMPLE 2. If z is a point inside the circle centered at the origin and with radius 2, so that $|z| < 2$, then

$$|z^3 + 3z^2 - 2z + 1| \le |z|^3 + 3|z|^2 + 2|z| + 1 < 25.$$

EXERCISES

1. Locate the numbers $z_1 + z_2$ and $z_1 - z_2$ vectorially when
   (a) $z_1 = 2i, z_2 = \dfrac{2}{3} - i$; (b) $z_1 = (-\sqrt{3}, 1), z_2 = (\sqrt{3}, 0)$;
   (c) $z_1 = (-3, 1), z_2 = (1, 4)$; (d) $z_1 = x_1 + iy_1, z_2 = x_1 - iy_1$.
2. Use properties of conjugates and moduli to show that
   (a) $\overline{\bar{z} + 3i} = z - 3i$; (b) $\overline{iz} = -i\bar{z}$; (c) $\overline{(2+i)^2} = 3 - 4i$;
   (d) $|(2\bar{z} + 5)(\sqrt{2} - i)| = \sqrt{3}|2z + 5|$.
3. Verify inequalities (3), Sec. 3, involving Re z, Im z, and $|z|$.
4. Prove that $\sqrt{2}|z| \ge |\text{Re}\, z| + |\text{Im}\, z|$.
5. Verify properties (6) and (7) of $\bar{z}$ in Sec. 3.
6. Use the property $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ to show that (a) $\overline{z_1 z_2 z_3} = \bar{z}_1 \bar{z}_2 \bar{z}_3$; (b) $\overline{(z^4)} = (\bar{z})^4$.
7. Verify property (12) of moduli in Sec. 3.
8. Use results in Sec. 3 to show that when z_2 and z_3 are nonzero,
   (a) $\overline{\left(\dfrac{z_1}{z_2 z_3}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2 \bar{z}_3}$; (b) $\left|\dfrac{z_1}{z_2 z_3}\right| = \dfrac{|z_1|}{|z_2||z_3|}$.
9. With the aid of inequalities in Sec. 4, show that when $|z_3| \ne |z_4|$,
   $$\left|\frac{z_1 + z_2}{z_3 + z_4}\right| \le \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$
10. In each case, sketch the set of points determined by the given condition:
    (a) $|z - 1 + i| = 1$; (b) $|z + i| \le 3$; (c) $\text{Re}(\bar{z} - i) = 2$; (d) $|2z - i| = 4$.
11. Apply inequalities in Secs. 3 and 4 to show that
    $$|\text{Im}(1 - \bar{z} + z^2)| < 3 \quad \text{when} \quad |z| < 1.$$
12. By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and then using inequality (5), Sec. 4, show that if z lies on the circle $|z| = 2$, then
    $$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \le \frac{1}{3}.$$
13. It is shown in Sec. 2 that if $z_1 z_2 = 0$, then at least one of the numbers z_1 and z_2 must be zero. Give an alternative proof, based on the corresponding result for real numbers, using identity (11), Sec. 3.
14. Prove that
    (a) z is real if and only if $\bar{z} = z$;
    (b) z is either real or pure imaginary if and only if $(\bar{z})^2 = z^2$.
15. Use mathematical induction to show that when $n = 2, 3, \ldots$,
    (a) $\overline{z_1 + z_2 + \cdots + z_n} = \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_n$; (b) $\overline{z_1 z_2 \cdots z_n} = \bar{z}_1 \bar{z}_2 \cdots \bar{z}_n$.
16. Let $a_0, a_1, a_2, \ldots, a_n$ $(n \ge 1)$ denote *real* numbers, and let z be any complex number. With the aid of the results in Exercise 15, show that
    $$\overline{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n} = a_0 + a_1 \bar{z} + a_2 \bar{z}^2 + \cdots + a_n \bar{z}^n.$$
17. Show that the equation $|z - z_0| = R$ of a circle, centered at z_0 with radius R, can be written
    $$|z|^2 - 2\,\text{Re}(z\bar{z}_0) + |z_0|^2 = R^2.$$

18. Using expressions (9), Sec. 3, for Re z and Im z, show that the hyperbola $x^2 - y^2 = 1$ can be written

$$z^2 + \bar{z}^2 = 2.$$

19. Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2, give a geometric argument that

(*a*) the equation $|z - 4i| + |z + 4i| = 10$ represents an ellipse whose foci are $(0, \pm 4)$;

(*b*) the equation $|z - 1| = |z + i|$ represents the line through the origin whose slope is -1.

5. POLAR COORDINATES AND EULER'S FORMULA

Let r and θ be polar coordinates of the point (x, y) that corresponds to a *nonzero* complex number $z = x + iy$. Since $x = r\cos\theta$ and $y = r\sin\theta$, z can be written in *polar form* as

$$z = r(\cos\theta + i\sin\theta). \tag{1}$$

If $z = 0$, the coordinate θ is undefined.

In complex analysis, the real number r is not allowed to be negative and is the length of the radius vector for z; that is, $r = |z|$. The real number θ represents the angle, measured in radians, that z makes with the positive real axis when z is interpreted as a radius vector (Fig. 5). As in calculus, θ has an infinite number of possible values, including negative ones, that differ by integral multiples of 2π. Those values can be determined from the equation $\tan\theta = y/x$, where the quadrant containing the point corresponding to z must be specified. Each value of θ is called an *argument* of z, and the set of all such values is denoted by $\arg z$. The *principal value* of $\arg z$, denoted by $\operatorname{Arg} z$, is that unique value Θ such that $-\pi < \Theta \le \pi$. Note that

$$\arg z = \operatorname{Arg} z + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{2}$$

Also, when z is a negative real number, $\operatorname{Arg} z$ has value π, not $-\pi$.

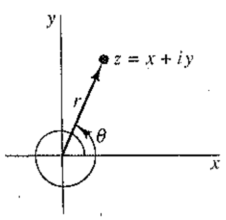

FIGURE 5

EXAMPLE 1. The complex number $-1 - i$, which lies in the third quadrant, has principal argument $-3\pi/4$. That is,

$$\operatorname{Arg}(-1 - i) = -\frac{3\pi}{4};$$

and it follows that

$$\arg(-1 - i) = -\frac{3\pi}{4} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Using the symbol $e^{i\theta}$, or $\exp(i\theta)$, which is defined by *Euler's formula* for any real value of θ as

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{3}$$

we can write the polar form (1) more compactly in *exponential form* as

$$z = re^{i\theta}. \tag{4}$$

The choice of the symbol $e^{i\theta}$ will be motivated later on in Sec. 23. Its use in Sec. 6 will, however, suggest that it is a natural choice.

EXAMPLE 2. The number $-1 - i$ in Example 1 has exponential form

$$-1 - i = \sqrt{2}\exp\left[i\left(-\frac{3\pi}{4}\right)\right]. \tag{5}$$

With the agreement that $e^{-i\theta} = e^{i(-\theta)}$, this can also be written $-1 - i = \sqrt{2}e^{-i3\pi/4}$. Expression (5) is, of course, only one of an infinite number of possibilities for the exponential form of $-1 - i$:

$$-1 - i = \sqrt{2}\exp\left[i\left(-\frac{3\pi}{4} + 2n\pi\right)\right] \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{6}$$

Consider now a point $z = re^{i\theta}$, lying on a circle centered at the origin and with radius r (Fig. 6). As θ is increased, z moves around the circle in the counterclockwise direction. In particular, when θ is increased by 2π, we arrive at the original point; and the same is true when θ is decreased by 2π. It is, therefore, evident from Fig. 6 that *two nonzero complex numbers*

$$z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2 e^{i\theta_2}$$

are equal if and only if

$$r_1 = r_2 \quad \text{and} \quad \theta_1 = \theta_2 + 2n\pi,$$

where n is some integer $(n = 0, \pm 1, \pm 2, \ldots)$.

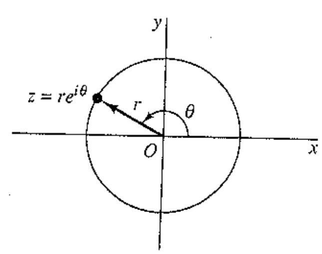

FIGURE 6

Note, too, that values of $e^{i\theta}$ are immediate from Fig. 6, without reference to Euler's formula (3), when $r = 1$ and θ is some integral multiple of $\pi/2$. It is, for instance, geometrically obvious that $e^{i\pi} = -1$, $e^{-i\pi/2} = -i$, and $e^{-i4\pi} = 1$.

Figure 6, with $r = R$, also shows that the equation

$$z = Re^{i\theta} \qquad (0 \le \theta \le 2\pi) \tag{7}$$

is a parametric representation of the circle $|z| = R$, centered at the origin with radius R. As the parameter θ in Fig. 6 increases from $\theta = 0$ over the interval $0 \le \theta \le 2\pi$, the point z starts from the positive real axis and traverses the circle once in the counterclockwise direction. More generally, the circle $|z - z_0| = R$, whose center is z_0 and whose radius is R, has the parametric representation

$$z = z_0 + Re^{i\theta} \qquad (0 \le \theta \le 2\pi). \tag{8}$$

This can be seen vectorially (Fig. 7) by noting that a point z traversing the circle $|z - z_0| = R$ once in the counterclockwise direction corresponds to the sum of the fixed vector z_0 and a vector of length R whose angle of inclination θ varies from $\theta = 0$ to $\theta = 2\pi$.

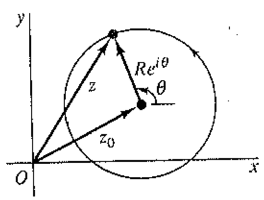

FIGURE 7

6. PRODUCTS AND QUOTIENTS IN EXPONENTIAL FORM

Simple trigonometry tells us that $e^{i\theta}$ has the familiar additive property of the exponential function in calculus:

$$\begin{aligned} e^{i\theta_1}e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = e^{i(\theta_1+\theta_2)}. \end{aligned}$$

Thus, if $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, the product z_1z_2 has exponential form

$$z_1z_2 = r_1r_2e^{i\theta_1}e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}. \tag{1}$$

Moreover,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\cdot\frac{e^{i\theta_1}e^{-i\theta_2}}{e^{i\theta_2}e^{-i\theta_2}} = \frac{r_1}{r_2}\cdot\frac{e^{i(\theta_1-\theta_2)}}{e^{i0}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}. \tag{2}$$

Because $1 = 1e^{i0}$, it follows from expression (2) that the inverse of any nonzero complex number $z = re^{i\theta}$ is

$$z^{-1} = \frac{1}{z} = \frac{1}{r}e^{-i\theta}. \tag{3}$$

Expressions (1), (2), and (3) are, of course, easily remembered by applying the usual algebraic rules for real numbers and e^x.

Another important result that can be obtained formally by applying rules for real numbers is

$$z^n = r^ne^{in\theta} \qquad (n = 0, \pm1, \pm2, \ldots). \tag{4}$$

It is easily verified for positive values of n by mathematical induction. To be specific, we first note that it becomes $z = re^{i\theta}$ when $n = 1$. Next, we assume that it is valid when $n = m$, where m is any positive integer. In view of expression (1) for the product of two nonzero complex numbers in exponential form, it is then valid for $n = m + 1$:

$$z^{m+1} = zz^m = re^{i\theta}r^me^{im\theta} = r^{m+1}e^{i(m+1)\theta}.$$

Expression (4) is thus verified when n is a positive integer. It also holds when $n = 0$, with the convention that $z^0 = 1$. If $n = -1, -2, \ldots$, on the other hand, we define z^n in terms of the multiplicative inverse of z by writing

$$z^n = (z^{-1})^m, \qquad \text{where} \qquad m = -n = 1, 2, \ldots.$$

Then, since expression (4) is valid for positive integral powers, it follows from the exponential form (3) of z^{-1} that

$$z^n = \left[\frac{1}{r}e^{i(-\theta)}\right]^m = \left(\frac{1}{r}\right)^m e^{im(-\theta)} = \left(\frac{1}{r}\right)^{-n} e^{i(-n)(-\theta)} = r^ne^{in\theta} \quad (n = -1, -2, \ldots).$$

Expression (4) is now established for all integral powers.

Observe that if $r = 1$, expression (4) becomes

$$(e^{i\theta})^n = e^{in\theta} \qquad (n = 0, \pm1, \pm2, \ldots). \tag{5}$$

When written in the form

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \qquad (n = 0, \pm1, \pm2, \ldots), \tag{6}$$

this is known as *de Moivre's formula.*

Expression (4) can be useful in finding powers of complex numbers even when they are given in rectangular form and the result is desired in that form.

EXAMPLE 1. In order to put $(\sqrt{3} + i)^7$ in rectangular form, one need only write

$$(\sqrt{3} + i)^7 = (2e^{i\pi/6})^7 = 2^7e^{i7\pi/6} = (2^6e^{i\pi})(2e^{i\pi/6}) = -64(\sqrt{3} + i).$$

We turn now to an important identity involving arguments (Sec. 5) of products:

$$\arg(z_1z_2) = \arg z_1 + \arg z_2. \tag{7}$$

It is to be interpreted as saying that if values of two of these three (multiple-valued) arguments are specified, then there is a value of the third such that the equation holds.

We start the verification of statement (7) by letting θ_1 and θ_2 denote any values of $\arg z_1$ and $\arg z_2$, respectively. Expression (1) then tells us that $\theta_1 + \theta_2$ is a value of $\arg(z_1 z_2)$. (See Fig. 8.) If, on the other hand, values of $\arg(z_1 z_2)$ and $\arg z_1$ are specified, those values correspond to particular choices of n and n_1 in the expressions

$$\arg(z_1 z_2) = (\theta_1 + \theta_2) + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and

$$\arg z_1 = \theta_1 + 2n_1\pi \qquad (n_1 = 0, \pm 1, \pm 2, \ldots).$$

Since

$$(\theta_1 + \theta_2) + 2n\pi = (\theta_1 + 2n_1\pi) + [\theta_2 + 2(n - n_1)\pi],$$

equation (7) is evidently satisfied when the value

$$\arg z_2 = \theta_2 + 2(n - n_1)\pi$$

is chosen. Verification when values of $\arg(z_1 z_2)$ and $\arg z_2$ are specified follows by symmetry.

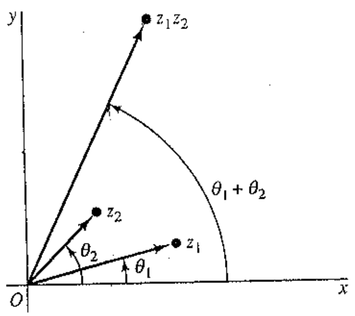

FIGURE 8

Statement (7) is sometimes valid when *arg* is replaced everywhere by *Arg* (see Exercise 6). But, as the following example illustrates, that is *not always* the case.

EXAMPLE 2. When $z_1 = -1$ and $z_2 = i$,

$$\text{Arg}(z_1 z_2) = \text{Arg}(-i) = -\frac{\pi}{2} \quad \text{but} \quad \text{Arg}\, z_1 + \text{Arg}\, z_2 = \pi + \frac{\pi}{2} = \frac{3\pi}{2}.$$

If, however, we take the values of $\arg z_1$ and $\arg z_2$ just used and select the value $\arg(z_1 z_2) = 3\pi/2$, we find that equation (7) *is* satisfied.

Another statement, analogous to statement (7), is that

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2. \tag{8}$$

It can be verified with the aid of expression (2).

EXERCISES

1. Find the principal argument Arg z when

(*a*) $z = \dfrac{-2}{1 + \sqrt{3}i}$; (*b*) $z = \dfrac{i}{-2 - 2i}$; (*c*) $z = (\sqrt{3} - i)^6$.

Ans. (*a*) $2\pi/3$; (*b*) $-3\pi/4$; (*c*) π.

2. By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

(*a*) $i(1 - \sqrt{3}i)(\sqrt{3} + i) = 2(1 + \sqrt{3}i)$; (*b*) $5i/(2 + i) = 1 + 2i$;

(*c*) $(-1 + i)^7 = -8(1 + i)$; (*d*) $(1 + \sqrt{3}i)^{-10} = 2^{-11}(-1 + \sqrt{3}i)$.

3. Show that

(*a*) $|e^{i\theta}| = 1$; (*b*) $\overline{e^{i\theta}} = e^{-i\theta}$; (*c*) $e^{i\theta_1} e^{i\theta_2} \cdots e^{i\theta_n} = e^{i(\theta_1 + \theta_2 + \cdots + \theta_n)}$ $(n = 2, 3, \ldots)$.

4. Solve the equation $|e^{i\theta} - 1| = 2$ for θ $(0 \le \theta < 2\pi)$ and verify the solution geometrically.

Ans. π.

5. Use de Moivre's formula (Sec. 6) to derive the following trigonometric identities:

(*a*) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$; (*b*) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

6. Show that if $\text{Re}\, z_1 > 0$ and $\text{Re}\, z_2 > 0$, then

$$\text{Arg}(z_1 z_2) = \text{Arg}\, z_1 + \text{Arg}\, z_2,$$

where $\text{Arg}(z_1 z_2)$ denotes the principal value of $\arg(z_1 z_2)$, etc.

7. Verify the statement (Sec. 6)

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2.$$

8. According to Sec. 3, the difference $z - z_0$ of two distinct complex numbers can be interpreted vectorially. (See Fig. 9, where θ denotes the angle of inclination of the vector representing $z - z_0$). By translating the vector for $z - z_0$ so that it is a radius vector, show that the values of $\arg(\overline{z - z_0})$ are the same as the values of $-\arg(z - z_0)$. Use the same method to show that

$$\text{Arg}(\overline{z - z_0}) = -\text{Arg}(z - z_0)$$

if and only if $z - z_0$ is not a negative real number.

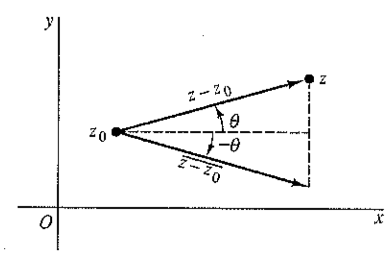

FIGURE 9

9. Given that $z_1 z_2 \neq 0$, use the exponential forms of z_1 and z_2 to prove that

$$\text{Re}(z_1 \bar{z}_2) = |z_1||z_2|$$

if and only if $\theta_1 - \theta_2 = 2n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$, where $\theta_1 = \arg z_1$ and $\theta_2 = \arg z_2$.

10. Given that $z_1 z_2 \neq 0$ and using the result in Exercise 9, modify the algebraic derivation of the triangle inequality in Sec. 4 to show that

$$|z_1 + z_2| = |z_1| + |z_2|$$

if and only if $\theta_1 - \theta_2 = 2n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$, where $\theta_1 = \arg z_1$ and $\theta_2 = \arg z_2$. Interpret this statement geometrically.

11. Let z be a nonzero complex number and n a negative integer $(n = -1, -2, \ldots)$. Also, write $z = re^{i\theta}$ and $m = -n = 1, 2, \ldots$. Using the expressions $z^m = r^m e^{im\theta}$ and $z^{-1} = (1/r)e^{i(-\theta)}$, verify that $(z^m)^{-1} = (z^{-1})^m$ and hence that the definition $z^n = (z^{-1})^m$ in Sec. 6 could have been written alternatively as $z^n = (z^m)^{-1}$.

12. Prove that two nonzero complex numbers z_1 and z_2 have the same moduli if and only if there are complex numbers c_1 and c_2 such that $z_1 = c_1 c_2$ and $z_2 = c_1 \overline{c_2}$.

Suggestion: Note that

$$\exp\left(i\frac{\theta_1 + \theta_2}{2}\right)\exp\left(i\frac{\theta_1 - \theta_2}{2}\right) = \exp(i\theta_1)$$

and [see Exercise 3(*b*)]

$$\exp\left(i\frac{\theta_1 + \theta_2}{2}\right)\overline{\exp\left(i\frac{\theta_1 - \theta_2}{2}\right)} = \exp(i\theta_2).$$

13. Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1),$$

and then use it to derive *Lagrange's trigonometric identity:*

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \qquad (0 < \theta < 2\pi).$$

Suggestion: As for the first identity, write $S = 1 + z + z^2 + \cdots + z^n$ and consider the difference $S - zS$. To derive the second identity, write $z = e^{i\theta}$ in the first one.

14. Use mathematical induction to establish the binomial formula for complex numbers:

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k} z_2^k \qquad (n = 1, 2, \ldots),$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (k = 0, 1, 2, \ldots, n)$$

and where it is agreed that $0! = 1$.

15. Use mathematical induction to verify de Moivre's formula (Sec. 6)

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

when n is a positive integer $(n = 1, 2, \ldots)$.

16. (*a*) Use the binomial formula (Exercise 14) and de Moivre's formula (see Exercise 15) to write

$$\cos n\theta + i\sin n\theta = \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta\,(i\sin\theta)^k \qquad (n = 1, 2, \ldots).$$

Then define the integer m by means of the equations

$$m = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{ is odd,} \end{cases}$$

and use the above sum to obtain the expression [compare Exercise 5(*a*)]

$$\cos n\theta = \sum_{k=0}^{m} \binom{n}{2k} (-1)^k \cos^{n-2k}\theta \sin^{2k}\theta \qquad (n = 1, 2, \ldots).$$

(*b*) Write $x = \cos\theta$ and suppose that $0 \leq \theta \leq \pi$, in which case $-1 \leq x \leq 1$. Point out how it follows from the final result in part (*a*) that each of the functions

$$T_n(x) = \cos(n\cos^{-1} x) \qquad (n = 0, 1, 2, \ldots)$$

is a polynomial of degree n in the variable x.*

7. ROOTS OF COMPLEX NUMBERS

The expression $z^n = r^n e^{in\theta}$ in Sec. 6 for integral powers of complex numbers $z = re^{i\theta}$ is useful in finding the nth roots of any nonzero complex number $z_0 = r_0 e^{i\theta_0}$, where n has one of the values $n = 2, 3, \ldots$. The method starts with the observation that an nth root of z_0 is a nonzero number $z = re^{i\theta}$ such that $z^n = z_0$, or

$$r^n e^{in\theta} = r_0 e^{i\theta_0}.$$

Now, according to the statement in italics near the end of Sec. 5,

$$r^n = r_0 \qquad \text{and} \qquad n\theta = \theta_0 + 2k\pi,$$

where k is any integer $(k = 0, \pm 1, \pm 2, \ldots)$. So $r = \sqrt[n]{r_0}$, where this radical denotes the unique *positive* nth root of the positive real number r_0, and

$$\theta = \frac{\theta_0 + 2k\pi}{n} = \frac{\theta_0}{n} + \frac{2k\pi}{n} \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Consequently, the complex numbers

$$z = \sqrt[n]{r_0}\exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

are the nth roots of z_0. We are able to see immediately from this exponential form of the roots that they all lie on the circle $|z| = \sqrt[n]{r_0}$ about the origin and are equally spaced every $2\pi/n$ radians, starting with argument θ_0/n. Evidently, then, all of the *distinct* roots are obtained when $k = 0, 1, 2, \ldots, n-1$, and no further roots arise with other values of k. We let c_k $(k = 0, 1, 2, \ldots, n-1)$ denote these distinct roots and write

$$(1) \qquad c_k = \sqrt[n]{r_0}\exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \qquad (k = 0, 1, 2, \ldots, n-1).$$

*These polynomials are called Chebyshev polynomials and are prominent in approximation theory.

The number $\sqrt[n]{r_0}$ is the length of each of the radius vectors representing the n roots. The first root c_0 has argument θ_0/n; and the two roots when $n = 2$ lie at the opposite ends of a diameter of the circle $|z| = \sqrt[n]{r_0}$, the second root being $-c_0$. When $n \geq 3$, the roots lie at the vertices of a regular polygon of n sides inscribed in that circle.

We shall let $z_0^{1/n}$ denote the *set* of nth roots of z_0. If, in particular, z_0 is a positive real number r_0, the symbol $r_0^{1/n}$ denotes the entire set of roots; and the symbol $\sqrt[n]{r_0}$ in expression (1) is reserved for the one positive root. When the value of θ_0 that is used in expression (1) is the principal value of $\arg z_0$ $(-\pi < \theta_0 \leq \pi)$, the number c_0 is referred to as the *principal root*. Thus when z_0 is a positive real number r_0, its principal root is $\sqrt[n]{r_0}$.

Finally, a convenient way to remember expression (1) is to write z_0 in its most general exponential form (compare Example 2 in Sec. 5),

$$z_0 = r_0 \exp[i(\theta_0 + 2k\pi)] \qquad (k = 0, \pm 1, \pm 2, \ldots),$$

and to *formally* apply laws of fractional exponents involving real numbers, keeping in mind that there are precisely n roots:

$$\begin{aligned} z_0^{1/n} &= \{r_0 \exp[i(\theta_0 + 2k\pi)]\}^{1/n} \\ &= \sqrt[n]{r_0} \exp\left[\frac{i(\theta_0 + 2k\pi)}{n}\right] \\ &= \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \qquad (k = 0, 1, 2, \ldots, n-1). \end{aligned}$$

EXAMPLE 1. In order to determine the nth roots of unity, we write

$$1 = 1 \exp[i(0 + 2k\pi)] \qquad (k = 0, \pm 1, \pm 2 \ldots)$$

and find that

$$1^{1/n} = \sqrt[n]{1} \exp\left[i\left(\frac{0}{n} + \frac{2k\pi}{n}\right)\right] = \exp\left(i\frac{2k\pi}{n}\right) \qquad (k = 0, 1, 2, \ldots, n-1). \tag{2}$$

When $n = 2$, these roots are, of course, ± 1. When $n \geq 3$, the regular polygon at whose vertices the roots lie is inscribed in the unit circle $|z| = 1$, with one vertex corresponding to the principal root $z = 1$ $(k = 0)$.

If we write

$$\omega_n = \exp\left(i\frac{2\pi}{n}\right), \tag{3}$$

it follows from property (5), Sec. 6, of $e^{i\theta}$ that

$$\omega_n^k = \exp\left(i\frac{2k\pi}{n}\right) \qquad (k = 0, 1, 2, \ldots, n-1).$$

Hence the distinct nth roots of unity just found are simply

$$1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}.$$

See Fig. 10, where the cases $n = 3, 4$, and 6 are illustrated. Note that $\omega_n^n = 1$. Finally, it is worthwhile observing that if c is any particular nth root of a nonzero complex number z_0, the set of nth roots can be put in the form

$$c, c\omega_n, c\omega_n^2, \ldots, c\omega_n^{n-1}.$$

This is because multiplication of any nonzero complex number by ω_n increases the argument of that number by $2\pi/n$, while leaving its modulus unchanged.

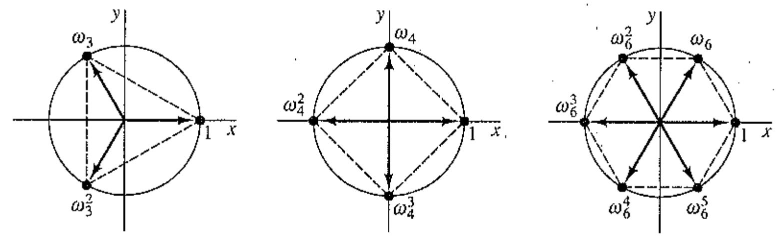

FIGURE 10

EXAMPLE 2. Let us find all values of $(-8i)^{1/3}$, or the three cube roots of $-8i$. One need only write

$$-8i = 8 \exp\left[i\left(-\frac{\pi}{2} + 2k\pi\right)\right] \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

to see that the desired roots are

$$c_k = 2 \exp\left[i\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right] \qquad (k = 0, 1, 2). \tag{4}$$

They lie at the vertices of an equilateral triangle, inscribed in the circle $|z| = 2$, and are equally spaced around that circle every $2\pi/3$ radians, starting with the principal root (Fig. 11)

$$c_0 = 2 \exp\left[i\left(-\frac{\pi}{6}\right)\right] = 2\left(\cos\frac{\pi}{6} - i \sin\frac{\pi}{6}\right) = \sqrt{3} - i.$$

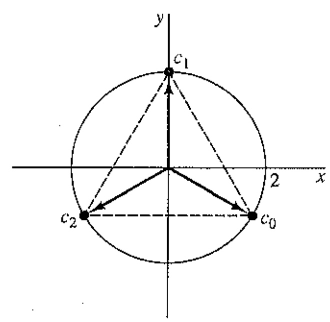

FIGURE 11

Without any further calculations, it is then evident that $c_1 = 2i$; and, since c_2 is symmetric to c_0 with respect to the imaginary axis, we know that $c_2 = -\sqrt{3} - i$.

These roots can, of course, be written

$$c_0, c_0\omega_3, c_0\omega_3^2, \qquad \text{where} \qquad \omega_3 = \exp\left(i\frac{2\pi}{3}\right).$$

(See the remarks at the end of Example 1.)

EXERCISES

1. Find the square roots of (*a*) $2i$; (*b*) $1 - \sqrt{3}i$, and express them in rectangular coordinates.

   Ans. (*a*) $\pm(1+i)$; (*b*) $\pm\dfrac{\sqrt{3}-i}{\sqrt{2}}$.

2. In each case, find all of the roots in rectangular coordinates, exhibit them geometrically, and point out which is the principal root:
   (*a*) $(-1)^{1/3}$; (*b*) $(-16)^{1/4}$; (*c*) $8^{1/6}$; (*d*) $(-8-8\sqrt{3}i)^{1/4}$.

   Ans. (*b*) $\pm\sqrt{2}(1+i), \pm\sqrt{2}(1-i)$; (*c*) $\pm\sqrt{2}, \pm\dfrac{1+\sqrt{3}i}{\sqrt{2}}, \pm\dfrac{1-\sqrt{3}i}{\sqrt{2}}$;

   (*d*) $\pm(\sqrt{3}-i), \pm(1+\sqrt{3}i)$.

3. Let $z = re^{i\theta}$ be any nonzero complex number and n a negative integer $(n = -1, -2, \ldots)$. Then define $z^{1/n}$ by means of the equation $z^{1/n} = (z^{-1})^{1/m}$, where $m = -n$. By showing that the m values of $(z^{1/m})^{-1}$ and $(z^{-1})^{1/m}$ are the same, verify that $z^{1/n} = (z^{1/m})^{-1}$. (Compare Exercise 11, Sec. 6.)

4. (*a*) Let a denote any fixed real number, and show that the two square roots of $a + i$ are

   $$\pm\sqrt{A}\exp\left(i\frac{\alpha}{2}\right),$$

   where $A = \sqrt{a^2+1}$ and $\alpha = \operatorname{Arg}(a+i)$.

   (*b*) With the aid of the trigonometric identities

   $$\cos^2\left(\frac{\alpha}{2}\right) = \frac{1+\cos\alpha}{2}, \qquad \sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{2},$$

   show that the square roots obtained in part (*a*) can be written

   $$\pm\frac{1}{\sqrt{2}}\left(\sqrt{A+a} + i\sqrt{A-a}\right).$$

5. According to Sec. 7, the three cube roots of a nonzero complex number z_0 can be written $c_0, c_0\omega_3, c_0\omega_3^2$, where c_0 is the principal cube root of z_0 and

   $$\omega_3 = \exp\left(i\frac{2\pi}{3}\right) = \frac{-1+\sqrt{3}i}{2}.$$

   Show that if $z_0 = -4\sqrt{2} + 4\sqrt{2}i$, then $c_0 = \sqrt{2}(1+i)$ and the other two cube roots are, in rectangular form, the numbers

   $$c_0\omega_3 = \frac{-(\sqrt{3}+1)+(\sqrt{3}-1)i}{\sqrt{2}}, \qquad c_0\omega_3^2 = \frac{(\sqrt{3}-1)-(\sqrt{3}+1)i}{\sqrt{2}}.$$

6. Find the four roots of the equation $z^4 + 4 = 0$ and use them to factor $z^4 + 4$ into quadratic factors with real coefficients.

   Ans. $(z^2+2z+2)(z^2-2z+2)$.

7. Show that if c is any nth root of unity other than unity itself, then

   $$1 + c + c^2 + \cdots + c^{n-1} = 0.$$

   Suggestion: Use the first identity in Exercise 13, Sec. 6.

8. (*a*) Prove that the usual formula solves the quadratic equation

   $$az^2 + bz + c = 0 \qquad (a \neq 0)$$

   when the coefficients a, b, and c are complex numbers. Specifically, by completing the square on the left-hand side, derive the quadratic formula

   $$z = \frac{-b + (b^2 - 4ac)^{1/2}}{2a},$$

   where both square roots are to be considered when $b^2 - 4ac \neq 0$.

   (*b*) Use the result in part (*a*) to find the roots of the equation $z^2 + 2z + (1-i) = 0$.

   Ans. (*b*) $\left(-1+\dfrac{1}{\sqrt{2}}\right)+\dfrac{i}{\sqrt{2}}, \quad \left(-1-\dfrac{1}{\sqrt{2}}\right)-\dfrac{i}{\sqrt{2}}$.

8. REGIONS IN THE COMPLEX PLANE

In this section, we are concerned with sets of complex numbers, or points in the z plane, and their closeness to one another. Our basic tool is the concept of an ε *neighborhood*

$$(1) \qquad |z - z_0| < \varepsilon$$

of a given point z_0. It consists of all points z lying inside but not on a circle centered at z_0 and with a specified positive radius ε (Fig. 12). When the value of ε is understood or is immaterial in the discussion, the set (1) is often referred to as just a neighborhood. Occasionally, it is convenient to speak of a *deleted neighborhood*

$$(2) \qquad 0 < |z - z_0| < \varepsilon,$$

consisting of all points z in an ε neighborhood of z_0 except for the point z_0 itself.

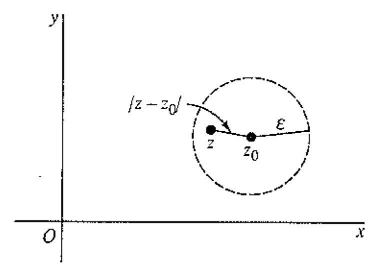

FIGURE 12

A point z_0 is said to be an *interior point* of a set S whenever there is some neighborhood of z_0 that contains only points of S; it is called an *exterior point* of S when there exists a neighborhood of it containing no points of S. If z_0 is neither of these, it is a *boundary point* of S. A boundary point is, therefore, a point all of whose neighborhoods contain points in S and points not in S. The totality of all boundary points is called the *boundary* of S. The circle $|z| = 1$, for instance, is the boundary of each of the sets

(3) $$|z| < 1 \quad \text{and} \quad |z| \le 1.$$

A set is *open* if it contains none of its boundary points. It is left as an exercise to show that a set is open if and only if each of its points is an interior point. A set is *closed* if it contains all of its boundary points; and the *closure* of a set S is the closed set consisting of all points in S together with the boundary of S. Note that the first of the sets (3) is open and that the second is the closure of both of those sets.

Some sets are, of course, neither open nor closed. For a set to be not open, there must be a boundary point that is contained in the set; and if a set is not closed, there exists a boundary point not contained in the set. Observe that the punctured disk $0 < |z| \le 1$ is neither open nor closed. The set of all complex numbers is, on the other hand, both open and closed since it has no boundary points.

An open set S is *connected* if each pair of points z_1 and z_2 in it can be joined by a polygonal line, consisting of a finite number of line segments joined end to end, that lies entirely in S. The open set $|z| < 1$ is connected. The annulus $1 < |z| < 2$ is, of course, open and it is also connected (see Fig. 13). An open set that is connected is called a *domain*. Note that any neighborhood is a domain. A domain together with some, none, or all of its boundary points is referred to as a *region*.

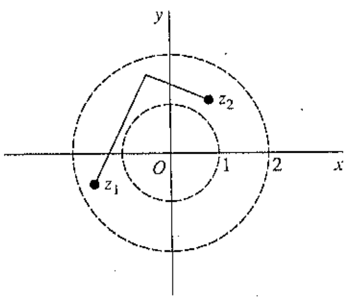

FIGURE 13

A set S is *bounded* if every point of S lies inside some circle $|z| = R$; otherwise, it is *unbounded*. Both of the sets (3) are bounded regions, and the half plane $\text{Re}\, z \ge 0$ is unbounded.

A point z_0 is said to be an *accumulation point* of a set S if each deleted neighborhood of z_0 contains at least one point of S. It follows that if a set S is closed, then it contains each of its accumulation points. For if an accumulation point z_0 were not in S, it would be a boundary point of S; but this contradicts the fact that a closed set

contains all of its boundary points. It is left as an exercise to show that the converse is, in fact, true. Thus, a set is closed if and only if it contains all of its accumulation points.

Evidently, a point z_0 is *not* an accumulation point of a set S whenever there exists some deleted neighborhood of z_0 that does not contain points of S. Note that the origin is the only accumulation point of the set $z_n = i/n$ $(n = 1, 2, \ldots)$.

EXERCISES

1. Sketch the following sets and determine which are domains:
(*a*) $|z - 2 + i| \le 1$; (*b*) $|2z + 3| > 4$;
(*c*) $\text{Im}\, z > 1$; (*d*) $\text{Im}\, z = 1$;
(*e*) $0 \le \arg z \le \pi/4$ $(z \ne 0)$; (*f*) $|z - 4| \ge |z|$.
Ans. (*b*), (*c*) are domains.

2. Which sets in Exercise 1 are neither open nor closed?
Ans. (*e*).

3. Which sets in Exercise 1 are bounded?
Ans. (*a*).

4. In each case, sketch the closure of the set:
(*a*) $-\pi < \arg z < \pi$ $(z \ne 0)$; (*b*) $|\text{Re}\, z| < |z|$; (*c*) $\text{Re}\left(\frac{1}{z}\right) \le \frac{1}{2}$; (*d*) $\text{Re}(z^2) > 0$.

5. Let S be the open set consisting of all points z such that $|z| < 1$ or $|z - 2| < 1$. State why S is not connected.

6. Show that a set S is open if and only if each point in S is an interior point.

7. Determine the accumulation points of each of the following sets:
(*a*) $z_n = i^n$ $(n = 1, 2, \ldots)$; (*b*) $z_n = i^n/n$ $(n = 1, 2, \ldots)$;
(*c*) $0 \le \arg z < \pi/2$ $(z \ne 0)$; (*d*) $z_n = (-1)^n(1 + i)(n - 1)/n$ $(n = 1, 2, \ldots)$.
Ans. (*a*) None; (*b*) 0; (*d*) $\pm(1 + i)$.

8. Prove that if a set contains each of its accumulation points, then it must be a closed set.

9. Show that any point z_0 of a domain is an accumulation point of that domain.

10. Prove that a finite set of points $z_1, z_2, \ldots, z_n$ cannot have any accumulation points.

CHAPTER 2

ANALYTIC FUNCTIONS

We now consider functions of a complex variable and develop a theory of differentiation for them. The main goal of the chapter is to introduce analytic functions, which play a central role in complex analysis.

9. FUNCTIONS OF A COMPLEX VARIABLE

Let S be a set of complex numbers. A *function* f defined on S is a rule that assigns to each z in S a complex number w. The number w is called the *value* of f at z and is denoted by $f(z)$; that is, $w = f(z)$. The set S is called the *domain of definition* of f.*

It is not always convenient to use different notation to distinguish between a given function and its values. For example, if f is defined on the half plane $\operatorname{Re} z > 0$ by means of the equation $w = 1/z$, it may also be referred to as the function $w = 1/z$, or simply the function $1/z$, where $\operatorname{Re} z > 0$.

It must be emphasized that both a domain of definition and a rule are needed in order for a function to be well defined. When the domain of definition is not mentioned, we agree that the largest possible set is to be taken. Thus, if we speak only of the function $1/z$, the domain of definition is understood to be the set of all nonzero points in the plane.

Suppose that $w = u + iv$ is the value of a function f at $z = x + iy$, so that

$$u + iv = f(x + iy).$$

*Although the domain of definition is often a domain as defined in Sec. 8, it need not be.

Each of the real numbers u and v depends on the real variables x and y, and it follows that $f(z)$ can be expressed in terms of a pair of real-valued functions of the real variables x and y:

$$f(z) = u(x, y) + iv(x, y). \tag{1}$$

If the polar coordinates r and θ, instead of x and y, are used, then

$$u + iv = f(re^{i\theta}),$$

where $w = u + iv$ and $z = re^{i\theta}$. In that case, we may write

$$f(z) = u(r, \theta) + iv(r, \theta). \tag{2}$$

EXAMPLE. If $f(z) = z^2$, then

$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + i2xy.$$

Hence

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

When polar coordinates are used,

$$f(re^{i\theta}) = (re^{i\theta})^2 = r^2 e^{i2\theta} = r^2 \cos 2\theta + ir^2 \sin 2\theta.$$

Consequently,

$$u(r, \theta) = r^2 \cos 2\theta \quad \text{and} \quad v(r, \theta) = r^2 \sin 2\theta.$$

If, in either equation (1) or (2), the function v is always zero, then the number $f(z)$ is always real. An example of such a *real-valued function of a complex variable* is

$$f(z) = |z|^2 = x^2 + y^2 + i0.$$

If n is zero or a positive integer and if $a_0, a_1, a_2, \ldots, a_n$ are complex constants, where $a_n \neq 0$, the function

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

is a *polynomial* of degree n. Note that the sum here has a finite number of terms and that the domain of definition is the entire z plane. Quotients $P(z)/Q(z)$ of polynomials are called *rational functions* and are defined at each point z where $Q(z) \neq 0$. Polynomials and rational functions constitute elementary, but important, classes of functions of a complex variable.

A generalization of the concept of function is a rule that assigns more than one value to a point z in the domain of definition. These *multiple-valued functions* occur in the theory of functions of a complex variable, just as they do in the case of real variables. When multiple-valued functions are studied, usually just one of the possible values assigned to each point is taken, in a systematic manner, and a (single-valued) function is constructed from the multiple-valued function. Suppose, for example, that z is any nonzero complex number $z = re^{i\theta}$. We know from Sec. 7

that $z^{1/2}$ has the two values $z^{1/2} = \pm\sqrt{r}e^{i\theta/2}$, where θ is the principal value ($-\pi < \theta \le \pi$) of arg z. But, if we choose only the positive value of $\pm\sqrt{r}$ and write

$$f(z) = \sqrt{r}e^{i\theta/2} \qquad (r > 0, -\pi < \theta < \pi),$$

we see that this (single-valued) function f is well defined on the indicated domain. Since zero is the only square root of zero, we also write $f(0) = 0$. The function f is, then, well defined on the domain consisting of the entire complex plane except for the ray $\theta = \pi$, which is the negative real axis.

10. MAPPINGS

Properties of a real-valued function of a real variable are often exhibited by the graph of the function. But when $w = f(z)$, where z and w are complex, no such convenient graphical representation of the function f is available because each of the numbers z and w is located in a plane rather than on a line. One can, however, display some information about the function by indicating pairs of corresponding points $z = (x, y)$ and $w = (u, v)$. To do this, it is generally simpler to draw the z and w planes separately.

When a function f is thought of in this way, it is often referred to as a *mapping*, or transformation. The *image* of a point z in the domain of definition S is the point $w = f(z)$, and the set of images of all points in a set T that is contained in S is called the image of T. The image of the entire domain of definition S is called the *range* of f. The *inverse image* of a point w is the set of all points z in the domain of definition of f that have w as their image. The inverse image of a point may contain just one point, many points, or none at all. The last case occurs, of course, when w is not in the range of f.

Terms such as *translation*, *rotation*, and *reflection* are used to convey dominant geometric characteristics of certain mappings. In such cases, it is sometimes convenient to consider the z and w planes to be the same. For example, the mapping

$$w = z + 1 = (x + 1) + iy,$$

where $z = x + iy$, can be thought of as a translation of each point z one unit to the right. Since $i = e^{i\pi/2}$, the mapping

$$w = iz = r\exp\left[i\left(\theta + \frac{\pi}{2}\right)\right],$$

where $z = re^{i\theta}$, rotates the radius vector for each nonzero point z through a right angle about the origin in the counterclockwise direction; and the mapping

$$w = \bar{z} = x - iy$$

transforms each point $z = x + iy$ into its reflection in the real axis.

More information is usually exhibited by sketching images of curves and regions than by simply indicating images of individual points. In the following examples, we illustrate this with the transformation $w = z^2$. Geometric interpretations of functions as mappings will be developed more extensively in Chap. 8.

EXAMPLE 1. According to the example in Sec. 9, the mapping $w = z^2$ can be thought of as the transformation

$$u = x^2 - y^2, \qquad v = 2xy \tag{1}$$

from the xy plane to the uv plane. This form of the mapping is especially useful in finding the images of certain hyperbolas.

It is easy to show, for instance, that each branch of a hyperbola $x^2 - y^2 = c_1$ ($c_1 > 0$) is mapped in a one to one manner onto the vertical line $u = c_1$. We start by noting from the first of equations (1) that $u = c_1$ when (x, y) is a point lying on either branch. When, in particular, it lies on the right-hand branch, the second of equations (1) tells us that $v = 2y\sqrt{y^2 + c_1}$. Thus the image of the right-hand branch can be expressed parametrically as

$$u = c_1, \qquad v = 2y\sqrt{y^2 + c_1} \qquad (-\infty < y < \infty);$$

and it is evident that the image of a point (x, y) on that branch moves upward along the entire line as (x, y) traces out the branch in the upward direction (Fig. 14). Likewise, since the pair of equations

$$u = c_1 \qquad v = -2y\sqrt{y^2 + c_1} \qquad (-\infty < y < \infty)$$

furnishes a parametric representation for the image of the left-hand branch of the hyperbola, the image of a point going *downward* along the entire left-hand branch is seen to move up the entire line $u = c_1$.

It is left as an exercise to show that each branch of a hyperbola $2xy = c_2$ ($c_2 > 0$) is transformed into the line $v = c_2$, as indicated in Fig. 14. The cases in which c_1 and c_2 are negative are also treated in the exercises.

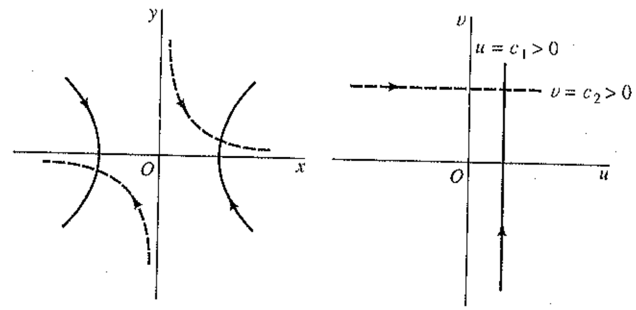

FIGURE 14
$w = z^2$.

EXAMPLE 2. Let us use equations (1) to show that the image of the vertical strip $0 \le x \le 1, y \ge 0$, shown in Fig. 15, is the closed semiparabolic region indicated there.

When $0 < x_1 < 1$, the point (x_1, y) moves up a vertical half line, labeled L_1 in Fig. 15, as y increases from $y = 0$. The image traced out in the uv plane has,

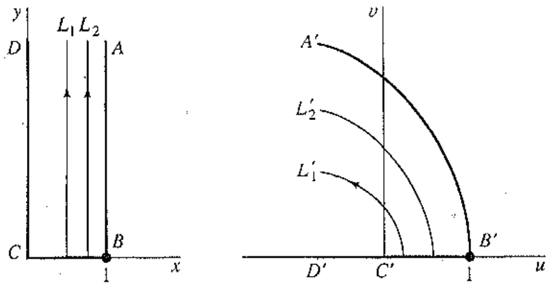

FIGURE 15
$w = z^2$.

according to equations (1), the parametric representation

$$(2) \qquad u = x_1^2 - y^2, \qquad v = 2x_1 y \qquad (0 \le y < \infty).$$

Using the second of these equations to substitute for y in the first one, we see that the image points (u, v) must lie on the parabola

$$(3) \qquad v^2 = -4x_1^2(u - x_1^2),$$

with vertex at $(x_1^2, 0)$ and focus at the origin. Since v increases with y from $v = 0$, according to the second of equations (2), we also see that as the point (x_1, y) moves up L_1 from the x axis, its image moves up the top half L_1' of the parabola from the u axis. Furthermore, when a number x_2 larger than x_1, but less than 1, is taken, the corresponding half line L_2 has an image L_2' that is a half parabola to the right of L_1', as indicated in Fig. 15. We note, in fact, that the image of the half line BA in that figure is the top half of the parabola $v^2 = -4(u - 1)$, labeled $B'A'$.

The image of the half line CD is found by observing from equations (1) that a typical point $(0, y)$, where $y \ge 0$, on CD is transformed into the point $(-y^2, 0)$ in the uv plane. So, as a point moves up from the origin along CD, its image moves left from the origin along the u axis. Evidently, then, as the vertical half lines in the xy plane move to the left, the half parabolas that are their images in the uv plane shrink down to become the half line $C'D'$.

It is now clear that the images of all the half lines between and including CD and BA fill up the closed semiparabolic region bounded by $A'B'C'D'$. Also, each point in that region is the image of only one point in the closed strip bounded by $ABCD$. Hence we may conclude that the semiparabolic region is the image of the strip and that there is a one to one correspondence between points in those closed regions. (Compare Fig. 3 in Appendix 2, where the strip has arbitrary width.)

EXAMPLE 3. We saw in the example in Sec. 9 that

$$w = z^2 = r^2 e^{i2\theta}$$

when $z = re^{i\theta}$. Hence if $w = \rho e^{i\phi}$, we have $\rho e^{i\phi} = r^2 e^{i2\theta}$; and the statement in italics near the end of Sec. 5 tells us that

$$\rho = r^2, \qquad \phi = 2\theta + 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Evidently, then, the image of any nonzero point z is found by squaring the modulus of z and doubling a value of $\arg z$.

Observe that points $z = r_0 e^{i\theta}$ on a circle $r = r_0$ are transformed into points $w = r_0^2 e^{i2\theta}$ on the circle $\rho = r_0^2$. As a point on the first circle moves counterclockwise from the positive real axis to the positive imaginary axis, its image on the second circle moves counterclockwise from the positive real axis to the negative real axis (see Fig. 16). So, as all possible positive values of r_0 are chosen, the corresponding arcs in the z and w planes fill out the first quadrant and the upper half plane, respectively. The transformation $w = z^2$ is, then, a one to one mapping of the first quadrant $r \ge 0, 0 \le \theta \le \pi/2$ in the z plane onto the upper half $\rho \ge 0, 0 \le \phi \le \pi$ of the w plane, as indicated in Fig. 16. The point $z = 0$ is, of course, mapped onto the point $w = 0$.

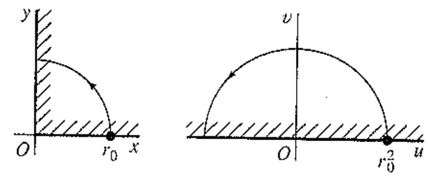

FIGURE 16
$w = z^2$.

The transformation $w = z^2$ also maps the upper half plane $r \ge 0, 0 \le \theta \le \pi$ onto the entire w plane. However, in this case, the transformation is not one to one since both the positive and negative real axes in the z plane are mapped onto the positive real axis in the w plane.

When n is a positive integer greater than 2, various mapping properties of the transformation $w = z^n$, or $\rho e^{i\phi} = r^n e^{in\theta}$, are similar to those of $w = z^2$. Such a transformation maps the entire z plane onto the entire w plane, where each nonzero point in the entire w plane is the image of n distinct points in the z plane. The circle $r = r_0$ is mapped onto the circle $\rho = r_0^n$; and the sector $r \le r_0, 0 \le \theta \le 2\pi/n$ is mapped onto the disk $\rho \le r_0^n$, but not in a one to one manner.

EXERCISES

1. For each of the functions below, describe the domain of definition that is understood:

(a) $f(z) = \dfrac{1}{z^2 + 1}$; (b) $f(z) = \operatorname{Arg}\left(\dfrac{1}{z}\right)$;

(c) $f(z) = \dfrac{z}{z + \bar{z}}$; (d) $f(z) = \dfrac{1}{1 - |z|^2}$.

Ans. (a) $z \ne \pm i$; (c) $\operatorname{Re} z \ne 0$.

2. Write the function $f(z) = z^3 + z + 1$ in the form $f(z) = u(x, y) + iv(x, y)$.

Ans. $(x^3 - 3xy^2 + x + 1) + i(3x^2 y - y^3 + y)$.

3. Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the fact (Sec. 3) that

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i}$$

to express $f(z)$ in terms of z, and simplify the result.

Ans. $\bar{z}^2 + 2iz$.