Topology Review Problems for the Comprehensive Examination

June 20, 2007

Topics Covered: The comprehensive examination in topology will be based on basic concepts and theorems of general topology, which are typically covered in the first semester topology course Math 7510. The topics include the notions of topological and metric spaces, open and closed sets, bases, sequences and perhaps some more general notion of convergence (nets or filters), subspaces, products, continuous functions, quotient maps and quotient spaces, homeomorphisms, connectedness, compactness, local properties, separation properties, and complete metric spaces. A good reference for this material is *Topology* by James Munkres, Chapters 2, 3, 4, 5.1, 7.1, 7.3, and 7.7. There are also numerous other elementary topology texts that treat these subjects.

Definition 1. Let X be a set. A collection $\mathcal{B}$ of subsets of X is a *basis for a topology* or simply a *basis*, if

1. for B_1, $B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$;

2. for all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.

A subset U of X is *open* if for each $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Problem 1. (i) Show that the collection of open sets defined from the basis $\mathcal{B}$ in Definition 1 forms a topology $\mathcal{T}$, that is, satisfies
(a) The null set $\emptyset$ and the set X are open sets.

(b) The union of any collection of open sets is again open.
(c) The intersection of any finite number of open sets is open.
In this case the topology $\mathcal{T}$ is said to be *generated* by the basis $\mathcal{B}$.

(ii) Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{B} \subseteq \mathcal{T}$ satisfy the condition that if $x \in U \in \mathcal{T}$, then there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Show that $\mathcal{B}$ is a basis in the sense of Definition 1 and that $\mathcal{B}$ generates $\mathcal{T}$.

Problem 2. (i) If d is a metric on a set X, $\epsilon > 0$ is a real number, and $p \in X$, then the *open ϵ-ball* with center p is defined by

$$N_\epsilon(p) := \{x \in X | d(x, p) < \epsilon\}.$$

Show that the collection $\{N_\epsilon(p) | p \in X,\ \epsilon > 0\}$ forms a basis for a topology, called the *metric topology*. (ii) Prove or give a counterexample: For each $x \in X$ and $\epsilon > 0$, $\overline{N_\epsilon(x)} = \{y \in X | d(x, y) \leq \epsilon\}$. (iii) If D is a dense subspace of a metric space (X, d), show that the collection of all open balls

$$N_{\frac{1}{n}}(x),\ x \in D,\ n \in \mathbb{N},$$

forms a basis that generates the metric topology on X. (Hint: Use (ii) of Problem 1.)

Problem 3. Let X be a metric space. Prove the following are equivalent.
(1) X is separable (has a countable dense subset).
(2) X is second countable (has a countable basis for the topology).
(3) X is Lindelöf (each open cover has a countable subcover).

Definition 2. A topological space X is *regular* if given A closed and $x \notin A$, there exists open sets U, V such that $A \subseteq U$, $x \in V$, and $U \cap V = \emptyset$. The space X is *normal* if given disjoint closed sets A and B, there exist open sets U, V, such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$. A space is T_0 if given $x \neq y$, there exists an open set U that contains one of them and misses the other, T_1 if singleton sets are closed sets, T_2 if it is Hausdorff, T_3 if it is regular and Hausdorff, and T_4 if it is normal and Hausdorff.

Problem 4. Show that a metric space is T_4.

Problem 5. (i) Let (X, d_1) and (Y, d_2) be metric spaces. Show that

$$d\big((x_1, y_1), (x_2, y_2)\big) := d_1(x_1, x_2) + d_2(y_1, y_2)$$

defines a metric on $X \times Y$, and that the metric topology agrees with the product topology.
(ii) (More difficult) Let (X_n, d_n) be metric spaces for $n \in \mathbb{N}$, and assume that each metric d_n is bounded by 1. Define a metric d on $\prod_n X_n$, show that it is a metric, and prove that the product topology agrees with the metric topology.
(iii) Show that the countable product of metrizable spaces is metrizable.

Problem 6. Prove or disprove:
(i) If C is a connected subset of X and if $C \subseteq A \subseteq \overline{C}$, then A is connected.
(ii) If C is connected, then its interior C° is connected.

Problem 7. Let C and each of C_i, $i \in I$ be connected subsets of X. If $C \cap C_i \neq \emptyset$ for each $i \in I$, prove that $C \cup \bigcup_i C_i$ is connected.

Problem 8. A *bounded linear continuum* is a totally ordered set $(X, \leq)$ with smallest element a and largest element b such that the order $\leq$ is complete (satisfies the least upper bound property)and order dense (given $x < z$ in X, there exists $y \in X$ such that $x < y < z$). A basis for the *order topology* on X is given by all sets of the form (u, v), where $u < v$, $[a, v)$, where $a < v$, and $(u, b]$, where $u < b$. (a) Show that X is compact. (b) Show that X is connected. (c) Using (b), explain why the real line $\mathbb{R}$ is connected.

Problem 9. Given a point x in a topological space X, define the component $C(x)$ of x, show that it exists and is unique, and that it is closed.

Problem 10. Show that a compact Hausdorff space is normal.

Problem 11. Prove: A function $f\colon (X, d) \to (Y, \delta)$ between metric spaces is continuous if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$ is a convergent sequence in X.

Problem 12. Let $\{X_i : i \in I\}$ be a collection of topological spaces, and let $\emptyset \neq A_i \subseteq X_i$ for each i. In the product space $\prod_i X_i$ equipped with the product topology, show that
(i) if each A_i is closed in X_i, then $A := \prod_i A_i$ is closed in $\prod_i X_i$;
(ii) for general subspaces $A_i \subseteq X_i$, $\overline{\prod_i A_i} = \prod_i \overline{A_i}$.

Problem 13. Let $f : X \to \mathbb{R}$ be a continuous function.
(i) If X is compact, show that f has a maximum value which it attains.

(ii) If X is connected and if $f(x) < c < f(z)$, show that there exists $y \in X$ such that $f(y) = c$.
(iii) If X is compact and connected, show that $f(X)$ is a point or a closed interval.

Problem 14. Let X, Y, Z be topological spaces, let $f : X \to Y$ be a quotient mapping onto Y, and let $h : X \to Z$ be continuous. If $h(x_1) = h(x_2)$ whenever $f(x_1) = f(x_2)$, show that there exists a unique function $g : Y \to Z$ such that $g \circ f = h$ and that g is continuous.

$$\begin{array}{ccc} Y & \overset{\exists! g}{\dashrightarrow} & Z \\ \uparrow^{f} & \nearrow_{h} & \\ X & & \end{array}$$

Problem 15. Let $f_n \colon X \to \mathbb{R}$, $n \in \mathbb{N}$ be a sequence of continuous functions, and let $f \colon X \to \mathbb{R}$. Prove or give a counterexample: (i) If the sequence f_n converges uniformly to f on X, then f must also be continuous. (ii) If the sequence f_n converges to f pointwise, then f is continuous.

Problem 16. Let $\mathcal{D}$ be a descending family ($D_1, D_2 \in \mathcal{D} \Rightarrow \exists D_3 \in \mathcal{D}$ s.t. $D_3 \subseteq D_1 \cap D_2$) of non-empty closed subsets of a Hausdorff space X, and let $F := \bigcap\{D | D \in \mathcal{D}\}$. Prove or disprove each of the following: (i) If each $D \in \mathcal{D}$ is connected, then F is connected. (ii) if each $D \in \mathcal{D}$ is compact, then $F \neq \emptyset$. (iii) if each $D \in \mathcal{D}$ is compact and U is an open set containing F, then $\exists D \in \mathcal{D}$ s.t. $D \subseteq U$. (iv) if each $D \in \mathcal{D}$ is compact and connected, then F is compact and connected.

Problem 17. Let $X (\neq \emptyset)$ be a compact Hausdorff space, and let $F : X \to X$ be a continuous map. Let $A = \bigcap_n F^n(X)$. Show that
(a) A is closed.
(b) A is compact.
(c) $A \neq \emptyset$.
(d) $F(A) \subseteq A$.
(e) $A \subseteq F(A)$.

Problem 18. Let X be the set $\{(t, 0) : 0 < t \leq 1\} \cup \{(1/n, t) : n \in \mathbb{N}, 0 \leq t \leq 1\}$. Note that the origin $(0, 0)$ is *not* included.

Define $d : X \times X \to \mathbb{R}$ by $d(x, y)$ is equal to the length of the shortest arc in X from x to y.
(i) Argue that the metric toplogy τ_d agrees with the relative topology inherited from the euclidean plane $\mathbb{R}^2$.
(ii) Determine whether (X, d) is
(a) connected
(b) compact
(c) locally compact
(d) bounded
(e) totally bounded ($\forall \epsilon > 0$, X is covered by finitely many $N_\epsilon(x)$)
(f) complete.
Your answers don't need to be complete proofs, but should give some indication how you reached your conclusions.

Problem 19. Use the Tychonoff theorem to prove that a product of Hausdorff spaces $\prod X_\alpha$ is locally compact if and only if all but finitely many of the factors are compact and the remainder are locally compact.

Problem 20. Let $f : X \to Y$ be a continuous function from a compact metric space (X, d) to a metric space (Y, ρ). Show that f is uniformly continuous, i.e., given $\epsilon > 0$, there exists $\delta > 0$ such that $\rho\big(f(u), f(v)\big) < \epsilon$ whenever $d(u, v) < \delta$.

Problem 21. Consider the linear continuum $X := [0, 1] \times [0, 1]$ with the lexicographic order and order topology.
(i) Is X connected? compact? first countable? separable? metrizable? Support each of your answers (you may use standard theorems, where appropriate).
(ii) Describe a simple basis for the subspace topology on $[0, 1] \times \{0\}$.

Problem 22. (i) Let $\pi_j : \prod_i X_i \to X_j$ be the projection into the j-th factor. Show that the product topology on $\prod_i X_i$ is the coarsest topology making all π_j continuous.
(ii) Let $f \colon X \to \prod_i X_i$. Show that f is continuous if and only if all the coordinate functions $f_i := \pi_i \circ f$ are continuous.

Problem 23. The diameter of a set A in a metric space (X, d) is defined by

$$\operatorname{diam}(A) = \sup\{d(x, y) |\ x, y \in A\}.$$

A is *bounded* if $\operatorname{diam}(A) < \infty$.
(a) Show that a compact subset A is closed and totally bounded.
(b) Show that a totally bounded subset E is bounded.
(c) Suppose you know that closed intervals $[a, b]$ in $\mathbb{R}$ are compact and you know the Tychonoff theorem. Prove that a closed and bounded subset of $\mathbb{R}^n$ is compact.

Problem 24. Let (X, d) be a complete metric space and let $\{A_n\}$ be a nested sequence of closed non-empty sets such that $\lim_{n\to\infty} \operatorname{diam}(A_n) = 0$. Show that there exists $x \in X$ such that $\bigcap_n A_n = \{x\}$.

Problem 25. (i) Show that if $S, A \subseteq X$ and S is open, then $\overline{S \cap A} = \overline{S \cap \overline{A}}$; use this fact to show that the intersection of a dense open subset and a dense set is again dense.
(ii) Let (X, d) be a complete metric space and let $\{U_n | \, n \in \mathbb{N}\}$ be a sequence of dense open subsets. Show that $\bigcap_n U_n$ is dense in X. (This is one version of the Baire Category Theorem.)

Problem 26. (i) Let $Y = \prod_n [0, 1/2^n]$ with the product topology, and let $F : Y \to \mathbb{R}$ be defined by sending a sequence in Y to its sum. Show that F is continuous.
(ii) Let X be a normal space and let U be an open subset which can be written as a countable union of closed sets. Use Urysohn's Lemma to show that there exists a continuous function $f : X \to [0, 1]$ such that $f(x) > 0$ for $x \in U$ and $f(x) = 0$ for $x \in X \setminus U$.

Problem 27. Let $f : X \to Y$ be a function between Hausdorff spaces. The *graph* of f is defined by

$$Gr(f) = \{(x, y) \in X \times Y | \, y = f(x)\}.$$

(i) Show that if f is continuous and X is path-connected , then $Gr(f)$ is path-connected (in calculus for $X = Y = \mathbb{R}$, we say the graph can be traced without lifting the pencil). (ii) Show that if f is continuous, then $Gr(f)$ is closed. (iii) Show that if $\overline{f(X)}$ is compact and $Gr(f)$ is closed, then f is continuous.

Problem 28. Let (X, d) be a metric space.
(i) Show that a sequence $\{x_n\}$ can have a most one limit.
(ii) A point $y \in X$ is a *cluster point* of $\{x_n\}$ if given $\epsilon > 0$ and $n \in \mathbb{N}$, there

exists $m \geq n$ such that $d(x_m, y) < \epsilon$. Show that the set S of cluster points of $\{x_n\}$ is given by

$$S = \bigcap_n \overline{\{x_i|\ i \geq n\}}.$$

(iii) Show that if a Cauchy sequence clusters to a point p , then it converges to p.

Problem 29. The one-point compactification X^* of a non-compact space X consists of the points of X together with a point x^* not in X. The topology of X^* consists of the open sets of X and all subsets U of X^* such that $X^* \setminus U$ is a closed compact subset of X. (i) Prove that X^* is compact (ii) Prove that X^* is Hausdorff if and only if X is locally compact Hausdorff. (iii) Sketch a picture representing the one-point compactification of each of the following spaces: (a) $X_1 = \mathbb{R}$; (b) $X_2 = (0,1) \cup (1,2) \subset \mathbb{R}$; (c) $[0,1) \cup (1,2) \cup (2,3] \subset \mathbb{R}$; (d) $\mathbb{R}^2$.

Problem 30. Let X, Y be topological spaces, $X = A \cup B$, and $f : X \to Y$ a function such that $f|_A$ and $f|_B$ are continuous. Show that if both A and B are closed (resp. open), then f is continuous.

Problem 31. (i) Show that a regular T_0 space is Hausdorff, hence T_3. (ii) Show that if $f : X \to Y$ is closed, continuous, and surjective, X is normal, and Y is T_1, then Y is T_4.

Problem 32. Let $R \subseteq X \times X$ be an equivalence relation on X. Let X/R denote the set of equivalence classes and let $\rho : X \to X/R$ denote the map sending a point to its equivalence class. We endow X/R with the quotient topology. (i) Show that if X/R is Hausdorff, then R is closed in $X \times X$ (Hint: Show that $(\rho \times \rho)^{-1}(\Delta) = R$). (ii) Show that if R is closed in $X \times X$ and $A \subseteq X$ is compact, then $[R]A := \{y \in X |\, \exists x \in A \text{ s.t. } (y, x) \in R\}$ is closed. (iii) Show that if R is closed and X is compact Hausdorff, then ρ is a closed mapping. It then follows (see Problem 31) that that X/R is T_4, in particular T_2. (iv) Explain how to get a space homeomorphic to the unit circle from the unit interval $[0, 1]$ by the preceding construction.

Problem 33. A map $f : X \to Y$ is *monotone* if $f^{-1}(y)$ is connected for every $y \in Y$. Let $f : X \to Y$ be a surjective monotone quotient mapping. Show that if Y is connected, then X is connected.

Problem 34. (Banach Fixed-Point Theorem) Let (X, d) be a complete metric space, let $F : X \to X$, and suppose that there exists $0 < \lambda < 1$ such that $d\big(F(x), F(y)\big) \leq \lambda d(x, y)$ for all $x, y \in X$. Prove that F has a unique fixed point.

Problem 35. (i) Let $f \colon X \to Y$ be a function from a topological space X to a space Y. We say that f is continuous at $x \in X$ if for every open set V containining $f(x)$, there exists U open containing x such that $f(U) \subseteq V$, and we say that f is continuous if it is continuous at every $x \in X$. Show that f is continuous (in the preceding sense) if and only if for all $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$. (ii) Let $f \colon X \to Y$ be a continuous function from a compact metric space X to a metric space Y. Show that f is uniformly continuous (given $\epsilon > 0$, there exists $\delta > 0$ such that $d(x_1, x_2) < \delta$ implies that $d(f(x_1, x_2)) < \epsilon$). You may use freely standard theorems about compactness and continuous functions. (iii) Prove or disprove: the function $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}$ is uniformly continuous.

Problem 36. (i) Let X be a Hausdorff space with the property that for every $x \in X$, there exists a compact set K (depending on x) such that $x \in K^\circ$, the interior of K. (This is one, though probably not the best, way of defining a locally compact space.) Show that the space X is regular. (ii) Show that a metric space is normal.

Problem 37. Let (X, d) be a metric space. (i) Show that a sequence $\{x_n\}$ can have at most one point to which it converges. (We say that limits are unique.) (ii) A point $y \in X$ is a *cluster point* of $\{x_n\}$ if given $\epsilon > 0$ and $n \in \mathbb{N}$, there exists $m \geq n$ such that $d(x_m, y) < \epsilon$. Show that the set S of cluster points of $\{x_n\}$ is given by

$$S = \bigcap_n \overline{\{x_i \mid i \geq n\}}.$$

(iii) Show that if a Cauchy sequence clusters to a point p, then it converges to p. (iv) Prove or disprove: if a sequence has exactly one cluster point, then it must converge to that point.

Problem 38. (i) Let X be a set with distinct topologies $\mathcal{T}$ and $\mathcal{T}'$. Recall that two topologies on X are comparable if one is a subset of the other. Show that if $(X, \mathcal{T})$ and $(X, \mathcal{T}')$ are both compact Hausdorff, then $\mathcal{T}$ and $\mathcal{T}'$ are not comparable. (ii) Let X be a finite set. Show that there is exactly one Hausdorff topology on X.

Problem 39. Let $p\colon X \to Y$ be a closed, continuous surjective map. Show that if Y is compact and $p^{-1}(y)$ is compact for each $y \in Y$, then X is compact.

Problem 40. (i) Show that if $f\colon X \to Y$ is a continuous bijection from a compact space X to a Hausdorff space Y, then f is a homeomorphism. (ii) Give an example of topological spaces X and Y and a continuous bijection $f\colon X \to Y$ that is **not** a homeomorphism.

Problem 41. (i) Prove that a non-empty connected subset of a topological space X that is both open and closed is a connected component of X. (ii) In general, is a connected component of a topological space open? Is it necessarily closed? Justify your answers.

Problem 42. Let $f\colon [0,1] \to [0,1]$ be continuous (where $[0,1]$ is regarded as a subspace of the real line $\mathbb{R}$, with the usual topology). Show that there is a point $x \in [0,1]$ such that $f(x) = x$.

Problem 43. Find a function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point. Show that your example works.

Problem 44. Prove that a connected normal space with more than one point is uncountable. [Hint: use Urysohn's Lemma.]

Problem 45. (i) Show that any open interval (a, b) in the real line $\mathbb{R}$ is homeomorphic to $\mathbb{R}$. (ii) Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. (Quote without proof the appropriate basic results you use.)

Problem 46. Let X and Y be topological spaces, and $f\colon X \to Y$ a function. Recall that "f is open" means that if U is an open set in X, then $f(U)$ is open in Y. (i) If f is continuous, does it follow that f is open? (Proof or counterexample.) (ii) If f is open, does it follow that f is continuous? (Proof or counterexample.) (iii) Show that if $X = Y \times Z$ (a product of topological spaces, with the product topology) and f is the first projection, then f is open. (iv) Under the conditions and notation of (iii), if F is closed in $X = Y \times Z$, does it follow that $f(F)$ is closed in Y? (Proof or counterexample.)

Problem 47. Let $f\colon S^1 \to \mathbb{R}$ be a continuous function, where S^1 (the unit circle) is the set of all complex numbers of modulus 1. (i) Show that there is a point z of S^1 such that $f(z) = f(-z)$. (ii) Show that f cannot be onto. (Quote without proof the appropriate standard topological results.)

Problem 48. Recall that a metric d on a set X is called *bounded* if there is a positive real constant M such that $d(x, y) \leq M$ for any pair of points x, y in X. Show that given any metric δ on X, there is a bounded metric d on X that induces the same topology as δ.

Problem 49. Prove that a compact subset of a Hausdorff space is closed.

Problem 50. Prove that a closed subset of a compact space is compact

Problem 51. Let S denote the following union of three point-sets in the Euclidean plane $\mathbb{R}^2$:

$$S = \{(t, 0) \mid 0 < t \leq 1\} \cup \{(1/n, s) \mid n = 1, 2, 3, \ldots \text{ and } 0 \leq s \leq 1\} \cup \{(0, 1/2)\}.$$

State whether S ics connected or not, and prove your assertion.

Problem 52. Let $Y = \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$ and let Z be the graph of the function $y = \sin(\pi/x)$ for $0 < x \leq 1$. Is the union $Y \cup Z$ connected or disconnected in the standard topology on $\mathbb{R}^2$? Prove your answer.

Problem 53. Let $f \colon A \to B$ be a continuous function between topological spaces. (i) If A is compact, prove that $f(A)$ is compact. (ii) If A is connected, prove that $f(A)$ is connected.

Problem 54. (i) Show that a path-connected space is connected. (ii) Show that a connected, locally path-connected space is path-connected.