Resonances in Physics and Geometry

Maciej Zworski

Resonances are most readily associated with musical instruments or with the Tacoma bridge disaster. The latter is described in many physics and ODE books, and at the Ontario Science Center one can even find a model allowing one to find the destructive resonant frequency. The resonances I would like to write about are closely related but have their origins in quantum or electromagnetic scattering. To introduce them in a rough way, let us first recall the notion of *eigenvalues*. Eigenvalues of self-adjoint operators describe, among other things, the energies of bound states, states that exist forever if unperturbed. These do exist in real life; for instance, we can tell the composition of stars from our knowledge of atomic spectra. In most situations, however, states do not exist forever, and a more accurate model is given by a decaying state that oscillates at some rate. The decay might be caused by damping or by a possibility of escape to infinity. To describe these more realistic states, we use resonances. They have a very long tradition in mathematical physics, but they also appear naturally in pure mathematics. The last ten years brought many new ideas and new results into the subject. Old problems concerning the proximity of resonances to the real axis, their relation to quasi-modes, and their distribution for scattering by convex bodies have been solved. Upper bounds for counting functions of resonances have become well understood, and the new area of lower bounds has become active. New directions were opened by considering resonances in geometry, where in fact they make a very natural appearance. The purpose of this article is to motivate the study of resonances and to survey the recent advances.

The focus of the presentation is very personal, and many aspects of the enormous subject of resonances are bound to be neglected. Allusion to some other aspects of the subject is made at various places in the text, and some references, or pointers to lists of references, are included.

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Eigenvalues

To introduce resonances we first need to talk about eigenvalues. Perhaps the simplest case in which they arise is that of an oscillating string. Let $X = [0, \pi]$, and let $P = -\partial_x^2$ be the Laplacian on X acting on functions satisfying the Dirichlet boundary condition $u(0) = u(\pi) = 0$. The position at time t of the string fixed at the end points $u(t, x)$ is given by solutions of the wave equation

$$(1) \quad (-\partial_t^2 - P)u(t, x) = 0, \qquad u(t, 0) = u(t, \pi) = 0.$$

To solve this, the initial values of the position and velocity of the string are needed—$u(0, x)$, $\partial_t u(0, x)$. It is therefore advantageous to consider a system where instead of $u(t, x)$ we take

$$U(t, x) = \begin{pmatrix} u(t, x) \\ -i\partial_t u(t, x) \end{pmatrix}.$$

The wave equation (1) becomes

$$(2) \quad \begin{gathered} \frac{1}{i}\partial_t U = \mathcal{P}U, \qquad U(0, x) = \begin{pmatrix} u(0, x) \\ -i\partial_t u(0, x) \end{pmatrix}, \\ \mathcal{P} = \begin{pmatrix} 0 & 1 \\ P & 0 \end{pmatrix}, \qquad U(t, 0) = U(t, \pi) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{gathered}$$

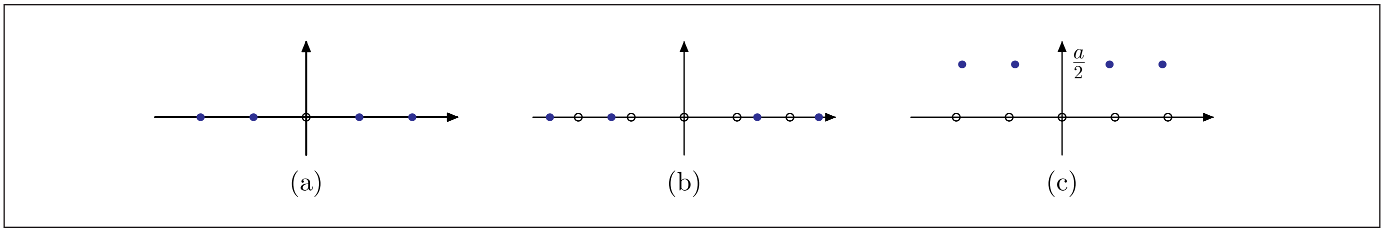

Figure 1. (a) The spectrum of $\mathcal{P}$ corresponding to $-\partial_x^2$. (b) The spectrum (•) of $\mathcal{P}$ corresponding to $-\partial_x^2 + V(x)$ (points ∘ show the spectrum in the unperturbed case $V = 0$). (c) The spectrum of $\mathcal{P}_a$ corresponding to the damped wave equation obtained by adding a term $-a\partial_t$, $a > 0$, constant, hence the constant imaginary part corresponding to the constant rate of damping.

At least formally the solution is simply given by putting $U(t,x) = \exp(it\mathcal{P})U(0,\cdot)(x)$. To actually write down the solution, we use the eigenvalues and the eigenfunctions of $\mathcal{P}$:

$$\mathcal{P}W = \lambda W, \qquad W|_{\partial X} = 0. \tag{3}$$

We say that λ is an *eigenvalue* of $\mathcal{P}$ or that $\lambda \in \sigma(\mathcal{P})$, where $\sigma(\mathcal{P})$ is the *spectrum* of $\mathcal{P}$. The solutions of (2) are known to be given by superpositions of $\exp(i\lambda t)W(x)$ where λ and W are as in (3). In the case of the string, we have $\lambda = n \in \mathbb{Z} \setminus \{0\}$, and $\sigma(\mathcal{P})$ is shown in Figure 1a. Nothing much changes when we consider a more general operator on $X : P = \partial_x^2 + V(x)$, $V(x) \geq 0$, $V \in C^\infty(X)$ (positivity and smoothness are assumed to avoid technical difficulties). The spectrum shifts due to the presence of V, and it is shown in Figure 1b. The eigenvalues stay real because P with the Dirichlet boundary condition is an unbounded self-adjoint operator on a suitable space. So is $\mathcal{P}$ when we take $\mathcal{H}$ to be the closure of $C_c^\infty(X) \times C_c^\infty(X)$ with respect to the inner product

$$\langle U, V\rangle_{\mathcal{H}} = \langle Pu_0, v_0\rangle + \langle u_1, v_1\rangle,$$
$$U = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \qquad V = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}.$$

Checking the Hermitian property $\langle \mathcal{P}W, V\rangle_{\mathcal{H}} = \langle W, \mathcal{P}V\rangle_{\mathcal{H}}$ is straightforward. Then we argue as we do in the case of matrices, and consequently $\lambda \in \mathbb{R}$. In these examples eigenvalues can also be characterized by saying that

$$\lambda \in \sigma(\mathcal{P}) \iff \lambda \text{ is a pole of } (\mathcal{P} - z)^{-1} : \mathcal{H} \to \mathcal{H}.$$

The *multiplicity* is defined as the rank of the residue.

We conclude this brief discussion by quoting perhaps the most famous general result about eigenvalues of operators on compact manifolds, the Weyl asymptotics:

$$\#\{\lambda \in \sigma(\mathcal{P}) : |\lambda| \leq r\} \sim 2\frac{\mathrm{vol}(\mathbb{B}^n)}{(2\pi)^n}\mathrm{vol}(X)r^n, \tag{4}$$

where $\mathbb{B}^n$ is the unit n-ball and in our case $n = 1$. The improvement of Weyl asymptotics and deeper understanding of the distribution of eigenvalues at high energies remain fascinating subjects closely related to such issues as "quantum chaos". In the last thirty years great breakthroughs were achieved by Lars Hörmander, J. J. Duistermaat, Victor Guillemin, and Victor Ivrii.

All this said, we remember that no string oscillates forever! This failure is of course due to some form of damping. Mathematically it can be introduced by adding a term $q(x)\partial_x$ to the operator P, as is done in standard ODE courses, or by adding a dissipative term $a(x)\partial_t$ in the wave equation. In each case the self-adjointness of P is destroyed. For the purposes of this discussion let us keep $P = -\partial_x^2 + V(x), V(x) \geq 0$, $X = [0, \pi]$, and the Dirichlet boundary conditions. Let us change the wave equation to:

$$(-\partial_t^2 + a(x)\partial_t - P)u(t,x), \qquad u|_{(0,\infty)\times\partial X} = 0, \quad t > 0, \tag{5}$$

where $a(x) \geq 0$. We can again conveniently rewrite this as a system

$$\frac{1}{i}\partial_t U = \mathcal{P}_a U, \qquad \mathcal{P}_a = \begin{pmatrix} 0 & 1 \\ P & ia(x) \end{pmatrix}. \tag{6}$$

Now, however, $\mathcal{P}_a$ is not self-adjoint on $\mathcal{H}$ unless $a(x) \equiv 0$. Nevertheless, the general picture remains the same: the eigenvalues and eigenfunctions of $\mathcal{P}_a$ are defined by:

$$\mathcal{P}_a W = \lambda W, \qquad W|_{\partial X} = 0,$$

and the solutions of (6) are given by superpositions of solutions of the form $U(t,x) = \exp(it\lambda)W(x)$. The rate of oscillations of U is given by the real part of λ, and the rate of decay by the imaginary part. In the simplest case $a(x) \equiv a > 0$

$$\lambda \in \sigma(\mathcal{P}_a) \iff \lambda(\lambda - ia) \in (\sigma(\mathcal{P}))^2.$$

The qualitative picture is very close to this one also in the variable $a(x)$ case (see Figure 1c), and in particular the Weyl asymptotics remain valid.

These types of non-self-adjoint perturbations corresponding to some dissipative effects have been successfully studied, most recently in spectacular work of Gilles Lebeau. To learn more, the reader may consult [6] and references given there. The damped wave equation is mentioned here to illustrate the following important point:

Solutions are described as superpositions of states corresponding to complex numbers, where the real part describes the rate of oscillations and the imaginary part the rate of exponential decay. They are eigenvalues of the non-self-adjoint operator $\mathcal{P}_a$ or, equivalently, poles of $(\mathcal{P}_a - z)^{-1} : \mathcal{H} \to \mathcal{H}$.

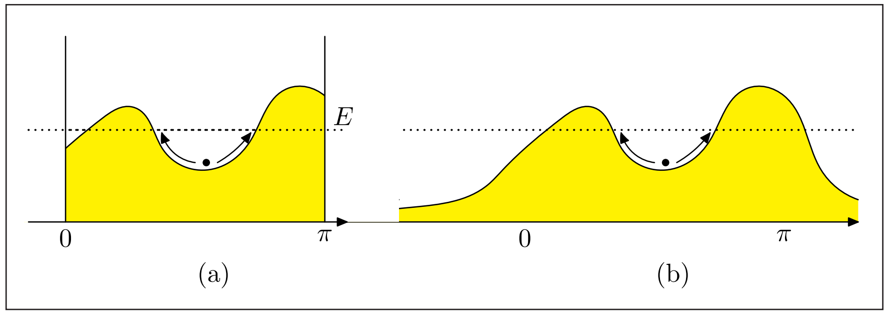

Figure 2. (a) A potential well on a finite interval. (b) A potential on $\mathbb{R}$: same classical picture, but a very different quantum picture.

The notion of resonance mentioned in the opening of this article is closely related to this. Suppose we solve $(\mathcal{P}_a - \mathrm{Re}\lambda)u = f$ where λ is an eigenvalue of $\mathcal{P}_a$ with $\mathrm{Im}\lambda$ small. This can be done, since $\mathrm{Re}\lambda$ is not an eigenvalue. However, with the right choice of the forcing term f, $(\mathcal{P}_a - \mathrm{Re}\lambda)^{-1}f$ can be enormous, as $(\mathcal{P}_a - z)^{-1}$ has a pole nearby. So, with the resonant frequency $\mathrm{Re}\lambda$ we hit a *resonance* which decays at a rate given by $\mathrm{Im}\lambda$.

Resonances

A more dramatic change of the situation occurs when, instead of damping, some escape to infinity is allowed. Mathematically that means that instead of considering $P = -\partial_x^2 + V(x)$ on $X = [0, \pi]$, we will consider it on $X = \mathbb{R}$. We will take $V(x) \geq 0$ and assume that $V \in C_c^\infty(\mathbb{R})$. See Figure 2 for a meaningful example. When we consider the corresponding wave equations, we cannot describe the solutions as sums of solutions coming from propagating eigenfunctions of P. In fact, we can easily see that P cannot have any square integrable eigenfunctions: if $Pu = Eu$, then we have

$$E \int |u(x)|^2 dx = \int (-\partial_x^2 + V(x))u(x)\overline{u(x)}\, dx$$
$$= \int (|\partial_x u(x)|^2 + V(x)|u(x)|^2)\, dx > 0,$$

and hence $E > 0$. But then, outside of the support of $V(x)$, $u(x)$ has to be a combination of $\exp(\pm i\sqrt{E}x)$. It is square integrable only when it is identically zero, and hence, by the uniqueness theorem for ODEs, $u(x)$ is itself identically zero.

Why should we expect any similarity with the picture involving eigenvalues? To answer this in a somewhat fuzzy way, we have to enter the fuzzy world of quantum mechanics. Classical mechanics describes a particle by specifying its position x and its momentum ξ. The motion is then governed by the classical equations of motions, such as conservation of energy:

$$(7) \qquad \textbf{Classical:} \quad \xi^2 + V(x) = E,$$

as we see in the rolling ball of Figure 2. Quantum mechanics describes a particle in a stationary state by a *wave function*, $u(x)$, and $|u(x)|^2 dx$ gives the probability density of finding the particle in a given region. In particular, $u(x)$ should be square integrable. The wave function should satisfy the quantized version of (7):

$$(8) \quad \textbf{Quantum:} \quad \left(\left(\frac{h}{i}\partial_x\right)^2 + V(x)\right)u(x) = Eu(x),$$

obtained by replacing ξ with $(h/i)\partial_x$. Here h is the Planck constant. When h is very small compared to everything else (so to speak!), we expect the quantum picture to be close to the classical picture: that is, we take the *semiclassical* view of the world. Previously, in P, the Planck constant was 1, but that of course means that everything else had to be appropriately large to justify the semiclassical approximation. When confined to $X = [0, \pi]$, the set of E's corresponding to eigenvalues of P is discrete: the energy levels are quantized. As h gets smaller, the levels get denser, and the classical situation of having every energy level allowed is approached.

When we are on $X = \mathbb{R}$ and the energy is at the level shown in Figure 2b, the classical picture remains unchanged from the case $X = [0, \pi]$. The rolling ball does not know about the world behind the "mountains". Thinking semiclassically, we expect a corresponding quantum state to exist. However, we have seen that the quantized equation (8) does not have any square integrable solutions. Hence, that state cannot be a usual wave function. The trouble is due to "tunneling" through the barrier created by the mountains. The particle bounces back and forth at some frequency, but it decays at some rate due to tunneling and the consequent escape to infinity. Here *tunneling* means interaction between the well and infinity and is measured by the rate of decay.[1] Any understanding of this has to be in a fundamental way dynamical: that is, it has to involve evolution in time. We moved from a spectral problem to a "scattering problem", with the potential as the "scatterer".

Although the situation discussed above might sound frighteningly fuzzy, it has a very elegant

[1] *The mathematical theory of tunneling is rather well understood, especially in the context of symmetry breaking for low-lying states. The article of Barry Simon [8] and the references in [3] describe this theory. There tunneling refers to interactions between different potential wells and to the way the exponentially small effects of those interactions are measured.*

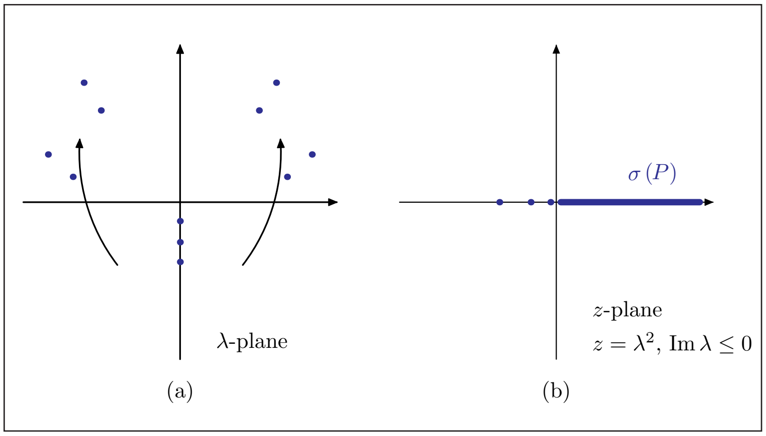

Figure 3. (a) Meromorphic continuation of the resolvents $\mathcal{R}(\lambda) = (\mathcal{P} - \lambda)^{-1}$ and $R(\lambda) = (P - \lambda^2)^{-1}$; the poles in the lower half-plane correspond to negative eigenvalues of P. (b) The spectrum of P: the z-plane, $z = \lambda^2$.

mathematical description valid in many interesting cases. Let us leave quantum mechanics for the moment and, rather than study the Schrödinger equation, go back to the wave equation. Thus we consider the operator $\mathcal{P}$ given by (2), now on $X = \mathbb{R}$. Since the eigenvalues appeared as poles of the resolvent, then to seek analogous objects it is natural to consider the *resolvent* of $\mathcal{P}$, $\mathcal{R}(z) = (\mathcal{P} - z)^{-1}$. As an operator on $\mathcal{H}$ it is bounded at each $z \in \mathbb{C} \setminus \mathbb{R}$ and makes no sense on $\mathbb{R}$, which is the continuous spectrum of $\mathcal{P}$. If the source space is made smaller and the target space bigger, much more than boundedness on $\mathbb{R}$ can be achieved:

> As an operator $\mathcal{R}(z) = (\mathcal{P} - z)^{-1}$ from $\mathcal{H}_c$ to $\mathcal{H}_{\text{loc}}$, $\mathcal{R}(z)$ continues meromorphically from $\operatorname{Im} z < 0$ to $\mathbb{C}$.

Here by $\mathcal{H}_c$ we mean the elements of $\mathcal{H}$ that are zero outside some compact set, and by $\mathcal{H}_{\text{loc}}$, functions that are locally in $\mathcal{H}$. The meromorphic continuation has poles, and by the analogy with the characterization of eigenvalues, we would like to consider those poles as the replacement of discrete spectral data for problems on noncompact domains (in this discussion $\mathbb{R}$). We denote the set of poles of $\mathcal{R}(z)$ by $\operatorname{Res}(P)$, where P is the original operator from which $\mathcal{P}$ was constructed (here it is $P = -\partial_x^2 + V(x)$, $V \in C_c^\infty$, $V \geq 0$). See Figure 3.

That the elements of $\operatorname{Res}(P)$ are what we are looking for is shown by their appearance in the solutions of the wave equation: if

$$(-\partial_t^2 - P)u = 0, \qquad u|_{t=0}, \qquad \partial_t u|_{t=0} \in C_c^\infty,$$

then for any fixed compact set $K \subset X$,

$$u(t,x) \sim \sum_{\lambda \in \operatorname{Res}(P)} e^{it\lambda} w_\lambda(x), \qquad x \in K, \qquad t \longrightarrow \infty. \tag{9}$$

Hence, in a fuzzy way we have a similarity with the case of eigenvalues, in which the solutions are given by superpositions of propagated eigenfunctions. We see that in (9)

$\operatorname{Re}\lambda$ corresponds to the rate of oscillations,

$\operatorname{Im}\lambda$ corresponds to the rate of decay.

The assumption that $V(x) \geq 0$ can be eliminated quite easily.The only complication comes from the possible presence of (honest) negative eigenvalues of P which then produce imaginary eigenvalues of $\mathcal{P}$ (the definition of $\mathcal{H}$ has to be modified, and $\mathcal{P}$ is no longer self-adjoint). The meromorphic continuation can also be described purely in terms of the resolvent of P. We consider

$$R(\lambda) = (P - \lambda^2)^{-1} \; : \; L_c^2 \longrightarrow L_{\text{loc}}^2,$$

so that it coincides with the resolvent bounded on L^2 for $\operatorname{Im}\lambda < 0$ (except at poles corresponding to $\lambda^2 \in \sigma(P)$). It then continues meromorphically to $\mathbb{C}$ as shown on Figure 3a. In Figure 3b we exhibit also the "honest" spectrum of P in the z-plane, $z = \lambda^2$.

The definition of resonances as poles of the meromorphically continued resolvent remains valid, with various modifications, in many situations. It is valid in the exact same way as described above for compactly supported perturbations of the Laplacian in $\mathbb{R}^n$, n odd. It is valid also for n even, but then we have to continue $\mathcal{R}(z)$ to the Riemann surface for $\log z$ rather than $\mathbb{C}$. Perhaps the most interesting class of compactly supported perturbations comes from considering scattering by compact obstacles. In that case the wave-equation point of view towards resonances was introduced in the seminal work of Peter Lax and Ralph Phillips [5] in the 1960s. Many general results can now be formulated in an abstract notion of "black box scattering", in which one does not have to consider specific aspects of the perturbation. That formalism was introduced by Johannes Sjöstrand and the author and has now been extended to include classes of long-range perturbations. References and a review of results are provided in [9] and [11].

As in the case of eigenvalues, it is natural to study *large-energy asymptotics*, that is, to look for analogues of the Weyl law (4). Since the resonances are distributed in the complex plane, it is not clear what is the most appropriate way of counting them; the issues involved in that will hopefully become clearer below. The study of global upper bounds on the number of resonances—that is, of counting them in discs $\{z \in \mathbb{C} \; : \; |z| \leq r\}$—was initiated by Richard Melrose, who proved the sharp polynomial bound $\mathcal{O}(r^n)$ in the case of obstacle scattering in $\mathbb{R}^n$, n odd. In the last ten years, that work inspired many results on global and local, upper and lower bounds on the number of resonances. The main results were obtained by Johannes Sjöstrand, Georgi Vodev, Laurent Guillopé, and the author. See [7], [2], [9], and [11] for discussion and references.

Despite significant advances, the state of knowledge is hardly satisfactory. In standard Euclidean scattering, asymptotics for the counting function of resonances are known only in dimension one and in some specific radial cases such as scattering by the sphere (see Figure 8).

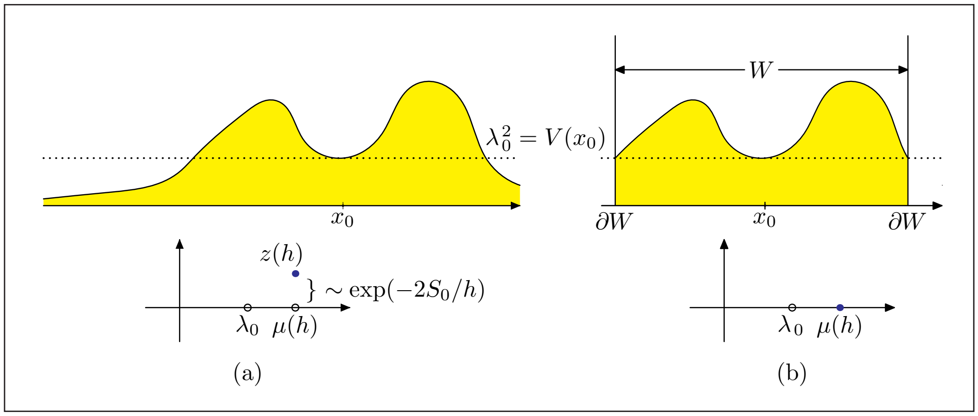

Figure 4. (a) A potential well and a resonance generated by the eigenvalue of a reference Dirichlet boundary problem. (b) A reference problem and its eigenvalue close to the ground state of the well.

Two Examples

So far we have discussed in some detail the simplest but already nontrivial example of potential scattering on the line. Let us now give two more involved examples that have long traditions in mathematical physics and in geometric analysis.

The first example is a semiclassical potential well in $\mathbb{R}^n$. Let $V(x)$ be an analytic potential which looks like the potential shown in Figure 4; we skip here a detailed description of the technical assumptions on V. Roughly speaking, it is supposed to have a nondegenerate local minimum at x_0, and that point should be the only place where a particle moving according to the classical equation, $|\xi|^2 + V(x) = E$, could be trapped. We are then interested in resonances of the semiclassical Schrödinger operator

$$P(h) = -h^2\Delta + V(x), \qquad \Delta = \sum_{i=1}^{n} \partial_{x_i}^2,$$

near λ_0 with $\lambda_0^2 = V(x_0)$, that is, near the energy corresponding to the bottom of the well. Under further assumptions a typical resonance is given by

$$z(h) = \mu(h) + \mathcal{O}(1)\exp(-2S_0/h),$$
$$\operatorname{Im} z(h) = r(h)\exp(-2S_0/h), \qquad r(h) \sim h^{C_1},$$

where $\mu(h) \to \lambda_0$ is an eigenvalue of the operator $P(h)$ restricted to W, shown in Figure 4b, with zero boundary conditions at ∂W and S_0 is the "Agmon width" of the barrier. The latter measures the rate of tunneling through a potential barrier and is closely related to weights appearing in "Carleman estimates" that are used to show unique continuation of solutions to PDEs. In fact, unique continuation is due to tunneling, or depending on one's perspective, tunneling is a quantitative form of unique continuation: a solution cannot be completely localized to a compact set.

In this example the resonances can be defined through meromorphic continuation of the resolvent. Typically, however, we can continue only to a neighborhood of the real axis. That is done using the method of "complex scaling" introduced by Jacques Aguilar, Jean-Michel Combes, and Eric Balslev in the early 1970s: through a deformation of the operator, the resonances become eigenvalues of a non-self-adjoint problem.[2] This method was developed by many authors, and it was raised to the level of high art through the use of analytic microlocal analysis by Bernard Helffer and Johannes Sjöstrand [3]. The example described above has been extensively studied, and [3], [4] can be consulted for the history and results.

The second example is pure "pure mathematics". Let X be the modular surface obtained as the quotient of the hyperbolic upper half-plane by the modular group: $X = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$, where $\mathbb{H}^2 = \mathrm{SL}(2,\mathbb{R})/SO(2)$. The famous fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ is shown in Figure 5. The quotient, X, is a noncompact surface with a cusp at infinity and two conic singularities. It inherits the hyperbolic metric from $\mathbb{H}^2$, and consequently we have a natural Laplace operator, Δ_X. The resolvent $(-\Delta_X - \zeta)^{-1}$ turns out to be bounded on $\mathbb{C} \setminus (\{0\} \cup [1/4, \infty))$, and we then consider

$$R(\lambda) = (P - \lambda^2)^{-1}, \qquad P = -\Delta_X - \frac{1}{4},$$
$$\operatorname{Im}\lambda < 0, \qquad \lambda \neq -\frac{i}{2}.$$

As before, $R(\lambda)$ continues meromorphically to $\mathbb{C}$, and its poles, which are eigenvalues and resonances, have a classical interpretation; most of the poles lie exactly on $\mathbb{R}$ and are honest L^2 eigenvalues embedded in the continuous spectrum. See Figure 5. They correspond to "cusp forms" of analytic number theory. The remaining poles are honest resonances (that is, they have nonzero imaginary parts), and they are given by the solutions of

[2] *The operator-theoretical approach to complex scaling is comprehensively reviewed by Peter Hislop and I. Michael Sigal, themselves important contributors to the subject, in [4]. For a brief exposition of the geometrical approach in the spirit of Helffer-Sjöstrand, the reader may consult Sec. 5b of [11].*

$$\zeta(1+2i\lambda)=0, \qquad \lambda \notin i\mathbb{R},$$

where ζ is the Riemann zeta function, $\zeta(s)=\sum_{n=1}^{\infty} n^{-s}$. The reason for this comes from another characterization of resonances: they are the poles of the meromorphic continuation of the "scattering matrix". In the case of modular surface, the scattering matrix can be expressed explicitly in terms of the zeta function.

The scattering-theoretical interpretation of the theory of automorphic forms was initiated by Ludvig Faddeev and continued by Peter Lax and Ralph Phillips. Their insight has been useful in more complicated situations, as shown by the work of Werner Müller and then of Lizhen Ji on the trace class conjecture. It also inspired Yves Colin de Verdière to show that a generic C_c^∞ perturbation of the metric on X will destroy all embedded eigenvalues and turn them into resonances. That motivated the work of Ralph Phillips and Peter Sarnak on dissolving embedding eigenvalues when an arithmetic surface is deformed to another surface of constant curvature (which cannot be done for our X, but is possible in other situations). It is now believed that most of the eigenvalues become resonances under such deformations.

Classical Dynamical Structure and Distribution of Resonances

Our quantum mechanical motivation for introducing resonances rested on the semiclassical assumption that the existence of classical states should imply the existence of corresponding quantum states. That classical-quantum correspondence is very subtle and is one of the central issues of spectral and scattering theories. In the crudest form it already manifests itself in the Weyl law (4). If we are interested in eigenvalues of an operator $P=-\Delta+V(x)$ that is the quantization of $p=|\xi|^2+V(x)$, then, very roughly speaking,

$$\#\{\lambda^2 \in \sigma(P) : |\lambda| \le r\} \sim \mathrm{vol}\{(x,\xi)\in T^*X : p(x,\xi)\le r^2\},$$

as we have seen in (4) for Schrödinger operators on compact manifolds.

In the case of resonances a new wealth of phenomena is seen. In addition to situations in which resonances behave as perturbed eigenvalues (see Figure 4), there are many situations in which there is no eigenvalue analogue. In particular, the dynamical structure of the scatterer may manifest itself directly in the counting function for resonances. Since resonances are supposed to correspond to states that eventually escape and since their lifespan (which is the inverse of the rate of decay) should be related to the inverse of their imaginary parts, a very heuristic and in fact not quite correct analogue of the Weyl law is

$$(10)\quad \#\left\{\lambda \in \mathrm{Res}\,(P) : r \le |\lambda| \le 2r, 0 \le \mathrm{Im}\,\lambda \le \frac{r}{T}\right\}$$

$$\sim \mathrm{vol}\left\{\begin{array}{l}(x,\xi)\in T^*X \text{ such that } r^2 \le p(x,\xi)\le (2r)^2 \\ \text{and a trajectory through } (x,\xi) \text{ of the flow of} \\ \text{the Hamiltonian } p \text{ spends time } T \text{ near the} \\ \text{perturbation}\end{array}\right\},$$

where r and T are supposed to be large and where T can depend on r. Since this is appallingly vague, let us mention that a better formulation is obtained in the semiclassical, small h regime with the use of the "escape function" of Helffer-Sjöstrand, which is also known as the "Lyapunoff function". In fact, Johannes Sjöstrand obtained related upper bounds for analytic semiclassical operators in $\mathbb{R}^n$, and that discovery was followed by some similar bounds for hyperbolic surfaces that are illustrated in Figure 6b. References can be found in [9] and [2].

From the dynamical interpretation of the imaginary parts as the rates of decay of the corresponding states, we see that only resonances near the real axis are truly meaningful. How "near" is of course dependent on the problem, but as indicated already in (10), small conic neighborhoods are really the farthest we should look for detailed information. Knowing what happens farther away can, however, be important; in particular, it can be useful to know that there are not too many resonances far from the axis (so that they do not

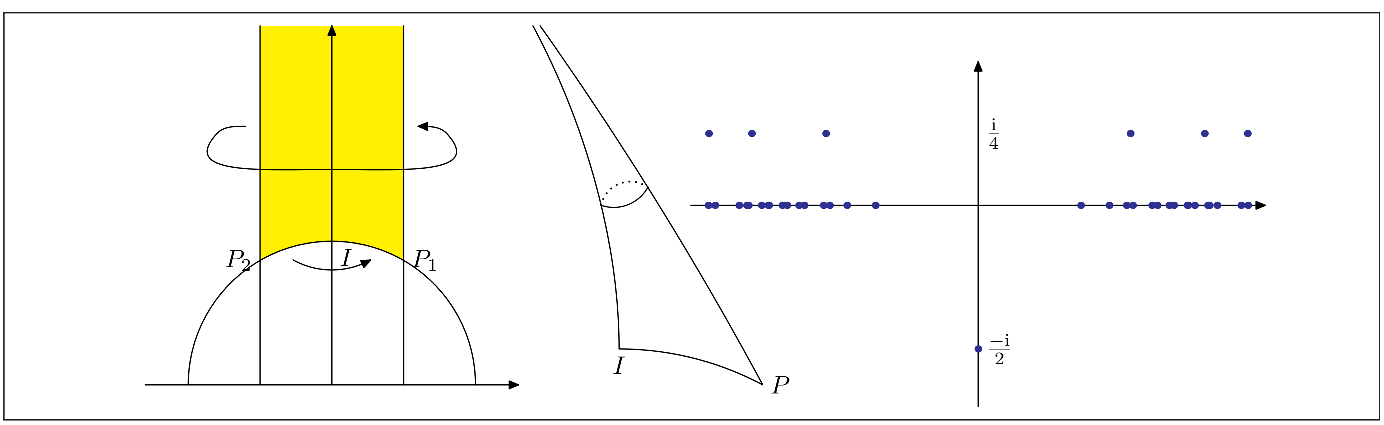

Figure 5. The fundamental domain of the modular group and the resonances of the modular surface.

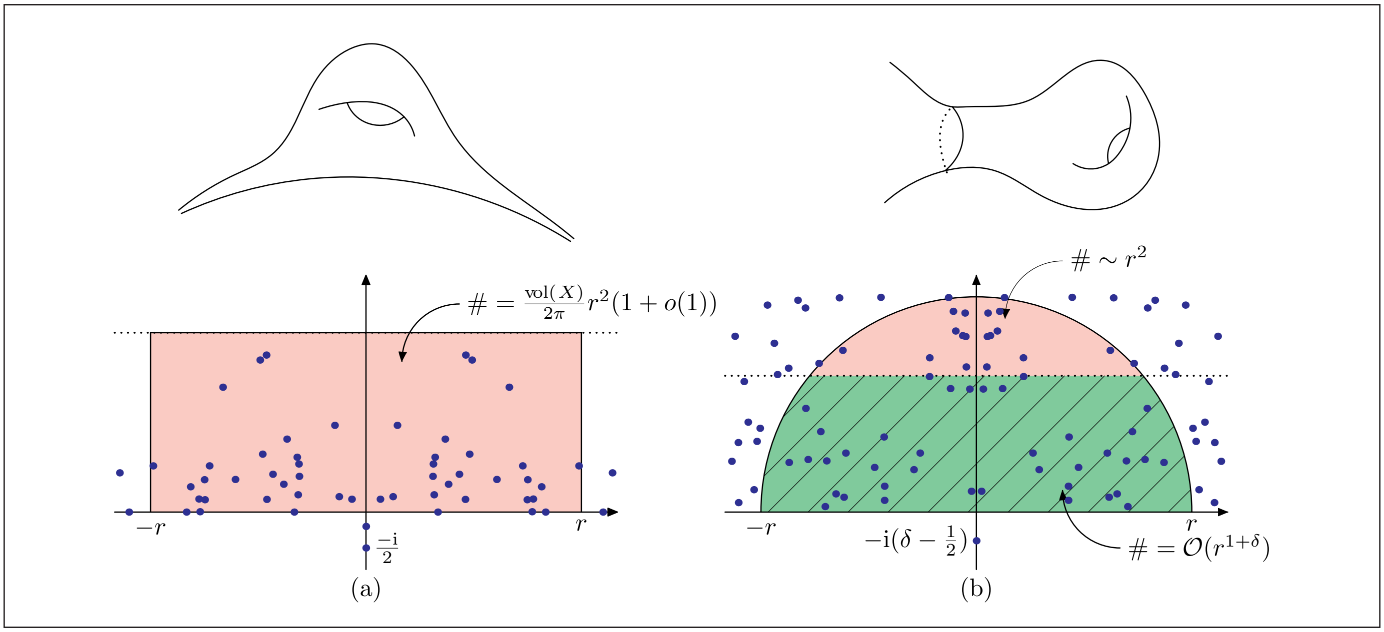

Figure 6. (a) Resonances for a finite volume hyperbolic surface: they are confined to a horizontal strip and satisfy the usual Weyl law. (b) Resonances for an infinite volume surface with no cusps: they are scattered all over the upper half-plane; the counting function in a disc of radius r is bounded from above and below by multiples of r^2; when we count only in a strip the number of resonances is bounded by a multiple of $r^{1+\delta}$, where δ is a number less than one with a dynamical intepretation.

affect the behavior on the real axis). Also, for low-energy resonances, a restriction to conic neighborhoods is irrelevant.

In the opposite direction, we can ask how near the resonances can approach the real axis. In situations where infinity is "small" (see Figures 5 and 6a) there are no restrictions, and resonances can mix with embedded eigenvalues. When infinity is "large", as in Euclidean obstacle problems or as in Figure 6b, the resonances have to be separated from the real axis. A fundamental result obtained very recently by Nicolas Burq [1] says that for $|\mathrm{Re}\lambda| > C_0$, the resonances have to satisfy $\mathrm{Im}\lambda > \exp(-C_1|\mathrm{Re}\lambda|)$ for some constants C_0 and C_1. The proof is based on "Carleman estimates", that is, on quantitative understanding of tunneling (see the discussion of the example shown in Figure 4 above). One could say that not only can states not live forever, but they cannot live for an arbitrarily long time. Of course, a lifespan that is exponential in energy is more than any one of us can hope for!

Figure 6 illustrates some of the issues discussed above in the case of two-dimensional hyperbolic surfaces, $X = \Gamma\backslash\mathbb{H}^2$. In Figure 6a we look at the finite-volume case where the resonances are confined to a strip. Most of them lie very close to the real axis, and they satisfy the usual Weyl law (4), with $n = 2$. This is now classical and was established by Atle Selberg in the 1950s. In Figure 6b we take X to be an infinite volume surface with no cusps. The resonances are now scattered through the entire half-plane, and we have global bounds

$$\begin{aligned} r^2/C - C &\leq \{\lambda \in \mathrm{Res}(-\Delta_X - 1/4) : |\lambda| \leq r\} \\ &\leq Cr^2 + C, \end{aligned} \tag{11}$$

which are a special case of bounds obtained in [2] for a more general class of surfaces. It is the existence of the lower bound that makes this particularly interesting, as these are rarely known when infinity is "large". In strips parallel to the real axis the number of resonances is much smaller. The author has recently shown that in a strip the number of resonances with $|\lambda| \leq r$ is bounded by $Cr^{1+\delta}$, where δ is the dimension of the "limit set" of the Γ. This is related to the dynamical interpretation of δ given by Dennis Sullivan and is an indication of the validity of (a modified version of) (10). We should also note that in the case of exact quotients, such as shown in Figure 6b, the resonances have an interesting reinterpretation as zeros of the meromorphic continuation of the dynamical zeta function, and that has been recently studied by S. J. Patterson and Peter Perry.

We conclude with the description of two important results in obstacle scattering. In both of them, resonances behave in ways that do not have eigenvalue analogues.

If we consider a scatterer consisting of two strictly convex bodies, then the dynamical structure is very simple. The only trapped orbit coming from reflections is the hyperbolic trajectory obtained by bouncing along the ray, minimizing the distance between the two bodies. See Figure 7. It is hyperbolic, since any small perturbation will result in fast escape to infinity. Mitsuru Ikawa has shown that this closed hyperbolic orbit generates

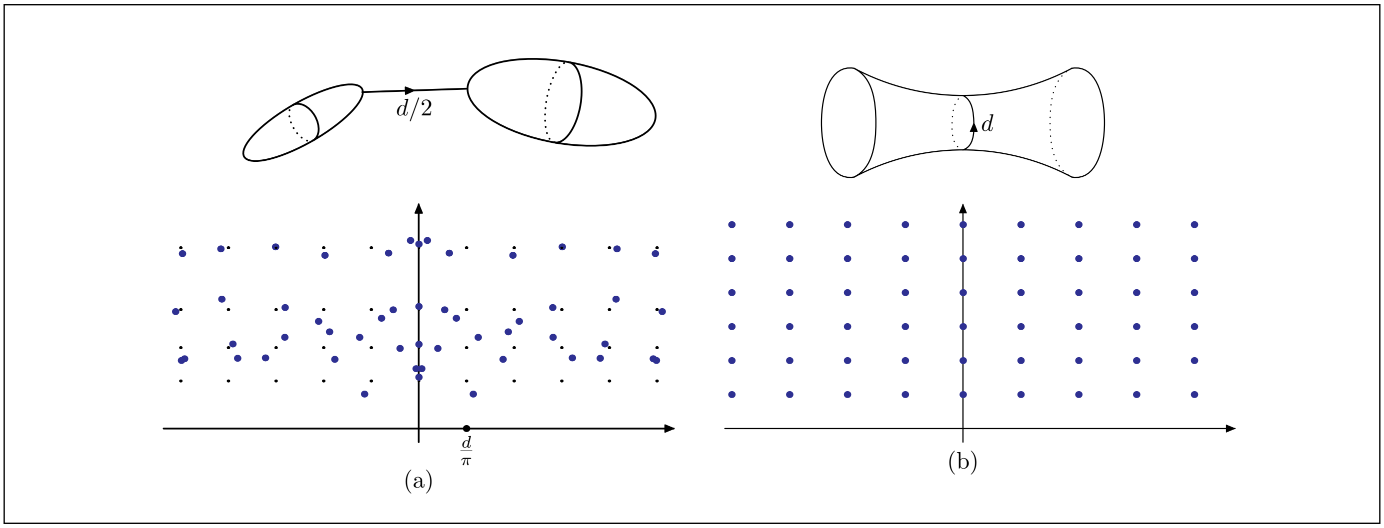

Figure 7. (a) Resonances associated to two strictly convex bodies: in every fixed strip, the resonances become closer to points on the lattice as the real part increases. (b) Resonances for a hyperbolic cylinder: all resonances lie exactly on a lattice. The underlying dynamical structure, exactly one hyperbolic closed orbit, is the same in the two examples.

a string of resonances parallel to the real axis with no other resonances below them. That was later extended by Christian Gérard, who described all resonances in a strip. The great significance of this result lies in the fact that it described quantum objects (a lattice of resonances) associated to a single hyperbolic orbit. Later, a simpler and exact model for this classical-quantum correspondence, based on the hyperbolic cylinder, was given by Charles Epstein and Laurent Guillopé (see Figure 7b), while a physically more relevant semiclassical version was developed by Gérard-Sjöstrand. The point is that the presence of a single hyperbolic periodic orbit generates a lattice of resonances no matter what situation we are in.

Another significance of Ikawa's result lies in the fact that it disproves the *Lax-Phillips conjecture,*[3] which stated that in case of any classical trapping (such as existence of one closed orbit) there should exist a sequence of resonances converging to the real axis. The conjecture was consequently modified; and, in particular, when one assumed the existence of an *elliptic* closed orbit, it was proved by Plamen Stefanov and Georgi Vodev. A quantitative version was then found by Siu-Hung Tang and the author, and in 1998 that was improved further by Stefanov; see [10] and references given there. Ideally, we should finally reach a statement resembling the modified Weyl law (10). At the moment, these results are based on the understanding of the relation between resonances and "quasimodes", which are approximate eigenvalues.

Let us also mention that the first result on the existence of resonances associated to two convex bodies was obtained by Claude Bardos, Jean-Claude Guillot, and James Ralston in 1982 using their "Poisson formula" for resonances. That formula, improved and extended to other settings, has since become one of the most powerful tools for proving existence of resonances; further information may be found in [9] and references given there.

The second result, recently obtained by Johannes Sjöstrand and the author, describes resonances for scattering by strictly convex obstacles. This is illustrated in Figure 8: the curves are of the form $\mathrm{Im}\lambda = K_j^{\pm}|\mathrm{Re}\lambda|^{1/3} \pm C$, and the pinching condition that guarantees the existence of j bands is

$$\frac{\max_{S\partial\mathcal{O}} Q}{\min_{S\partial\mathcal{O}} Q} < \gamma(j), \qquad \gamma(1) = 2.31186\cdots,$$
$$\gamma(j) \to 1, \qquad j \to \infty,$$

where Q is the second fundamental form and $S\partial\mathcal{O}$ is the unit tangent bundle of $\partial\mathcal{O}$.

The association of cubic curves with resonances of convex bodies has a long tradition originating in diffraction theory. Our result has been preceded by many related results in applied mathematics and in microlocal analysis, in particular by Lebeau's work on propagation of Gevrey singularities. Heuristically, the resonances for convex bodies are created by waves creeping along the geodesics on the boundary and losing energy at a rate depending on the curvature. Consequently, the precise distribution depends in a subtle way on the dynamics of the geodesic flow of the surface and its relation to the curvature. However, those subtle effects are mostly present in the distribution

[3] *More like a suspicion than a conjecture, to be quite fair (see the end of Sec. V.3 of [5]). A statement made more clearly as a conjecture concerned propagation of singularities for boundary value problems (see (A) and (B) in Sec. V.3 of [5]), and it was proved in the works of Andersson, Melrose, Sjöstrand, and Taylor in the late 1970s. It was also motivated by the study of resonances, as it implied that logarithmic neighborhoods of $\mathbb{R}$ are free of resonances for nontrapping obstacles.*

of imaginary parts of the resonances. The crude heuristic picture suggests that as far as the real parts are concerned, the distribution should be governed by the same rules as those for eigenvalues of the surface. Our result justifies this claim. The pinching condition for the curvature needs to be imposed to eliminate interference between different bands.

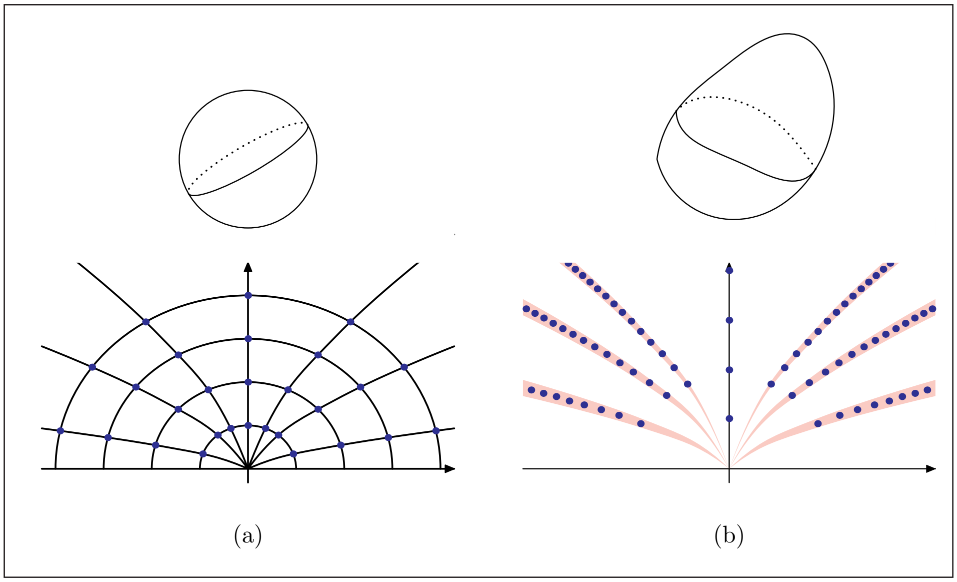

Figure 8. (a) Resonances for the sphere; the resonances on the lth ring have multiplicity $2l+1$. (b) Resonances for a convex body $\mathcal{O}$ satisfying the pinched curvature assumption. The resonances in each band (on a cubic curve in case of the sphere) satisfy the same Weyl law as the eigenvalues of the Laplacian on $\partial\mathcal{O}$.

Open Problems

In this article introduction of technical terminology was rather systematically avoided, and therefore a precise formulation of open problems is a somewhat difficult task. The existence of many problems should, however, already be clear. Continuing in the same spirit of vagueness, we can make them a little bit more precise. Some of them are present already in most basic settings, and their solutions may be elementary. Other problems involve extending the existing knowledge to more complicated situations. We may ask for:

- Global lower bounds of the form (11) on the number of resonances; at the moment very few unconditional bounds are known. To put this in perspective, until the work of Plamen Stefanov in 1998, the sphere was the only obstacle for which the optimal lower bound was known; it is still unknown for an arbitrary obstacle.
- Local lower bounds related to finer aspects of the dynamical structure: a modification of (10). At present only the "one hyperbolic orbit" examples and their extensions provide lower bounds corresponding to finer upper bounds.
- The modified Lax-Phillips conjecture of Ikawa stating that there should be a strip with infinitely many resonances for the Dirichlet Laplacian on the exterior of several convex bodies. Ikawa proved this for the Neumann Laplacian.
- Understanding of meromorphic continuation of the resolvent on manifolds. In addition to manifolds with simple structure at infinity (some of which were discussed above), the best understood general class consists of "conformally compact manifolds" studied by Rafe Mazzeo and Richard Melrose. They generalize surfaces of the type shown in Figure 6. Even there, the method of complex scaling is not properly understood, nor are the upper bounds. For other natural classes of manifolds the situation is even less clear.
- Generalization of existing methods and results (upper bounds, Poisson formulæ) to situations where there are singularities at infinity. The natural directions are provided by higher-rank symmetric spaces and by the quantum N-body problem.

The Riemann hypothesis could also have been added, since it can be formulated in terms of resonances (see Figure 5). It should, however, be remembered that in their book on automorphic scattering, Lax and Phillips had a chapter titled "How Not to Prove the Riemann Hypothesis". So it is better to leave it out.

Acknowledgments

I would like to dedicate this article to the memory of Professor Ralph Phillips, 1913–1998.

The article originated from a discussion after a talk on resonances for convex bodies I gave at a seminar in Berlin in May 1998. By chance, two editors of these *Notices* were in the audience—Beth Ruskai and Victor Guillemin. I learned most of what I know about eigenvalues from Victor, and to be mischievous, I made a statement that "eigenvalues are yet another expression of humanity's narcissistic desire for immortality." That led to a lively discussion in which it was suggested that I write an article on "Narcissism and self-adjointness". What I could write is, of course, much more prosaic.

I would like to thank Tanya Christiansen, Laurent Guillopé, Anders Holst, John Lott, Andras Vasy, and in particular Jared Wunsch and Steve

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