GROWTH FUNCTIONS OF THE BRAID GROUP

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1. Background

Definition 1. Given a group presentation $G = \langle X|R \rangle$, let $l(g)$ be the length of the word g with respect to the generating set X, the *growth function* is defined by

$$\rho_{G,X}(t) = \sum_{n=0}^{\infty} |\{g : l(g) = n\}| \cdot t^n = \sum_{g \epsilon G} t^{l(g)}$$

Definition 2. A *finite state automaton* is a quintuple (S, Σ, s_0, Y, μ) where

- S is a finite set called the *state set*
- Σ is a finite set called the *alphabet*
- $s_0 \epsilon S$ is the *initial state*
- $Y \subset S$ are the *accept states*
- $\mu : S \times \Sigma \to S$ is the *transition function*

Definition 3. Let M be a finite state automaton with n states, the *transition matrix* A of M is the $n \times n$ matrix with entries $a_{i,j}$ equal to $|\{a \epsilon \Sigma : \mu(s_j, a) = s_i\}|$. Graphically, this is the number of directed edges from s_j to s_i.

Definition 4. Let μ^* be a function $\mu^* : S \times \Sigma^* \to S$ where Σ^* is a monoid. Define μ^* recursively by

$$\mu^*(s_j, w) = \begin{cases} \mu(s_j, a_1) & \text{if } w = a_1 \epsilon \Sigma \\ \mu^*(\mu(s_j, a_1), a_2 ... a_n) & \text{if } w = a_1 a_2 ... a_n \epsilon \Sigma^* \end{cases}$$

We can think of this μ^* as a transition function for words instead of just letters; given a word w, $\mu^*(s_j, w)$ tells us what state the automaton is in after accepting this word w.

Lemma 1. *Let M be a finite state automaton with n states and A be its transition matrix, then A^r has entries $c_{i,j}$ equal to the number of words of length r that begin at s_j and end at s_i. Formally,*

$$c_{i,j} = |\{w \epsilon \Sigma^* : w = a_1 a_2 ... a_r \text{ and } \mu^*(s_j, w) = s_i\}|$$

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Proof. By construction the lemma holds for r equal to one. Assume the lemma holds for $r-1$ then $A^r = A^{r-1}A$ thus each $c_{i,j}$ of A^r is the dot product of the ith row of A^{r-1} and the jth column of A. The ith row of A^{r-1} contains $b_{i,k} = |\{w\epsilon\Sigma^* : w = a_1a_2...a_{r-1}$ and $\mu^*(s_k, w) = s_i\}|$ while the jth column of A contains $a_{k,j} = |\{a\epsilon\Sigma : \mu(s_j, a) = s_k\}|$. Thus their dot product $c_{i,j} = |\{aw\epsilon\Sigma^* : aw = aa_1a_2...a_{r-1}$ and $\mu^*(s_j, aw) = s_i\}|$. □

Theorem 2. [3, 299] *Let M be a finite state automaton with n states and A be its transition matrix, the growth function of M is given by $\rho_M(t) = v(I - tA)^{-1}u^T$ where $v, u\epsilon\mathbb{Z}^n$ with v being the characteristic function of Y and u the characteristic function of s_0.*

Proof. From the previous lemma, we know A^r has entries $b_{i,j}$ equal to the number of words of length exactly r from s_j to s_i. Then $c_i(M) = vA^iu^T$ is the number of strings of length i beginning at s_0 and ending at some accept state, this is simply the number of words of length r accepted by M. Thus

$$\rho_M(t) = \sum_{n=0}^{\infty} c_i t^i = \sum_{n=0}^{\infty} v(tA)^i u^T = v\Big\{\sum_{n=0}^{\infty}(tA)^i\Big\}u^T = v(I - tA)^{-1}u^T$$

□

2. Examples

In the following examples we have ignored the failure states. This simplifies the graphical representation of a finite state automaton, and does not impact our calculations since the fail state is not an accept state and given any letter a, $\mu(s_{fail}, a) = s_{fail}$ thus no word that is accepted by an automaton ends at or goes through the fail state. Also note that in the following diagrams the state label s_i has been reduced to i, and the letters x and x^{-1} have been replaced with x and X.

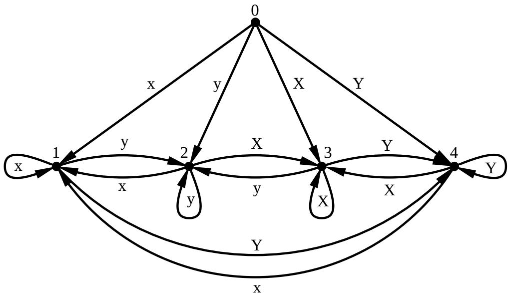
Example 1. Let F_n be the free group on n generators. In F_n, there are $2n$ words of length one (the generators and their inverses). And there are $(2n-1)(2n)$ words of length two, since given a word $w = x_1^{\epsilon_1}x_2^{\epsilon_2}...x_i^{\epsilon_i}$ of length i we can construct a word of length $i+1$ by $wx_{i+1}^{\epsilon_{i+1}}$ where $x_{i+1}^{\epsilon_{i+1}} \neq (x_i^{\epsilon_i})^{-1}$. This allows us to construct $(2n-1)$ words of length $i+1$ given a single root. Thus

$$\rho_{F_n}(t) = 1 + (2n)t + (2n-1)(2n)t^2 + ... + c_it^i + (2n-1)c_it^{i+1} + ...$$

$$\begin{aligned}\rho_{F_n}(t) - (2n-1)t\rho_{F_n}(t) &= [1 + (2n)t + ... + (2n-1)c_it^{i+1} + ...] \\ &- [(2n-1)t + ... + (2n-1)c_it^{i+1} + ...] \\ \rho_{F_n}(t)(1-(2n-1)t) &= 1 + t\end{aligned}$$

$$\rho_{F_n}(t) = \frac{1+t}{1-(2n-1)t}$$

Example 2. $F_2 = \langle x, y\rangle$ gives rise to the following automaton.



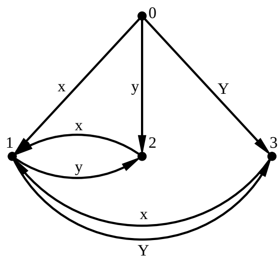
Thus using theorem 2 we have

$$\rho_{F_2}(t) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - t \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rho_{F_2}(t) = 1 + \frac{4t}{1-3t} = \frac{1+t}{1-3t} = \frac{1+t}{1-(2n-1)t}$$

This agrees with the more general formula proven in example 1.

Example 3. $PSL_2\mathbb{Z} = \langle x, y | x^2, y^3 \rangle$ is described in the following automaton since $x = x^{-1}$.



From this we find that each word ending in x can be lengthened by y or y^{-1}, but words ending in y or y^{-1} can only be lengthened by x. Giving the following recursive relationship where c_i are the number of words of length i and $c_{i,x}$ are the words of length i ending in x.

n	$c_i = c_{i,x} + c_{i,(y,y^{-1})}$	$c_{i,x}$	$c_{i,(y,y^{-1})}$
0	1	0	0
1	3	1	2
2	4	2	2
3	6	2	4
4	8	4	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	$\vdots$	$c_{m-1,(y,y^{-1})}$	$2c_{m-1,x}$
m	$2c_{m-2}$	$2c_{m-2,x}$	$2c_{m-2,(y,y^{-1})}$

From this recursive relationship we can find the growth function.

$$\begin{aligned}\rho_{PSL_2\mathbb{Z}}(t) &= 1+3t+4t^2+6t^3+...+2c_{i-2}t^i+... \\ &= 1+(3t+4t^2)(1+2t^2+4t^4+8t^6+...+(2t^2)^i+...) \\ &= 1+(3t+4t^2)(\frac{1}{1-2t^2}) \\ &= \frac{1+3t+2t^2}{1-2t^2}\end{aligned}$$

Which agrees with the following application of theorem 2.

$$\rho_{PSL_2\mathbb{Z}}(t) = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - t \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rho_{PSL_2\mathbb{Z}}(t) = 1 + \frac{2t+2t^2}{1-2t^2} + \frac{t+2t^2}{1-2t^2} = \frac{1+3t+2t^2}{1-2t^2}$$

3. The Braid Group

The braid group B_n is defined to be the group generated by $t_1, t_2...t_{n-1}$ subject to the following relations.

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$
$$t_i t_j = t_j t_i \text{ if } |i-j| > 1$$

Note that t_i is a positive crossing of the $i+1$th strand over the ith strand. A braid written as a product of these positive generators is a positive braid. A permutation braid is a positive braid in which two strands cross at most once. Our aim is to write every positive braid as a unique product of permutation braids. Though it should be noted that permutation braids can be written in multiple ways, for example $t_1t_2t_1 = t_2t_1t_2$.

Definition 5. A positive braid is written in *right greedy canonical form* if it is a product of permutation braids $w_1w_2...w_n$ such that the tail set of w_{k-1} is contained in the initial set of w_k. Geometrically, if two strands that are adjacent at the boundary of w_{k-1} and w_k cross in w_{k-1} they also cross in w_k.

Theorem 3. [2, 193] *Right greedy canonical form is unique.*

To examine the growth function of the braid group, we must first choose permutation braid word representatives denoted P. This set P of permutation braid words must be prefix closed. Then we construct a finite state automaton M that recognizes right greedy canonical form. M must keep track of two permutation braids w_{prev} and w_{curr}. Both w_{prev} and w_{curr} must be elements of P. M will have the following three states.

- *Accept states* will be pairs (w_{prev}, w_{curr}) such that the pair is in right greedy canonical form.

- *Placeholders* will be pairs (w_{prev}, w_{curr}) where w_{curr} is the prefix of some permutation braid $b \epsilon P$ such that (w_{prev}, b) is in right greedy canonical form.
- The *failure state* will contain all other pairs.

In the initial state, w_{prev} and w_{curr} are both the trivial braid. When M is in the state (w_{prev}, w_{curr}) and sees the character t_i, it goes to the state $(w_{prev}, w_{curr}t_i)$ if $w_{curr}t_i \epsilon P$ and to (w_{curr}, t_i) otherwise.

4. Δ-Prime Braids

Definition 6. The *fundamental word* of B_n denoted by

$$\Delta = (t_1)(t_2t_1)(t_3t_2t_1)...(t_{n-1}t_{n-2}...t_2t_1)$$

Geometrically this is the braid in which each strand crosses every other strand exactly once. A positive braid word w is Δ-*prime* if there is no positive word u such that $w = u\Delta$

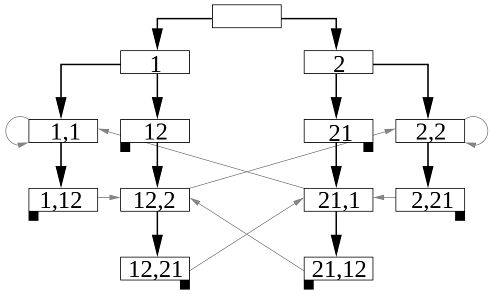
Proposition 1. [1, 203] *Given our set P of permutation braid words we can write every element of B_n in the following form $s\Delta^m$ where s is a positive Δ-prime braid word and $m \epsilon \mathbb{Z}$.*

Let $X_\Delta = \{\Delta, \Delta^{-1}, t_1, t_2, ...t_{n-1}\}$, and note that the braid relations are all length preserving in the monoid X_Δ^*. Thus all normal forms in X_Δ^* are length minimal.

Proposition 2. [1, 203] *If we know the growth function, $\rho_{\Delta-prime}(t)$, of Δ-prime elements, then the growth function of B_n with generating set X_Δ can be found by $\rho_{B_n,X_\Delta} = \frac{1+t}{1-t}\rho_{\Delta-prime}(t)$*

This is pretty clear since given a single Δ-prime element s we can generate $\frac{1+t}{1-t}$ elements in X_Δ^* by multiplying by some power of Δ. But, how do we find the growth function for these Δ-prime braids? We simply take the automaton described in the previous section and send all states (w_{prev}, Δ) to the failure state. The following examples illustrate how this is done.

Example 4. Consider B_3 the braid group on two generators. Given our finite state automaton construction we get the following. Where accept states are the white rectangles, the failure state is represented by the small black rectangles, and there are no placeholders.



Using Theorem 2 and Proposition 2 we find

$$
\begin{aligned}
\rho_{P_3}(t) &= -\frac{t^2+t+1}{t^2+t-1} \\
\rho_{B_3,X_\Delta}(t) &= \frac{(t+1)(t^2+t+1)}{(t-1)(t^2+t-1)} \\
\rho_{B_3,X_\Delta}(t) &= 1+4t+10t^2+20t^3+36t^4+62t^5+...
\end{aligned}
$$

Example 5. The finite state automaton for B_3 only had fourteen states, while the automaton for B_4 has two-hundred and nine. Below is a text formatted automaton where a P denotes a placeholder, and F denotes the failure state.

$(,):$	$(,1)$	$(,2)$	$(,3)$

$(,1):$	$(1,1)$	$(,12)$	$(,13)$
$(,2):$	$(,21)$	$(2,2)$	$(,23)$
$(,3):$	$(3,1)P$	$(,32)$	$(3,3)$

$(1,1):$	$(1,1)$	$(1,12)$	$(1,13)$
$(,12):$	$(,121)$	$(12,2)$	$(,123)$
$(,13):$	$(13,1)P$	$(,132)$	$(13,3)F$
$(,21):$	$(21,1)$	$(21,2)F$	$(,213)$
$(2,2):$	$(2,21)$	$(2,2)$	$(2,23)$
$(,23):$	$(23,1)P$	$(,232)$	$(23,3)$
$(3,1)P:$	$(1,1)$	$(3,12)P$	$(3,13)$
$(,32):$	$(,321)$	$(32,2)$	$(32,3)F$
$(3,3):$	$(3,1)P$	$(3,32)$	$(3,3)$

$(1,12):$	$(1,121)$	$(12,2)$	$(1,123)$
$(1,13):$	$(13,1)P$	$(1,132)$	$(13,3)F$
$(,121):$	$(121,1)P$	$(121,2)F$	$(,1213)$
$(12,2):$	$(12,21)$	$(2,2)$	$(12,23)$
$(,123):$	$(123,1)P$	$(,1232)$	$(123,3)$
$(13,1)P:$	$(1,1)$	$(13,12)P$	$(13,13)$
$(,132):$	$(,1321)$	$(132,2)$	$(132,3)F$
$(21,1):$	$(1,1)$	$(21,12)$	$(21,13)$
$(,213):$	$(213,1)P$	$(,2132)$	$(213,3)F$
$(2,21):$	$(21,1)$	$(21,2)F$	$(2,213)$
$(2,23):$	$(23,1)P$	$(2,232)$	$(23,3)$
$(23,1)P:$	$(1,1)$	$(23,12)P$	$(23,13)$
$(,232):$	$(,2321)$	$(232,2)P$	$(232,3)F$
$(23,3):$	$(3,1)P$	$(23,32)$	$(3,3)$
$(3,12)P:$	$(3,121)F$	$(12,2)$	$(3,123)P$
$(3,13):$	$(13,1)P$	$(3,132)$	$(13,3)F$
$(,321):$	$(321,1)$	$(321,2)F$	$(321,3)F$
$(32,2):$	$(32,21)$	$(2,2)$	$(32,23)$
$(3,32):$	$(3,321)$	$(32,2)$	$(32,3)F$

(1, 121) :	(121, 1)P	(121, 2)F	(1, 1213)
(1, 123) :	(123, 1)P	(1, 1232)	(123, 3)
(1, 132) :	(1, 1321)	(132, 2)	(132, 3)F
(121, 1)P :	(1, 1)	(121, 12)P	(121, 13)F
(, 1213) :	(1213, 1)P	(, 12132)	(1213, 3)F
(12, 21) :	(21, 1)	(21, 2)F	(12, 213)
(12, 23) :	(23, 1)P	(12, 232)	(23, 3)
(123, 1)P :	(1, 1)	(123, 12)P	(123, 13)
(, 1232) :	(, 12321)	(1232, 2)P	(1232, 3)F
(123, 3) :	(3, 1)P	(123, 32)	(3, 3)
(13, 12)P :	(13, 121)F	(12, 2)	(13, 123)P
(13, 13) :	(13, 1)P	(13, 132)	(13, 3)F
(, 1321) :	(1321, 1)P	(1321, 2)F	(1321, 3)F
(132, 2) :	(132, 21)	(2, 2)	(132, 23)
(21, 12) :	(21, 121)	(12, 2)	(21, 123)
(21, 13) :	(13, 1)P	(21, 132)	(13, 3)F
(213, 1)P :	(1, 1)	(213, 12)P	(213, 13)
(, 2132) :	(, 21321)	(2132, 2)	(2132, 3)F
(2, 213) :	(213, 1)P	(2, 2132)	(213, 3)F
(2, 232) :	(2, 2321)	(232, 2)P	(232, 3)F
(23, 12)P :	(23, 121)F	(12, 2)	(23, 123)P
(23, 13) :	(13, 1)P	(23, 132)	(13, 3)F
(, 2321) :	(2321, 1)P	(2321, 2)F	(2321, 3)F
(232, 2)P :	(232, 21)P	(2, 2)	(232, 23)P
(23, 32) :	(23, 321)	(32, 2)	(32, 3)F
(3, 123)P :	(123, 1)P	(3, 1232)	(123, 3)
(3, 132) :	(3, 1321)	(132, 2)	(132, 3)F
(321, 1) :	(1, 1)	(321, 12)	(321, 13)
(32, 21) :	(21, 1)	(21, 2)F	(32, 213)
(32, 23) :	(23, 1)P	(32, 232)	(23, 3)
(3, 321) :	(321, 1)	(321, 2)F	(321, 3)F

(1, 1213) :	(1213, 1)P	(1, 12132)	(1213, 3)F
(1, 1232) :	(1, 12321)	(1232, 2)P	(1232, 3)F
(1, 1321) :	(1321, 1)P	(1321, 2)F	(1321, 3)F
(121, 12)P :	(121, 121)	(12, 2)	(121, 123)F
(1213, 1)P :	(1, 1)	(1213, 12)P	(1213, 13)
(, 12132) :	(, 121321)F	(12132, 2)P	(12132, 3)F
(12, 213) :	(213, 1)P	(12, 2132)	(213, 3)F
(12, 232) :	(12, 2321)	(232, 2)P	(232, 3)F
(123, 12)P :	(123, 121)F	(12, 2)	(123, 123)P
(123, 13) :	(13, 1)P	(123, 132)	(13, 3)F
(, 12321) :	(12321, 1)P	(12321, 2)F	(12321, 3)F
(1232, 2)P :	(1232, 21)P	(2, 2)	(1232, 23)P
(123, 32) :	(123, 321)	(32, 2)	(32, 3)F
(13, 123)P :	(123, 1)P	(13, 1232)	(123, 3)
(13, 132) :	(13, 1321)	(132, 2)	(132, 3)F
(1321, 1)P :	(1, 1)	(1321, 12)P	(1321, 13)F
(132, 21) :	(21, 1)	(21, 2)F	(132, 213)
(132, 23) :	(23, 1)P	(132, 232)	(23, 3)
(21, 121) :	(121, 1)P	(121, 2)F	(21, 1213)
(21, 123) :	(123, 1)P	(21, 1232)	(123, 3)
(21, 132) :	(21, 1321)	(132, 2)	(132, 3)F
(213, 12)P :	(213, 121)F	(12, 2)	(213, 123)P
(213, 13) :	(13, 1)P	(213, 132)	(13, 3)F
(, 21321) :	(21321, 1)P	(21321, 2)F	(21321, 3)F
(2132, 2) :	(2132, 21)	(2, 2)	(2132, 23)
(2, 2132) :	(2, 21321)	(2132, 2)	(2132, 3)F
(2, 2321) :	(2321, 1)P	(2321, 2)F	(2321, 3)F
(23, 123)P :	(123, 1)P	(23, 1232)	(123, 3)
(23, 132) :	(23, 1321)	(132, 2)	(132, 3)F
(2321, 1)P :	(1, 1)	(2321, 12)P	(2321, 13)
(232, 21)P :	(21, 1)	(21, 2)F	(232, 213)P
(232, 23)P :	(23, 1)P	(232, 232)	(23, 3)
(23, 321) :	(321, 1)	(321, 2)F	(321, 3)F
(3, 1232) :	(3, 12321)	(1232, 2)P	(1232, 3)F
(3, 1321) :	(1321, 1)P	(1321, 2)F	(1321, 3)F
(321, 12) :	(321, 121)	(12, 2)	(321, 123)
(321, 13) :	(13, 1)P	(321, 132)	(13, 3)F
(32, 213) :	(213, 1)P	(32, 2132)	(213, 3)F
(32, 232) :	(32, 2321)	(232, 2)P	(232, 3)F

$(1,12132):$	$(1,121321)F$	$(12132,2)P$	$(12132,3)F$
$(1,12321):$	$(12321,1)P$	$(12321,2)F$	$(12321,3)F$
$(121,121):$	$(121,1)P$	$(121,2)F$	$(121,1213)$
$(1213,12)P:$	$(1213,121)F$	$(12,2)$	$(1213,123)P$
$(1213,13):$	$(13,1)P$	$(1213,132)$	$(13,3)F$
$(12132,2)P:$	$(12132,21)P$	$(2,2)$	$(12132,23)P$
$(12,2132):$	$(12,21321)$	$(2132,2)$	$(2132,3)F$
$(12,2321):$	$(2321,1)P$	$(2321,2)F$	$(2321,3)F$
$(123,123)P:$	$(123,1)P$	$(123,1232)$	$(123,3)$
$(123,132):$	$(123,1321)$	$(132,2)$	$(132,3)F$
$(12321,1)P:$	$(1,1)$	$(12321,12)P$	$(12321,13)$
$(1232,21)P:$	$(21,1)$	$(21,2)F$	$(1232,213)P$
$(1232,23)P:$	$(23,1)P$	$(1232,232)$	$(23,3)$
$(123,321):$	$(321,1)$	$(321,2)F$	$(321,3)F$
$(13,1232):$	$(13,12321)$	$(1232,2)P$	$(1232,3)F$
$(13,1321):$	$(1321,1)P$	$(1321,2)F$	$(1321,3)F$
$(1321,12)P:$	$(1321,121)$	$(12,2)$	$(1321,123)F$
$(132,213):$	$(213,1)P$	$(132,2132)$	$(213,3)F$
$(132,232):$	$(132,2321)$	$(232,2)P$	$(232,3)F$
$(21,1213):$	$(1213,1)P$	$(21,12132)$	$(1213,3)F$
$(21,1232):$	$(21,12321)$	$(1232,2)P$	$(1232,3)F$
$(21,1321):$	$(1321,1)P$	$(1321,2)F$	$(1321,3)F$
$(213,123)P:$	$(123,1)P$	$(213,1232)$	$(123,3)$
$(213,132):$	$(213,1321)$	$(132,2)$	$(132,3)F$
$(21321,1)P:$	$(1,1)$	$(21321,12)P$	$(21321,13)F$
$(2132,21):$	$(21,1)$	$(21,2)F$	$(2132,213)$
$(2132,23):$	$(23,1)P$	$(2132,232)$	$(23,3)$
$(2,21321):$	$(21321,1)P$	$(21321,2)F$	$(21321,3)F$
$(23,1232):$	$(23,12321)$	$(1232,2)P$	$(1232,3)F$
$(23,1321):$	$(1321,1)P$	$(1321,2)F$	$(1321,3)F$
$(2321,12)P:$	$(2321,121)F$	$(12,2)$	$(2321,123)P$
$(2321,13):$	$(13,1)P$	$(2321,132)$	$(13,3)F$
$(232,213)P:$	$(213,1)P$	$(232,2132)P$	$(213,3)F$
$(232,232):$	$(232,2321)$	$(232,2)P$	$(232,3)F$
$(3,12321):$	$(12321,1)P$	$(12321,2)F$	$(12321,3)F$
$(321,121):$	$(121,1)P$	$(121,2)F$	$(321,1213)$
$(321,123):$	$(123,1)P$	$(321,1232)$	$(123,3)$
$(321,132):$	$(321,1321)$	$(132,2)$	$(132,3)F$
$(32,2132):$	$(32,21321)$	$(2132,2)$	$(2132,3)F$
$(32,2321):$	$(2321,1)P$	$(2321,2)F$	$(2321,3)F$

$(121,1213):$	$(1213,1)P$	$(121,12132)$	$(1213,3)F$
$(1213,123)P:$	$(123,1)P$	$(1213,1232)$	$(123,3)$
$(1213,132):$	$(1213,1321)$	$(132,2)$	$(132,3)F$
$(12132,21)P:$	$(21,1)$	$(21,2)F$	$(12132,213)P$
$(12132,23)P:$	$(23,1)P$	$(12132,232)$	$(23,3)$
$(12,21321):$	$(21321,1)P$	$(21321,2)F$	$(21321,3)F$
$(123,1232):$	$(123,12321)$	$(1232,2)P$	$(1232,3)F$
$(123,1321):$	$(1321,1)P$	$(1321,2)F$	$(1321,3)F$
$(12321,12)P:$	$(12321,121)F$	$(12,2)$	$(12321,123)P$
$(12321,13):$	$(13,1)P$	$(12321,132)$	$(13,3)F$
$(1232,213)P:$	$(213,1)P$	$(1232,2132)P$	$(213,3)F$
$(1232,232):$	$(1232,2321)$	$(232,2)P$	$(232,3)F$
$(13,12321):$	$(12321,1)P$	$(12321,2)F$	$(12321,3)F$
$(1321,121):$	$(121,1)P$	$(121,2)F$	$(1321,1213)$
$(132,2132):$	$(132,21321)$	$(2132,2)$	$(2132,3)F$
$(132,2321):$	$(2321,1)P$	$(2321,2)F$	$(2321,3)F$
$(21,12132):$	$(21,121321)F$	$(12132,2)P$	$(12132,3)F$
$(21,12321):$	$(12321,1)P$	$(12321,2)F$	$(12321,3)F$
$(213,1232):$	$(213,12321)$	$(1232,2)P$	$(1232,3)F$
$(213,1321):$	$(1321,1)P$	$(1321,2)F$	$(1321,3)F$
$(21321,12)P:$	$(21321,121)$	$(12,2)$	$(21321,123)F$
$(2132,213):$	$(213,1)P$	$(2132,2132)$	$(213,3)F$
$(2132,232):$	$(2132,2321)$	$(232,2)P$	$(232,3)F$
$(23,12321):$	$(12321,1)P$	$(12321,2)F$	$(12321,3)F$
$(2321,123)P:$	$(123,1)P$	$(2321,1232)$	$(123,3)$
$(2321,132):$	$(2321,1321)$	$(132,2)$	$(132,3)F$
$(232,2132)P:$	$(232,21321)$	$(2132,2)$	$(2132,3)F$
$(232,2321):$	$(2321,1)P$	$(2321,2)F$	$(2321,3)F$
$(321,1213):$	$(1213,1)P$	$(321,12132)$	$(1213,3)F$
$(321,1232):$	$(321,12321)$	$(1232,2)P$	$(1232,3)F$
$(321,1321):$	$(1321,1)P$	$(1321,2)F$	$(1321,3)F$
$(32,21321):$	$(21321,1)P$	$(21321,2)F$	$(21321,3)F$

$(121, 12132)$:	$(121, 121321)F$	$(12132, 2)P$	$(12132, 3)F$
$(1213, 1232)$:	$(1213, 12321)$	$(1232, 2)P$	$(1232, 3)F$
$(1213, 1321)$:	$(1321, 1)P$	$(1321, 2)F$	$(1321, 3)F$
$(12132, 213)P$:	$(213, 1)P$	$(12132, 2132)P$	$(213, 3)F$
$(12132, 232)$:	$(12132, 2321)$	$(232, 2)P$	$(232, 3)F$
$(123, 12321)$:	$(12321, 1)P$	$(12321, 2)F$	$(12321, 3)F$
$(12321, 123)P$:	$(123, 1)P$	$(12321, 1232)$	$(123, 3)$
$(12321, 132)$:	$(12321, 1321)$	$(132, 2)$	$(132, 3)F$
$(1232, 2132)P$:	$(1232, 21321)$	$(2132, 2)$	$(2132, 3)F$
$(1232, 2321)$:	$(2321, 1)P$	$(2321, 2)F$	$(2321, 3)F$
$(1321, 1213)$:	$(1213, 1)P$	$(1321, 12132)$	$(1213, 3)F$
$(132, 21321)$:	$(21321, 1)P$	$(21321, 2)F$	$(21321, 3)F$
$(213, 12321)$:	$(12321, 1)P$	$(12321, 2)F$	$(12321, 3)F$
$(21321, 121)$:	$(121, 1)P$	$(121, 2)F$	$(21321, 1213)$
$(2132, 2132)$:	$(2132, 21321)$	$(2132, 2)$	$(2132, 3)F$
$(2132, 2321)$:	$(2321, 1)P$	$(2321, 2)F$	$(2321, 3)F$
$(2321, 1232)$:	$(2321, 12321)$	$(1232, 2)P$	$(1232, 3)F$
$(2321, 1321)$:	$(1321, 1)P$	$(1321, 2)F$	$(1321, 3)F$
$(232, 21321)$:	$(21321, 1)P$	$(21321, 2)F$	$(21321, 3)F$
$(321, 12132)$:	$(321, 121321)F$	$(12132, 2)P$	$(12132, 3)F$
$(321, 12321)$:	$(12321, 1)P$	$(12321, 2)F$	$(12321, 3)F$

$(1213, 12321)$:	$(12321, 1)P$	$(12321, 2)F$	$(12321, 3)F$
$(12132, 2132)P$:	$(12132, 21321)$	$(2132, 2)$	$(2132, 3)F$
$(12132, 2321)$:	$(2321, 1)P$	$(2321, 2)F$	$(2321, 3)F$
$(12321, 1232)$:	$(12321, 12321)$	$(1232, 2)P$	$(1232, 3)F$
$(12321, 1321)$:	$(1321, 1)P$	$(1321, 2)F$	$(1321, 3)F$
$(1232, 21321)$:	$(21321, 1)P$	$(21321, 2)F$	$(21321, 3)F$
$(1321, 12132)$:	$(1321, 121321)F$	$(12132, 2)P$	$(12132, 3)F$
$(21321, 1213)$:	$(1213, 1)P$	$(21321, 12132)$	$(1213, 3)F$
$(2132, 21321)$:	$(21321, 1)P$	$(21321, 2)F$	$(21321, 3)F$
$(2321, 12321)$:	$(12321, 1)P$	$(12321, 2)F$	$(12321, 3)F$

$(12132, 21321)$:	$(21321, 1)P$	$(21321, 2)F$	$(21321, 3)F$
$(12321, 12321)$:	$(12321, 1)P$	$(12321, 2)F$	$(12321, 3)F$
$(21321, 12132)$:	$(21321, 121321)F$	$(12132, 2)P$	$(12132, 3)F$

This automaton is too large for a computer to calculate the inverse in Theorem 2, but we can still find individual coefficients. The following coefficients were found using the technique describe in Theorem 2, explicitly $c_i = vA^i u^T$.

$$\rho_{B_4,X_\Delta}(t) = 1 + 3t + 8t^2 + 20t^3 + 48t^4 + 119t^5 + 302t^6 + 763t^7 + 1932t^8 + ...$$

This disagrees with Brazil's explicit computation suggesting that his automaton needs correction.

5. Conclusion

Although we were able to find the growth function of B_{4,X_Δ} our technique does not scale well. B_4 has twenty-four permutation braids giving rise to a finite state automaton with over two-hundred states. B_4 has one hundred twenty permutation braids suggesting an automaton with about seven thousand states. This is too large for modern computers making it difficult to find a pattern in the braid group. Another technique that might scale better is that of rewriting rules.

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References

[1] Marcus Brazil, *Monoid growth functions for braid groups*, Internat. J. Algebra Comput. **1** (1991), no. 2, 201–205. MR **92h:**20063

[2] David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston, *Word processing in groups*, Jones and Bartlett Publishers, Boston, MA, 1992. MR **93i:**20036

[3] David B. A. Epstein, Anthony R. Iano-Fletcher, and Uri Zwick, *Growth functions and automatic groups*, Experiment. Math. **5** (1996), no. 4, 297–315. MR **98h:**20056

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