2012 F Math 7386 Problem Set 5

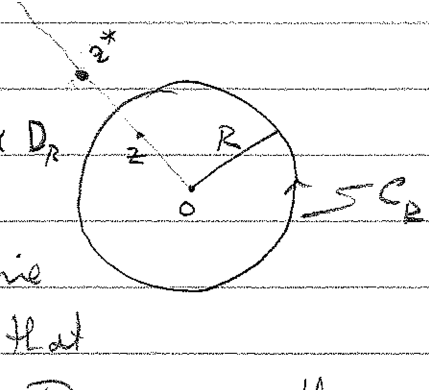
14. Denote a general point in $\mathbb{R}^n$ by $x = (x', x_n)$ $[x' = (x_1, \dots, x_{n-1})]$. Construct a Green function $G(x,y)$ for the "upper half space" $\{x = (x', x_n) : x_n > 0\}$ for the Laplace operator:

$$\begin{cases} \Delta_y G(x,y) = \delta(x-y), & x_n, y_n > 0 \\ G(x,y) = 0, & x_n > 0,\ y_n = 0 \end{cases}$$

by considering a combination of $N((x', x_n), y)$ and $N((x', -x_n), y)$ (a reflection about $\{(x', x_n) : x_n = 0\}$).

Using this G, derive the Poisson integral giving $u(x)$ $(x_n > 0)$ in terms of $\{u(y) : y_n = 0\}$, where u is harmonic in the upper half space and C' on the closed upper half space.

Do the same for the Helmholtz equation $(\Delta + k^2)u = 0$ in $\mathbb{R}^3_+$.



15. Derive the Poisson kernel for the disk D_R of radius R in the plane by methods of complex variables. For $z \in D_R$, define a point $z^* \in \bar{D}_R^{\,c}$ along the same ray as that of z such that $|z|/R = R/|z^*|$. Then use the formula $\frac{1}{2\pi i} \oint_{C_R} \frac{f(\zeta)}{\zeta - w}$, where $w = z$ and then where $w = z^*$, with f analytic in D_R and C' in $\bar{D}_R$.

16. Prove the <u>maximum principle</u> for harmonic functions using the mean-value theorem: If a harmonic function $u(x)$ in an open connected set $\Omega \subset \mathbb{R}^n$ admits a local maximum at a point $x \in \Omega$, then u is constant in Ω.

17. A function v in an open set $\Omega \subset \mathbb{R}^n$ is "subharmonic" if, for each open set B s.th. $\overline{B} \subset \Omega$ and each function u in $\overline{B}$, harmonic in B and s.th. that $v(y) = u(y)$ $\forall$ $y \in \partial B$ (the boundary of B), we have $v(x) \le u(x)$ $\forall$ $x \in B$.
Prove that, if v is subharmonic in a region $B(x_0, r_2) \setminus \overline{B(x_0, r_1)}$ between two spheres in $\mathbb{R}^n$, then the mean value

$$M_v(x_0, r) = \frac{1}{\omega_n} \int_{B(0,1)} u(x_0 + r\xi)\, d\xi$$

is convex with respect to the variable $\frac{1}{r^{n-2}}$ $(n \geq 3)$ or $\ln r$ $(n = 2)$ or r $(n = 1)$ in the interval (r_1, r_2).