SIGNED GRAPH LAPLACIANS

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Abstract. This paper uses chain complexes of based, finitely-generated $\mathbb{Z}$-modules to study the Laplacians of signed plane graphs. We extend a theorem of Lien and Watkins [6] regarding the Goeritz equivalence of the signed Laplacians of a signed plane graph and its dual by showing that it is possible to use only (± 1)-diagonal forms instead of the $(0, \pm 1)$-diagonal forms used by Lien and Watkins.

Introduction

Algebraic invariants derived from knot diagrams and medial diagrams have long been studied. Most of these studies have focused on the abstract graph theoretic properties while ignoring the plane embedding. However, Whitney showed in [8] that only 3-connected graphs have unique embeddings in the plane, giving some importance to the particular embedding. In this paper, we wish to study invariants which incorporate information from the plane embedding. We do this by constructing a chain complex corresponding to a signed plane graph and then examining the properties of its Laplacians.

1. Graphs and Chain Complexes

In this paper we permit graphs with multiple edges but disallow self-loops and vertices of degree one (i.e., vertices adjacent to only one other vertex). We make the second restriction to prevent small technical problems when dealing with the dual graph, since vertices of degree one create loops in the dual. See [7] for definitions and concepts from graph theory.

Fix an orientation for S^2 and let $G \subset S^2$ be a signed plane graph. Let G have ordered vertex set $V = \{v_1, v_2, \dots, v_n\}$, ordered edge set $E = \{e_1, e_2, \dots, e_m\}$, and ordered face set $F = \{f_1, f_2, \dots, f_r\}$. (By the Euler characteristic, we have that $r = 2 - n + m$.) The chosen orientation of S^2 induces an orientation on F, but we choose an arbitrary orientation for each edge in E, making G a signed plane digraph. When considering the oriented edges, we will refer to them as $\vec{e}_i$.

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Definition 1.1. Given an S^2 embedding of a signed digraph $G \subset S^2$, we can construct the *chain complex corresponding to* G (denoted $\mathcal{C}[G]$), a complex of free $\mathbb{Z}$-modules of finite rank, as follows:

$$0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

where C_0 is the free $\mathbb{Z}$-module on the vertices of G, C_1 is the free $\mathbb{Z}$-module on the edges of G, and C_2 is the free $\mathbb{Z}$-module on the faces of G. If $\vec{e} = (v_i, v_j) \in C_1$ (where our convention is that $\vec{e}$ is oriented *from* v_i *to* v_j), then $\partial_1(\vec{e}) = v_j - v_i$. Similarly, if $f \in C_2$, we define $\partial_2(f)$ to be the sum of the edges on the boundary of f with coefficient $+1$ if the orientation of the edge agrees with the induced orientation of f and coefficient -1 if it disagrees. When a particular ordered basis is important, we will denote the complex as $\mathcal{C}_{\mathcal{B}}[G]$, with $\mathcal{B}_0$, $\mathcal{B}_1$, and $\mathcal{B}_2$ as the bases of C_0, C_1, and C_2, respectively.

From $\mathcal{C}[G]$, we also construct its dual $\mathcal{C}^*[G]$, which has modules $C_i^* = \mathrm{Hom}(C_i, \mathbb{Z})$ and coboundary maps $\delta_i : C_i \to C_{i+1}$. The dual complex is shown in the following diagram.

$$0 \longleftarrow C_2^* \xleftarrow{\delta_1} C_1^* \xleftarrow{\delta_0} C_0^* \longleftarrow 0$$

A *bilinear form* $\langle\,,\,\rangle$ *on* C_1 is a mapping $C_1 \times C_1 \to \mathbb{Z}$ that is linear in each variable. That is,

$$\begin{aligned} \langle cx + y, z\rangle &= c\langle x, z\rangle + \langle y, z\rangle \\ \langle x, cy + z\rangle &= c\langle x, y\rangle + \langle x, z\rangle. \end{aligned}$$

We say that $\langle\,,\,\rangle$ is *symmetric* if $\langle x, y\rangle = \langle y, x\rangle$. The matrix of $\langle\,,\,\rangle$ is $A = (\langle e_i, e_j\rangle)$, where $\{e_i\}$ is an ordered basis. (Note that a change of basis corresponds to congruence of A, i.e., $A \longmapsto P^tAP$, where $P\{e_i\} = \{e_i\}$.) If A has integer entries, we call $\langle\,,\,\rangle$ *unimodular* if $\det A = \pm 1$. (When the entries come from another ring, we require that $\det A$ be a unit.) Equivalently, $\langle\,,\,\rangle$ is unimodular if the adjoint

$$\begin{aligned} \mathrm{Ad}\langle\,,\,\rangle : C_1 &\longrightarrow \mathrm{Hom}(C_1, \mathbb{Z}) \cong C_1^* \\ e &\longmapsto (f \mapsto \langle e, f\rangle_1) \end{aligned}$$

is an isomorphism.

Example 1.2. For $\mathcal{C}[G]$, there are two common symmetric bilinear forms that we wish to consider on C_1.

(a) The standard "dot product" on C_1 is defined by declaring the edges with chosen orientation $\{\vec{e}_i\}$ to be an orthonormal basis. Then

$$\langle \vec{e}_i, \vec{e}_j\rangle_1 = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

(b) If G is a signed graph, we set

$$\langle \vec{e}_i, \vec{e}_j\rangle_{1,\varepsilon} = \mathrm{sign}(\vec{e}_i)\delta_{ij}.$$

Note that the matrices of both these products are diagonal with all entries ± 1, making them unimodular. Therefore $\mathrm{Ad}\langle\ ,\ \rangle_{1(,\varepsilon)}$ is an isomorphism, and we have proved the following lemma.

Lemma 1.3. *As $\mathbb{Z}$-modules, C_1 and C_1^* are isomorphic.*

While the adjoint specifies an identification between C_1 and C_1^*, it is by no means a canonical isomorphism.

Definition 1.4. Given a signed plane digraph $G \subset S^2$, the *dual signed plane digraph of G* (denoted $\widehat{G}$) is constructed by

(1) choosing a vertex set $\widehat{V} = \{\widehat{v}_1, \widehat{v}_2, \ldots, \widehat{v}_r\}$ for $\widehat{G}$ in one-to-one correspondence with F;
(2) choosing a face set $\widehat{F} = (\widehat{f}_1, \widehat{f}_2, \ldots, \widehat{f}_n)$ in one-to-one correspondence with V;
(3) putting the edge set $\widehat{E}$ of $\widehat{G}$ in one-to-one correspondence with E such that an edge $\widehat{e} \in \widehat{E}$ has the opposite sign of the corresponding edge $e \in E$ and is oriented from the vertex corresponding to the face whose orientation disagrees with that of e to the vertex corresponding to the face whose orientation agrees with that of e.

Note that the chain complex of the dual of G is (with reordering) the dual of the chain complex of G (i.e., $\mathcal{C}[\widehat{G}] \cong \mathcal{C}^*[G]$).

Notation. We let ε_i denote $\langle e_i, e_i\rangle_\varepsilon$ for $e_i \in C_1$ and ε_i^* denote $\langle e_i^*, e_i^*\rangle_\varepsilon$ for $e_i^* \in C_1^*$. (Notice that, since the dual modules correspond to the edge sets of dual signed graphs, $\varepsilon_i^* = -\varepsilon_i$.)

2. Laplacians

Now we define the Laplacians both of a graph and of a chain complex, and then show the connections between the two of them.

Definition 2.1. The Laplacian matrix $L_0(G)$ of a graph G is

$$L_0(G) = D(G) - A(G),$$

where $D(G) = \mathrm{diag}(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$, the diagonal matrix of vertex degrees. The signed Laplacian matrix of a graph is

$$L_\varepsilon(G) = \sum_{\substack{e \in E(G)\\ \partial e = \{i,j\}}} \varepsilon_{ij}(E_{ii} - E_{ij} - E_{ji} + E_{jj}),$$

where E_{st} is the $n \times n$ matrix with exactly one nonzero entry in position (s,t) and ε_{ij} is $+1$ or -1, depending on the sign of the edge in question. (Note that the reference to G will be omitted when the context is clear.)

For the dual graph $\widehat{G}$, we will denote $L_\varepsilon(\widehat{G})$ by $\widehat{L}_\varepsilon$.

Since we have defined a chain complex corresponding to a graph, we consider now its combinatorial Laplacians and then show their relations to the graph Laplacians of definition 2.1.

Definition 2.2. Given a chain complex $\mathcal{C} = \{C_i, \partial_i\}$ where each module is endowed with a scalar product that makes its basis orthonormal, its *combinatorial Laplacians* $\Delta_i : C_i \to C_i$ are

$$\Delta_i = (\mathrm{Ad}\langle\, , \rangle_i)^{-1}\delta_{i-1}\,\mathrm{Ad}\langle\, , \rangle_{i-1}\partial_i + \partial_{i+1}(\mathrm{Ad}\langle\, , \rangle_{i+1})^{-1}\delta_i\,\mathrm{Ad}\langle\, , \rangle_i$$

If we use $\mathrm{Ad}\langle\, , \rangle_{1,\varepsilon}$ to identify C_1 and C_1^*, we write $\Delta_{i,\varepsilon}$.

Lemma 2.3. *Let G be a signed graph (with fixed but arbitrary edge orientation) embedded in S^2 and with corresponding chain complex $\mathcal{C}[G]$. Then*

1. $\Delta_0 = \partial_1(\mathrm{Ad}\langle\, , \rangle_1)^{-1}\delta_0\,\mathrm{Ad}\langle\, , \rangle_0 = L_0(G)$,
2. $\Delta_{0,\varepsilon} = \partial_1(\mathrm{Ad}\langle\, , \rangle_{1,\varepsilon})^{-1}\delta_0\,\mathrm{Ad}\langle\, , \rangle_0 = L_\varepsilon(G)$,
3. $\Delta_{2,\varepsilon} = \mathrm{Ad}\langle\, , \rangle_2\delta_2\,\mathrm{Ad}\langle\, , \rangle_{1,\varepsilon}\partial_1 = \widehat{L}_\varepsilon(G)$.

Proof. As a matrix, ∂_1 is a rank $C_0 \times$ rank C_1 matrix with rows corresponding to the vertices of G and columns corresponding to the edges of G, so it has a $+1$ and a -1 in every column, and $\delta_0 = \partial_1^t$. We commonly refer to ∂_1 as the oriented vertex-edge incidence matrix of G. Similarly, δ_1 is the oriented vertex-edge incidence matrix of $\widehat{G}$, and $\partial_2 = \delta_1^t$.

Since $\mathrm{Ad}\langle\, , \rangle_i$ is simply represented by the identity matrix, we can see that the first statement is proved in [2, p. 27]. Let $Q = \partial_1(\mathrm{Ad}\langle\, , \rangle_{1,\varepsilon})^{-1}\delta_0\,\mathrm{Ad}\langle\, , \rangle_0$ and denote its ij-th entry by q_{ij}. First consider a diagonal entry q_{ii}. It is the inner product of row i of ∂_1 with column i in $(\mathrm{Ad}\langle\, , \rangle_{1,\varepsilon})^{-1}\delta_0$. These two vectors have nonzero entries in the same locations, since they correspond to the same vertex and $\mathrm{Ad}\langle\, , \rangle_{1,\varepsilon}$ only multiplies entries by ± 1. Then for all j, $1 \le j \le m$, we have three possibilities. First, the j-th entry is zero, and contributes nothing to q_{ii}. Second, $\varepsilon_j = +1$, so j-th entries of the vectors are both either $+1$ or -1, and 1 is added to q_{ii}. Finally, $\varepsilon_j = -1$, so one of the vectors has $+1$ as its j-th entry and the other has -1, meaning that -1 is added to q_{ii}. Therefore $q_{ii} = \sum_{j=1}^m \varepsilon_j$.

Next consider an off-diagonal entry q_{ij} of Q, which is the inner product of row i of ∂_1 and column j of δ_0. This row-column pair corresponds to different vertices, so the vectors only have non-zero entries in the k-th position if v_i and v_j are the endpoints of e_k. If $\varepsilon_k = +1$ (i.e., e_k is a positive edge), then one of the vectors has a $+1$ in the k-th position and the other has a -1. Therefore, -1 is added to q_{ij}. If $\varepsilon_k = -1$ (i.e., e_k is a negative edge), then both vectors have either a $+1$ or a -1 in the k-th position. Hence, $+1$ is added to q_{ij}. Thus $q_{ij} = \sum(-\varepsilon_k)$, where the sum is taken over all edges between v_i and v_j. This matches with the definition of $L_\varepsilon(G)$.

The third statement is proved in the same manner as the second. □

Much attention has been given to $L_0(G)$ and (to a lesser degree) $L_\varepsilon(G)$. Despite this interest, the literature contains very little about Δ_1. Two major reasons for this are that graphs with fixed plane embeddings have been little studied, and the one-Laplacian involves a non-canonical choice of edge orientation. However, the following lemma can easily be proven.

Lemma 2.4. *Let G be a bipartite graph with n vertices and m edges with bipartition V_1 and V_2. Choose an orientation such that the edges of G are all directed from V_1 to V_2.*

(a) *Then in $\mathcal{C}[G]$, we have $(\mathrm{Ad}\langle\,,\,\rangle_1)^{-1}\delta_0\,\mathrm{Ad}\langle\,,\,\rangle_0\partial_1 = 2I_m + A(G^{\#})$, where $A(G^{\#})$ is the adjacency matrix of the line graph of G.*

(b) *If G is planar, we also have $\partial_2(\mathrm{Ad}\langle\,,\,\rangle_2)^{-1}\delta_1\,\mathrm{Ad}\langle\,,\,\rangle_1 = 2I_m + A(\widehat{G}^{\#})$.*

3. Goeritz Equivalence

In [3], Goeritz introduced the following equivalence relation on bilinear forms.

Definition 3.1. Let B_1 and B_2 be two bilinear forms on finitely generated free $\mathbb{Z}$-modules M_1 and M_2, respectively. Then B_1 is *Goeritz equivalent* to B_2 (denoted $B_1 \sim_G B_2$) if $(M_1, B_1 \oplus D_1)$ is isomorphic to $(M_2, B_2 \oplus D_2)$, where the D_i forms have a basis in which the form is diagonal with entries in $\{0, \pm 1\}$.

Here we define a second, slightly stronger relation as well.

Definition 3.2. Let B_1 and B_2 be two bilinear forms on finitely generated free $\mathbb{Z}$-modules M_1 and M_2, respectively. Then B_1 is *super-Goeritz equivalent* to B_2 (denoted $B_1 \sim_{SG} B_2$) if $(M_1, B_1 \oplus D_1)$ is isomorphic to $(M_2, B_2 \oplus D_2)$, where the D_i forms have a basis in which the form is diagonal with entries in $\{\pm 1\}$.

While the Laplacians are usually thought of as automorphisms or operators, we can use them to define bilinear forms on the modules of the chain complex corresponding to a graph.

Definition 3.3. Let $\mathcal{C}[G]$ be the chain complex corresponding a signed plane graph G. For each module C_i, define $B_{i,\varepsilon}(x, y)$ to be $\langle x, \Delta_{i,\varepsilon} y\rangle_{i,\varepsilon}$ for vectors x and y.

The following theorem is equivalent to the theorem Lien and Watkins proved in [6]. Here we have stated the theorem in terms of bilinear forms instead of using matrix terminology.

Theorem 3.4. *Let G be a signed plane graph with chain complex $\mathcal{C}[G]$. Then $(C_2, B_{2,\varepsilon}) \sim_G (C_0, B_{0,\varepsilon})$.*

Before proceeding, we need to define an idea from module theory and prove two statements about it.

Definition 3.5. Let M be a finitely-generated free $\mathbb{Z}$-module with bilinear form B. Then the *left radical of B*, denoted $\mathrm{Rad}\, B$, is

$$\{u \in M \mid B(u, M) = \{0\}\}\,.$$

Note that $\mathrm{Rad}\, B = \{0\}$ if and only if $\mathrm{Ad}\, B$ is injective. Also, a vector $\vec{r}$ is in the radical of a form with matrix B if and only if $\vec{r} \cdot B = \vec{0}$.

Lemma 3.6. *Let M and N be finitely-generated free $\mathbb{Z}$-modules and let*

$$\Phi : (M, B_M) \to (N, B_N)$$

be an isomorphism of modules with a bilinear form. Then

$$(M/\operatorname{Rad} B_M, B'_M) \cong (N/\operatorname{Rad} B_N, B'_N),$$

where $B'_M(\overline{x}, \overline{y}) = B_M(x, y)$ is well-defined by the definition of the radical.

Proof. First note that Φ preserves the bilinear form, and thus Φ maps $\operatorname{Rad} B_M$ to $\operatorname{Rad} B_N$. Let

$$\overline{\Phi} : (M/\operatorname{Rad} B_M, B'_M) \to (N/\operatorname{Rad} B_N, B'_N).$$

We claim that $\overline{\Phi}$ is an isomorphism. First, to check that $\overline{\Phi}$ is well-defined, we must show that $\Phi(m) - \Phi(m+r) \in \operatorname{Rad} B_N$ when $r \in \operatorname{Rad} B_M$. We have that

$$\begin{aligned} B_N(\Phi(m) - \Phi(m+r), n) &= B_N(\Phi(m) - \Phi(m) - \Phi(r), \Phi(m')) \\ &= -B_N(\Phi(r), \Phi(m')) \\ &= -B_M(r, m') \\ &= 0, \end{aligned}$$

since $r \in \operatorname{Rad} B_M$. Hence $\Phi(m) - \Phi(m+r) \in \operatorname{Rad} B_N$, and $\overline{\Phi}$ is well-defined.

Next we check that $\overline{\Phi}$ is injective. Let $\overline{x}, \overline{y} \in M/\operatorname{Rad} B_M$ and suppose that $\overline{\Phi}(\overline{x}) = \overline{\Phi}(\overline{y})$. Then $\overline{\Phi}(\overline{x}) - \overline{\Phi}(\overline{y}) = \overline{0}$, so $\Phi(x) - \Phi(y) \in \operatorname{Rad} B_N$. Let $z = \Phi(x) - \Phi(y)$ and set $z = \Phi(w)$, $w \in M$, since Φ is an isomorphism. Since Φ preserves radicals, we must have that $w \in \operatorname{Rad} B_M$. Now observe that

$$\begin{aligned} &\Phi(x) - \Phi(y) - \Phi(w) = 0 \\ \Rightarrow &\Phi(x - y - w) = 0 \\ \Rightarrow &x - y - w = 0 \\ \Rightarrow &x - w = y \\ \Rightarrow &\overline{x} = \overline{y}, \end{aligned}$$

since $w \in \operatorname{Rad} B_M$.

Finally we check that $\overline{\Phi}$ is surjective. Let $\overline{n}$ be an arbitrary equivalence class in $N/\operatorname{Rad} B_N$. Then members of $\overline{n}$ are of the form $n + r$, $r \in \operatorname{Rad} B_N$. Using Φ^{-1}, we have

$$\Phi^{-1}(n + r) = \Phi^{-1}(n) + \Phi^{-1}(r) = m + r',$$

for some $m, r' \in M$. However, r' must be in $\operatorname{Rad} B_M$, since Φ preserves radicals. Hence $m + r' \in M/\operatorname{Rad} B_M$ as required. Therefore $\overline{\Phi}$ is an isomorphism. □

Lemma 3.7. *The radical respects the direct sum. That is, if B_i and D_i are bilinear forms,*

$$\operatorname{Rad}(B_i \oplus D_i) = \operatorname{Rad}(B_i) \oplus \operatorname{Rad}(D_i).$$

Proof. Let $M = \begin{pmatrix} B_i & O \\ O & D_i \end{pmatrix}$ be the matrix of $B_i \oplus D_i$ and let us assume that B_i is an $n \times n$ matrix and D_i is an $m \times m$ matrix. Let $\vec{s} = (r_1, \ldots, r_n)$ and $\vec{t} = (r_{n+1}, \ldots, r_{n+m})$ and consider $\vec{r} = \vec{s} \oplus \vec{t} \in \operatorname{Rad}(C_i \oplus D_i)$. Then $\vec{r} \cdot M$ is necessarily $\vec{0}$. The first n entries of $\vec{r} \cdot M$ depend only on $\vec{s}$ and B_i, so $\vec{s} \cdot B_i = \vec{0}$. Hence $\vec{s} \in \operatorname{Rad}(B_i)$. The last m entries of $\vec{r} \cdot M$ depend similarly only on $\vec{t}$ and D_i, meaning that $\vec{t} \cdot D_i = \vec{0}$. Thus $\vec{t} \in \operatorname{Rad}(D_i)$. Therefore an element of $\operatorname{Rad}(B_i \oplus D_i)$ is formed from an element of $\operatorname{Rad}(B_i)$ and an element of $\operatorname{Rad}(D_i)$. Reversing the argument proves the opposite inclusion. □

Now we are in a position to state and prove the following theorem.

Theorem 3.8. *If G is a signed plane graph with chain complex $\mathcal{C}[G]$, then*

$$(C_2/\operatorname{Rad} B_{2,\varepsilon}, B'_{2,\varepsilon}) \sim_{SG} (C_0/\operatorname{Rad} B_{0,\varepsilon}, B'_{0,\varepsilon}),$$

where the $B'_{i,\varepsilon}$'s are bilinear forms on the quotient modules.

Proof. By theorem 3.4, $(C_2, B_{2,\varepsilon}) \sim_G (C_0, B_{0,\varepsilon})$. That is, there exist $(0, \pm 1)$-diagonal forms D_2, D_0 with

$$B_{2,\varepsilon} \oplus D_2 \cong B_{0,\varepsilon} \oplus D_0.$$

Lemma 3.6 implies that

$$(B_{2,\varepsilon} \oplus D_2)/\operatorname{Rad}(B_{2,\varepsilon} \oplus D_2) \cong (B_{0,\varepsilon} \oplus D_0)/\operatorname{Rad}(B_{0,\varepsilon} \oplus D_0).$$

By lemma 3.7, we have $\operatorname{Rad}(B_i \oplus D_i) = \operatorname{Rad}(B_i) \oplus \operatorname{Rad}(D_i)$. Now $D'_i \cong D_i/\operatorname{Rad}(D_i)$ is a (± 1)-diagonal form. Hence

$$\begin{aligned}(B_i \oplus D_i)/\operatorname{Rad}(B_i \oplus D_i) &\cong B_i/\operatorname{Rad}(B_i) \oplus D_i/\operatorname{Rad}(D_i) \\ &\cong B_i/\operatorname{Rad}(B_i) \oplus D'_i.\end{aligned}$$

Therefore the conclusion on super-Goeritz equivalence holds. □

4. Example

Consider the knot 8_{20}, which is illustrated with its medial graph in figure 1. Let $G = M(8_{20})$ as labelled in figure 2.

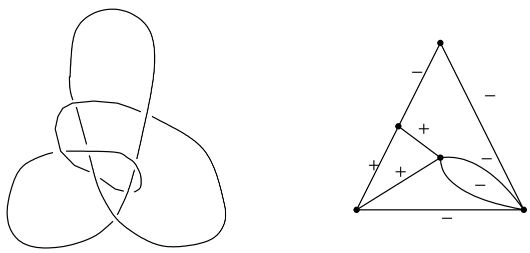

FIGURE 1. One diagram of the knot 8_{20}.

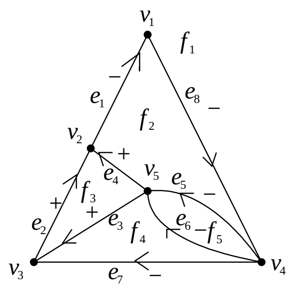

FIGURE 2. Labelled version of $M(8_{20})$.

Now we can construct $\mathcal{C}[G]$ and examine its boundary maps.

$$\partial_2 = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{pmatrix} = \delta_1^t$$

$$\partial_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \end{pmatrix} = \delta_0^t$$

Using the formulas of lemma 2.3, we can easily compute the following Laplacians.

$$L_0 = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & -1 & 4 & -2 \\ 0 & -1 & -1 & -2 & 4 \end{pmatrix}, \quad L_\varepsilon = \begin{pmatrix} -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 1 & -1 \\ 1 & 0 & 1 & -4 & 2 \\ 0 & -1 & -1 & 2 & 0 \end{pmatrix}$$

$$\widehat{L}_\varepsilon = \begin{pmatrix} 2 & -2 & 1 & -1 & 0 \\ -2 & 2 & 1 & 0 & -1 \\ 1 & 1 & -3 & 1 & 0 \\ -1 & 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix},$$

It is clear that the vector $(1,1,1,1,1)$ spans the radical of L_0, L_ε,and $\widehat{L}_\varepsilon$. Finally, we compute Δ_1 and $\Delta_{1,\varepsilon}$. Notice that $\Delta_{1,\varepsilon}$ is not a symmetric matrix. The loss of symmetry between entries a_{ij} and a_{ji} occurs if $\vec{e}_i$ and $\vec{e}_j$ are nonadjacent edges with opposite signs. Then $a_{ij} = -a_{ji}$. However, we note that $\Delta_{1,\varepsilon}$ is a self-adjoint matrix with respect to the form $\langle\, ,\, \rangle_{1,\varepsilon}$;

that is, $\langle x, \Delta_{1,\varepsilon}(y)\rangle_{1,\varepsilon} = \langle \Delta_{1,\varepsilon}(x), y\rangle_{1,\varepsilon}$. Considering matrices, this is simply $B_{1,\varepsilon}\Delta_{1,\varepsilon} = \Delta_{1,\varepsilon}^t B_{1,\varepsilon}$.

$$\Delta_1 = \begin{pmatrix} 4 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 4 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 4 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 4 & 0 \\ 1 & 1 & 0 & 1 & 0 & -1 & 0 & 4 \end{pmatrix}$$

$$\Delta_{1,\varepsilon} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & -2 & 2 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & -3 & -1 & 2 \\ 0 & 0 & 0 & 1 & -3 & 0 & -2 & 1 \\ 1 & 0 & 0 & 0 & -1 & -2 & 0 & 2 \\ 3 & -1 & 0 & -1 & 2 & 1 & 2 & 0 \end{pmatrix}$$

5. Further Problems and Questions

The preparation of this paper has lead to many more questions than it has answered, and some of the more prominent ones are included below.

Question 1. Is the super-Goeritz class of $(C_1, B_{1,\varepsilon})$ preserved under Reidemeister moves?

We expect the answer to question 1 to be "yes," since the work of Goeritz [3] and Kneser and Puppe [4] showed that the super-Goeritz classes of $(C_0, B_{0,\varepsilon})$ and $(C_2, B_{2,\varepsilon})$ of *knot* diagrams are preserved under Reidemeister moves. Kyle [5] noticed that only Goeritz equivalence is preserved under Reidemeister moves on *link* diagrams.

Question 2. Study the relationship between $(C_1, B_{1,\varepsilon})$ and

$$(C_2/\operatorname{Rad} B_{2,\varepsilon}, B'_{2,\varepsilon}) \oplus (C_0/\operatorname{Rad} B_{0,\varepsilon}, B'_{0,\varepsilon}).$$

Note that the Laplacian of the one-chains has no radical. In Appendix A, we show that, if we extend to rational coefficients for the one-chains, there is a duality with the one-cochains and a basis of the rational vector space of one-chains for which the associated Laplacian is a direct sum.

Question 3. Which based chain complexes arise from link diagrams?

This is such a simple question, but it is unlikely that it has a simple answer. An answer to this question will be a set of necessary and sufficient conditions for a chain complex to represent a plane graph.

The zero-Laplacian contains information on the critical group of a graph. This finite abelian group (defined in [1]) of order equal to the determinant of a principal submatrix of the zero-Laplacian formed by deleting a single row and corresponding column. This is the knot invariant discovered by Goeritz. A potential problem is to study the analogous critical group on the one-chain group $C_1(G)$.

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Appendix A. Rational One-Laplacian

The ordered edge set is not the only possible basis for C_1, and considering an alternative basis puts Δ_1 into a nicer form. However, we must extend the coefficients to include the rational numbers, making $\mathcal{C}[G]$ a chain complex of vector spaces over $\mathbb{Q}$, and choose yet a third duality isomorphism between C_1 and C_1^*.

Definition A.1. Let G be a graph with chain complex $\mathcal{C}[G]$. The *rational face-star basis* for C_1 is the ordered basis

$$\mathcal{B}_1' = (\partial_2(f_1), \partial_2(f_2), \ldots, \partial_2(f_{r-1}), \delta_0(v_1), \delta_0(v_2), \ldots, \delta_0(v_{n-1})).$$

Letting $\mathcal{B}_0'$ and $\mathcal{B}_2'$ be the standard vertex and face bases allows us to denote this modified complex by $\mathcal{C}_{\mathcal{B}'}[G]$.

Using the rational face-star basis, the map ∂_2 takes the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

since the first $r-1$ f_i's now map to basis elements and $\partial_2(f_r)$ can be expressed as $(-1)\sum_{i=1}^{r-1}\partial_2(f_i)$ because $\sum f_i$ is a 2-cycle representing the generator of $\mathrm{H}_2(S^2)\cong\mathbb{Z}$. Since the image of ∂_2 is the kernel of ∂_1, we also see that ∂_1 takes the form

$$\begin{pmatrix} 0 & \cdots & 0 & \\ \vdots & & \vdots & M \\ 0 & \cdots & 0 & \end{pmatrix},$$

where M is an $n\times(n-1)$ matrix.

The third duality isomorphism is defined by declaring the rational face-star basis to be an orthonormal basis for C_1 and using that fact to identify C_1 and C_1^*, it is clear that $(\mathrm{Ad}\langle\,,\,\rangle_1)^{-1}\delta_0\,\mathrm{Ad}\langle\,,\,\rangle_0\partial_1$ is an $m\times m$ matrix with all entries 0 except for the $(n-1)\times(n-1)$ submatrix in the lower right corner, and $\partial_2(\mathrm{Ad}\langle\,,\,\rangle_2)^{-1}\delta_1\,\mathrm{Ad}\langle\,,\,\rangle_1$ is an $m\times m$ matrix with all entries 0 except for the $(r-1)\times(r-1)$ submatrix in the upper left corner. Therefore, Δ_1 is in block form for this choice of orthonormal basis.

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