

The loss of stability of surface superconductivity

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(Received 9 September 2003; accepted 9 March 2004;

published online 10 June 2004)

The Ginzburg–Landau equations in a half-plane are considered in the large κ limit. We look at the reduced set of equations obtained in that limit. It is proved that the one-dimensional solution presented by Pan [Commun. Math. Phys. **228**, 327 (2002)] undergoes a bifurcation for an infinite number of applied magnetic field values which are lower than H_{C_2} . We also prove that each bifurcating mode is energetically preferable to the one-dimensional surface superconductivity solution, and thus, prove that the surface superconductivity becomes unstable for applied fields which are lower than H_{C_2} . © 2004 American Institute of Physics.

[DOI: 10.1063/1.1755859]

I. INTRODUCTION

Consider a planar superconducting body which is placed at a sufficiently low temperature (below the critical one) under the action of an applied magnetic field. Its energy is given by the Ginzburg–Landau energy functional which can be represented in the following dimensionless form:¹

$$E = \int_{\Omega} \left(-|\Psi|^2 + \frac{|\Psi|^4}{2} + |h - h_{\text{ex}}|^2 + \left| \frac{i}{\kappa} \nabla \Psi + A \Psi \right|^2 \right) dx_1 dx_2, \quad (1.1)$$

in which Ψ is the (complex) superconducting order parameter, such that $|\Psi|$ varies from $|\Psi|=0$ (when the material is at a normal state) to $|\Psi|=1$ (for the purely superconducting state). The magnetic vector potential is denoted by A (the magnetic field is, then, given by $h = \nabla \times A$), h_{ex} is the constant applied magnetic field, and κ is the Ginzburg–Landau parameter which is a material property. Superconductors for which $\kappa < 1/\sqrt{2}$ are called type I superconductors, and those for which $\kappa > 1/\sqrt{2}$ are called type II. Ω is a connected domain of superconductor, whose Gibbs free energy is given by E . Note that E is invariant to the gauge transformation

$$\Psi \rightarrow e^{i\kappa\eta} \Psi; \quad A \rightarrow A + \nabla \eta, \quad (1.2)$$

where η is any smooth function.

For sufficiently large magnetic fields it is well known, both from experimental observations² and both from theoretical predictions,³ that superconductivity is destroyed and the material must be in the normal state. If the applied magnetic field is then decreased there is a critical field where the material enters the superconducting phase once again. This field is called “the onset field” and is denoted by H_{C_3} .

It is well-known that at the bifurcation from the normal state, superconductivity remains concentrated near the boundary. Alternatively we can say that Ψ decays exponentially fast away from the boundaries as either κ or the size of Ω tend to infinity, which is the reason why the phenomenon has been termed surface superconductivity. This result has first been obtained for a half-plane,⁴ then also for disks,⁵ and for general smooth domains in \mathbb{R}^2 .^{6–9} It was extended later to weakly nonlinear cases in the large κ limit.¹⁰

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In the absence of boundaries the critical field at which superconductivity nucleates is denoted by H_{C_2} and is smaller than H_{C_3} ($H_{C_3} \approx 1.7\kappa$ whereas $H_{C_2} = \kappa$). Furthermore, the bifurcating modes are periodic lattices, named after Abrikosov¹¹⁻¹³ which have been observed experimentally.¹⁴ It has been conjectured, therefore, by Rubinstein¹⁵ that superconductivity remains concentrated near the boundary for $H_{C_2} < h_{ex} < H_{C_3}$. When $h_{ex} \approx H_{C_2}$ (either for κ large or for large domains) a bifurcation of Abrikosov's lattices far away from the wall was conjectured.¹⁵

Recently, it has been proved both in the large κ limit,^{16,17} and in the large domain limit¹⁸ that as long as $H_{C_2} < h_{ex} < H_{C_3}$ superconductivity remains concentrated near the boundaries. From a different direction, Sandier and Serfaty¹⁹ showed for the global minimizer of (1.1) that as $h_{ex} \rightarrow H_{C_2}$ from below and $\kappa \rightarrow \infty$, superconductivity vanishes in the domain's interior, away from the boundaries.

Despite the above-mentioned progress the transition from the surface superconductivity solution to the mixed state, where Abrikosov's lattices appear in the bulk of the material, has not been clarified yet. In particular, if the applied magnetic field is decreased below H_{C_2} it has not been proved yet that:

- (1) The surface superconductivity solution becomes unstable, i.e., it is not a local minimizer of E for $h_{ex} < H_{C_2}$.
- (2) The bifurcating mode is indeed the global minimizer and has to be periodic.

In the present contribution we prove, in the large κ limit, for a domain wall, that the surface superconductivity solution in a half-plane is not a local minimizer of E for $h_{ex} < H_{C_2}$, and hence cannot be stable. To this end we assume, just like Pan¹⁶ did, that the global minimizer is essentially one-dimensional in the boundary layer. In addition to the instability proof, we find the bifurcating modes and show, by an heuristic argument, that when properly superposed, Abrikosov's lattices can be formed. However, since linear superposition of modes is impossible, in view of the equation's nonlinearity, further research is necessary in that direction.

The Euler-Lagrange equations associated with (1.1), known as the steady state Ginzburg-Landau equations, are given in the form

$$\left(\frac{i}{\kappa}\nabla + A\right)^2 \Psi = \Psi(1 - |\Psi|^2), \tag{1.3a}$$

$$-\nabla \times \nabla \times A = \frac{i}{2\kappa}(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) + |\Psi|^2 A, \tag{1.3b}$$

and the natural boundary conditions by

$$\left(\frac{i}{\kappa}\nabla + A\right)\Psi \cdot \hat{n} = 0, \tag{1.4a}$$

$$h = h_{ex}. \tag{1.4b}$$

In Refs. 16 and 17 it is proved that as $\kappa \rightarrow \infty$, $h_{ex} - \kappa \gg 1/\kappa$ we have, near the boundary

$$\Psi(x_0 + \xi/\kappa) \xrightarrow{\kappa \rightarrow \infty} \psi(\xi) \quad \text{pointwise,}$$

where $x_0 \in \partial\Omega$, $\xi \in \mathbb{R}_+^2$,

$$\mathbb{R}_+^2 = \{(x_1, x_2); x_1 > 0\}$$

and ψ must satisfy

$$(i\nabla + x_1 \hat{i}_2)^2 \psi = \lambda \psi (1 - |\psi|^2) \quad \text{in } \mathbb{R}_+^2, \tag{1.5a}$$

$$\frac{\partial \psi}{\partial x_1} = 0 \quad \text{on } \partial \mathbb{R}_+^2, \tag{1.5b}$$

where

$$\lambda = \frac{\kappa}{h_{\text{ex}}}.$$

Let

$$\mathcal{H} = \{u \in H^1(z, \infty) \mid xu \in L^2(z, \infty)\}, \tag{1.6}$$

where z is a real number, and let

$$\beta(z) = \inf_{\phi \in \mathcal{H}} \frac{\int_z^\infty (|\phi'|^2 + x^2 |\phi|^2)}{\int_z^\infty |\phi|^2}. \tag{1.7}$$

The dependence of β on z has been studied in Refs. 20 and 21 afterward. In particular, it has been proved that there exist $z_1(\lambda)$ and $z_2(\lambda)$ such that $\lambda > \beta(z)$ if and only if $z_1 < z < z_2$, and that $z_2(1) = 0$ and $z_1(1) = -\infty$. It is also proved in Ref. 7 that

$$\beta_0 = \inf_{z \in \mathbb{R}} \beta(z) = \lim_{\kappa \rightarrow \infty} \frac{\kappa}{H_{C_3}} \approx 0.59.$$

The same result was also proved in Ref. 8.

Pan¹⁶ conjectured that any bounded solution of (1.5a) for $\beta_0 < \lambda \leq 1$ must be in the form

$$\psi = e^{i(\omega_0 x_2 + c)} f(x_1), \tag{1.8}$$

where ω_0 is a real number and $f(x_1, \lambda)$ satisfies in \mathbb{R}_+

$$-f'' + (x - \omega_0)^2 f = \lambda f (1 - f^2); \quad f'(0) = 0. \tag{1.9}$$

In Ref. 16 it is proved that if $\beta(-\omega_0) < \lambda \leq 1$ and

$$-z_2 < \omega_0 < -z_1$$

then there exists a solution for (1.9). Furthermore, it is proved in Ref. 16 the

$$f(x) \sim x^{-1-\lambda/2} e^{-1/2 x^2} \quad \text{as } x \rightarrow \infty. \tag{1.10}$$

The discussion in Ref. 16 was limited to the case $\lambda \leq 1$, since this is the regime where the surface superconductivity solution is expected to be the global minimizer of E . Nevertheless, it is not difficult to show that the above existence result and (1.10) still hold when $\lambda > 1$ for any $\omega_0 \geq 0$. We bring the proof in Appendix A.

Weaker conjectures can be made instead of assuming that (1.8) is the unique class of bounded solutions of (1.5a). Consider the energy functional

$$\mathcal{E}(\psi) = \int_{\mathbb{R}_+^2} |(i\nabla + x_1 \hat{i}_2) \psi|^2 + \lambda \left(\frac{1}{2} |\psi|^4 - |\psi|^2 \right), \tag{1.11}$$

where \hat{i}_2 is a unit vector in the x_2 direction, and let

$$H_{\text{mag}}^1(\Omega) = \{\phi \in L^2(\Omega) \mid \partial_{x_1} \phi \in L^2, \partial_{x_2} \phi + ix_1 \phi \in L^2\}. \tag{1.12}$$

It is well known¹⁶ that when $\lambda > \beta_0$ we have

$$\inf_{\phi \in H_{\text{mag}}^1(\mathbb{R}_+^2)} \mathcal{E}(\phi) = -\infty.$$

We therefore modify the definition of H_{mag}^1 so it would guarantee the existence of a global minimizer to \mathcal{E} in the modified space. We thus apply the transformation

$$x_1 \rightarrow x_1 - \omega_0; \quad \psi \rightarrow e^{-i\omega_0 x_2} \psi$$

to obtain

$$\mathcal{E}(\psi) = \int_{-\omega_0}^{\infty} dx_1 \int_{\mathbb{R}} dx_2 \left\{ |(i\nabla + x_1 \hat{i}_2) \psi|^2 + \lambda \left(\frac{1}{2} |\psi|^4 - |\psi|^2 \right) \right\}, \tag{1.13}$$

and define the space

$$\mathcal{P}_L^{\omega_0} = \{\phi \in H_{\text{mag}}^1([-\omega_0, \infty) \times \mathbb{R}) \mid \phi(x_1, x_2 + L) = \phi(x_1, x_2)\}. \tag{1.14}$$

We can now conjecture, just like Pan¹⁶ did, that

$$\psi = f(x_1, \lambda)$$

is the global minimizer of \mathcal{E} in $\mathcal{P}_L^{\omega_0}$, for every $L > 0$ and $\omega_0 \geq 0$.

We note that Pan¹⁶ studied the same problem for $\lambda > 1$ and found that the global minimizer of (1.11) in $\mathcal{P}_L^{\omega_0}$ decays exponentially fast away from the wall. Moreover, it is proved in Ref. 16 that the global minimizer of (1.1) in a smooth bounded domain must tend, as $\kappa \rightarrow \infty$, to a periodic solution whose period is of $O(\kappa)$.

Periodic solutions have already been studied in the absence of boundaries.¹¹⁻¹³ Periodicity was imposed in those works in both the x_1 and the x_2 directions. In this work we add the effect of a planar wall: We impose periodicity only in the direction which is parallel to the wall, whereas away from the wall we expect the solution to decay. This problem, which is still much simpler than the determination of the global minimizer of (1.1), is much closer to real situations than the problem in \mathbb{R}^2 .¹¹⁻¹³

The present contribution can be summarized by the following theorem.

Theorem 1.1: *There exists $n_0 \in \mathbb{N}$, which may depend on $\omega = 2\pi/L$ and ω_0 , and a sequence $\{\lambda_n\}_{n=n_0}^{\infty}$, such that*

- (1) *There exists a solution to (1.5a) in $\mathcal{P}_L^{\omega_0}$ which bifurcates from $\psi = f(x_1, \lambda)$, in some right semi-neighborhood of λ_n for every $n \geq n_0$.*
- (2) *For every $n \geq n_0$*

$$1 + C_1 \exp\left\{\frac{-(n\omega + \omega_0)^2}{2}\right\} < \lambda_n < 1 + C_2 \exp\left\{\frac{-(n\omega + \omega_0)^2}{2}\right\}, \tag{1.15}$$

where C_1 and C_2 are positive and independent of n .

- (3) *Denote the bifurcating solution in $\mathcal{P}_L^{\omega_0}$ by $\psi_n(x_1, \lambda)$. Then*

$$\mathcal{E}(\psi_n, \lambda) < \mathcal{E}(f, \lambda)$$

in some right semi-neighborhood of λ_n for every $n \geq n_0$.

In the next section we discuss the linearized equation and prove (1.15). Statements 1 and 3 are proved in Sec. III. Finally, in Sec. IV we briefly summarized the results obtained in Secs. II and III and list some related open problems.

II. LINEAR ANALYSIS

Consider the problem

$$(i\nabla + x_1 \hat{i}_2)^2 \psi = \lambda \psi (1 - |\psi|^2) \quad (x_1, x_2) \in (-\omega_0, \infty) \times \mathbb{R}, \tag{2.1a}$$

$$\psi_{x_1}(-\omega_0, x_2) = 0; \quad \psi(x_1, x_2 + L) = \psi(x_1, x_2). \tag{2.1b}$$

Let

$$u = \psi(x_1, x_2) - f(x_1, \lambda), \tag{2.2}$$

wherein f satisfies (1.9). Denote by \mathbb{X} the space

$$\mathbb{X} = \{u \in C^2\{[-\omega_0, \infty) \times \mathbb{R}\} \cap \mathcal{P}_L^{\omega_0} | u_{x_1}(-\omega_0, x_2) = 0\}$$

with the C^2 norm. Let $F: \mathbb{R}^+ \times \mathbb{X} \rightarrow C\{[-\omega_0, \infty) \times \mathbb{R}\}$ be the operator

$$F(\lambda, u) = (i\nabla + x_1 \hat{i}_2)^2 u - \lambda [u - f^2(2u + \bar{u}) - f(2|u|^2 + u^2) - |u|^2 u]. \tag{2.3}$$

Clearly, if $u \in \mathbb{X}$ satisfies $F(u, \lambda) = 0$ for some $\lambda > \beta_0$, then $\psi = u + f$ is a solution of (2.1). Furthermore, since $F(\lambda, 0) \equiv 0$ for all $\lambda > \beta_0$ we can consider the linear bifurcation of nontrivial solutions of $F(u, \lambda) = 0$ from $u \equiv 0$. Let F_u denote the Fréchet derivative of F . Then, the linearized form of $F(u, \lambda) = 0$ near $u \equiv 0$ is

$$F_u(0, \lambda) \phi = 0$$

or

$$(i\nabla + x_1 \hat{i}_2)^2 \phi - \lambda [\phi - f^2(2\phi + \bar{\phi})] = 0. \tag{2.4}$$

Our first result proves the existence of nontrivial solutions in \mathbb{X} for (2.4) and gives the corresponding critical values of λ .

Theorem 2.1: *There exists $n_0(\omega_0, \omega) \in \mathbb{N}$ and a sequence $\{\lambda_n\}_{n=n_0}^\infty$, such that when $\lambda = \lambda_n$ non trivial solutions of (2.4) exist. Furthermore, for all $n \geq n_0$ λ_n satisfies (1.15).*

Proof: Since we look for periodic solutions we multiply (2.4) by $e^{-in\omega x_2}$ where $n \in \mathbb{N}$ and integrate with respect to x_2 over $[-\pi/\omega, \pi/\omega]$ to obtain

$$-\hat{\phi}_n'' + [(x - n\omega)^2 - \lambda] \hat{\phi}_n + \lambda f^2(2\hat{\phi}_n + \overline{\hat{\phi}_{-n}}) = 0, \tag{2.5a}$$

$$-\hat{\phi}_{-n}'' + [(x + n\omega)^2 - \lambda] \hat{\phi}_{-n} + \lambda f^2(2\hat{\phi}_{-n} + \overline{\hat{\phi}_n}) = 0, \tag{2.5b}$$

$$\hat{\phi}_n'(-\omega_0) = \hat{\phi}_{-n}'(-\omega_0) = 0, \tag{2.5c}$$

where

$$\hat{\phi}_n(x_1) = \int_{-\pi/\omega}^{\pi/\omega} \phi(x_1, x_2) e^{-in\omega x_2} dx_2. \tag{2.6}$$

To prove the lower bound in (1.15) we need the following perturbation lemma.

Lemma 2.2: *Let $\mathcal{H}(\xi)$ be defined by (1.6), and let*

$$\alpha(\xi, g) = \inf_{\substack{\phi \in \mathcal{H} \\ \|\phi\|_{L^2(-\xi, \infty)} = 1}} \int_{-\xi}^\infty |\phi'|^2 + (x^2 + g(x + \xi)) |\phi|^2 dx,$$

where $g:[0,\infty)\rightarrow\mathbb{R}$ is continuous and decays as $x\rightarrow\infty$. Then,

$$\alpha = 1 + \int_{-\xi}^{\infty} g v^2 + \delta(\xi). \quad (2.7a)$$

In which v is the quasi-mode

$$v = c_{\xi} \chi(x + \xi) e^{-x^2/2} \quad (2.7b)$$

whose $L^2(-\xi, \infty)$ norm is unity, χ is a C^{∞} cutoff function satisfying

$$\chi = \begin{cases} 0, & 0 \leq x < \frac{1}{2} \\ 1, & 1 < x \end{cases} \quad (2.7c)$$

and, for sufficiently large ξ ,

$$\delta(\xi) \leq 2 \int_{-\xi}^{\infty} g^2 v^2 + C e^{-\xi^2}, \quad (2.7d)$$

where C is independent of ξ .

Proof: Denote by \mathcal{P} the operator

$$\mathcal{P} = -\frac{d^2}{dx^2} + x^2 + g,$$

and let

$$v = 1 + \int_{-\xi}^{\infty} g v^2.$$

Then,

$$(\mathcal{P} - v)v = c_{\xi} [-\chi'' + 2x\chi' + g\chi] e^{-x^2/2} - (v-1)v,$$

and hence,

$$\left| \int_{-\xi}^{\infty} v(\mathcal{P} - v)v \right| \leq \int_{-\xi}^{\infty} \left| -\frac{\chi''}{\chi} + 2x \frac{\chi'}{\chi} \right| v^2 \leq C e^{-\xi^2}. \quad (2.8)$$

Let $\{\mu_j\}_{j=0}^{\infty}$ denote the eigenvalues and $\{u_j\}_{j=0}^{\infty}$ the corresponding eigenmodes, whose L^2 norm is unity, of the following problem:

$$\mathcal{P}u_j = \mu_j u_j, \quad x > \xi,$$

$$u_j'(-\xi) = 0.$$

It is well known,²² that $\mu_j \uparrow \infty$ and that $\{u_j\}_{j=0}^{\infty}$ are square integrable and orthogonal. Let

$$\tilde{v} = v - a_0 u_0, \quad (2.9a)$$

where

$$a_0 = \int_{-\xi}^{\infty} v u_0. \quad (2.9b)$$

Substituting (2.9) in (2.8) we obtain (note that $\mu_0 = \alpha$)

$$a_0^2 |\alpha - \nu| \leq \int_{-\xi}^{\infty} |\tilde{v}(\mathcal{P} - \nu)\tilde{v}| + Ce^{-\xi^2}. \tag{2.10}$$

To estimate the first term on the right-hand side of (2.10) we make use of the following inequality:

$$\int_{-\xi}^{\infty} |(\mathcal{P} - \nu)\tilde{v}|^2 \leq \int_{-\xi}^{\infty} |(\mathcal{P} - \nu)v|^2 = \int_{-\xi}^{\infty} \left| -\frac{\chi''}{\chi} + 2x\frac{\chi'}{\chi} + g - (\nu - 1) \right|^2 v^2 \leq \int_{-\xi}^{\infty} g^2 v^2 + Ce^{-\xi^2}. \tag{2.11}$$

Since the distance of ν from the spectrum of \mathcal{P} in $\mathcal{H} \setminus \text{Span}(u_0)$ is $|\mu_1 - \nu|$ we have

$$(\mu_1 - \nu)^2 \int_{-\xi}^{\infty} |\tilde{v}|^2 \leq \int_{-\xi}^{\infty} |(\mathcal{P} - \nu)\tilde{v}|^2 \leq \int_{-\xi}^{\infty} |(\mathcal{P} - \nu)v|^2. \tag{2.12}$$

It is not difficult to show, using standard arguments from semi-classical analysis (cf. for instance theorem 3.4.1 in Ref. 23), that

$$\mu_1 \xrightarrow{\xi \rightarrow \infty} 3. \tag{2.13}$$

Hence, for sufficiently large ξ ,

$$\int_{-\xi}^{\infty} |\tilde{v}(\mathcal{P} - \nu)\tilde{v}| \leq \left[\int_{-\xi}^{\infty} |\tilde{v}|^2 \int_{-\xi}^{\infty} |(\mathcal{P} - \nu)\tilde{v}|^2 \right]^{1/2} \leq \int_{-\xi}^{\infty} |(\mathcal{P} - \nu)v|^2.$$

Substituting (2.10) in the above inequality yields

$$a_0^2 |\alpha - \nu| \leq \int_{-\xi}^{\infty} g^2 v^2 + Ce^{-\xi^2}. \tag{2.14}$$

By (2.12) and (2.13) we have

$$a_0^2 = 1 - \int_{-\xi}^{\infty} |\tilde{v}|^2 \geq 1 - \int_{-\xi}^{\infty} g^2 v^2 - Ce^{-\xi^2}$$

from which (2.7) can be easily obtained. □

We now continue the proof of Theorem 2.1. Let

$$\alpha_n(\lambda) =: \alpha(-n\omega - \omega_0, \lambda f^2). \tag{2.15}$$

Since, by (1.10)

$$\int_{-n\omega - \omega_0}^{\infty} f^2(x + n\omega + \omega_0) e^{-x^2} \geq C(n\omega + \omega_0)^{\lambda-1} \exp\{-\frac{1}{2}(n\omega + \omega_0)^2\},$$

we have, by (2.7),

$$\alpha_n \geq 1 + C(n\omega + \omega_0)^{\lambda-1} \exp\{-\frac{1}{2}(n\omega + \omega_0)^2\}. \tag{2.16}$$

We now define the functional

$$\begin{aligned} \mathcal{J}(\chi_n, \chi_{-n}) = & \int_{-\omega_0}^{\infty} |\chi'_n|^2 + (x - n\omega)^2 |\chi_n|^2 + |\chi'_{-n}|^2 + (x + n\omega)^2 |\chi_{-n}|^2 - \lambda [|\chi_n|^2 + |\chi_{-n}|^2 \\ & - f^2 (|\chi_n|^2 + |\chi_n + \bar{\chi}_{-n}|^2 + |\chi_{-n}|^2)]. \end{aligned} \tag{2.17}$$

Let $(\phi_n, \phi_{-n}) \in \mathcal{H} \times \mathcal{H}$ satisfy (2.5a) and (2.5b) and

$$\int_{-\omega_0}^{\infty} (|\phi_n|^2 + |\phi_{-n}|^2) = 1.$$

Multiplying (2.5a) by $\bar{\phi}_n$ and the complex conjugate of (2.5b) by ϕ_{-n} and integrating their sum over $[-\omega_0, \infty)$ we obtain

$$\mathcal{J}(\phi_n, \phi_{-n}) = 0.$$

However, from the definition of α_n it follows that

$$\int_{-\omega_0}^{\infty} |\phi'_n|^2 + (x - n\omega)^2 |\phi_n|^2 - \lambda(1 - f^2) |\phi_n|^2 \geq (\alpha_n - \lambda) \int_{-\omega_0}^{\infty} |\phi_n|^2.$$

Furthermore, for sufficiently large n , we have $(x + n\omega)^2 > (x - n\omega)^2$ for every $x \in [-\omega_0, \infty)$, and hence

$$\mathcal{J}(\phi_n, \phi_{-n}) \geq (\alpha_n - \lambda). \tag{2.18}$$

Consequently, the value of λ for which the minimal value of \mathcal{J} vanishes, must be greater than α_n . Therefore, by (2.16) the lower bound in (1.15) is proved.

To prove the upper bound we need, once again, to prove an auxiliary result:

Lemma 2.3: Let

$$\gamma_n(\lambda) = \inf_{\substack{(\chi_n, \chi_{-n}) \in \mathcal{H} \times \mathcal{H} \\ \|\chi_n\|_{L^2}^2 + \|\chi_{-n}\|_{L^2}^2 = 1}} \mathcal{J}(\chi_n, \chi_{-n}).$$

Then,

- (1) For every $\lambda \geq 1$, there exists a minimizer in $\mathcal{H} \times \mathcal{H}$.
- (2) γ_n is a continuous function of λ .

Proof: Since the proof is rather standard, we bring here only the main details and very briefly. Let $\{\phi_n^m, \phi_{-n}^m\}_{m=1}^{\infty}$ denote a minimizing sequence satisfying $\|\phi_n^m\|^2 + \|\phi_{-n}^m\|^2 = 1$ for all m . Obviously,

$$\int_a^{\infty} |\phi_n^m|^2 + |\phi_{-n}^m|^2 \leq \frac{C}{a^2},$$

otherwise $\limsup_{m \rightarrow \infty} \mathcal{J}(\phi_n^m, \phi_{-n}^m) = \infty$. It is easy to show that the minimizing sequence is bounded in $H^1 \times H^1$, and hence, there exists a subsequence which converges weakly to (ϕ_n, ϕ_{-n}) . Clearly,

$$1 \geq \int_{-\omega_0}^a |\phi_n|^2 + |\phi_{-n}|^2 \geq 1 - \frac{C}{a^2},$$

and hence $\|\phi_n\|^2 + \|\phi_{-n}\|^2 = 1$. To complete the proof of existence we need yet to show that E is weakly lower semicontinuous. This, however, is a very simple task. For instance,

$$\int_{-\omega_0}^{\infty} (\phi_n^m)' \overline{\phi_n'} \rightarrow \int_{-\omega_0}^{\infty} |\phi_n'|^2$$

in view of the weak convergence. Applying the Cauchy–Schwarz inequality we obtain

$$\liminf_{m \rightarrow \infty} \|(\phi_n^m)'\| \geq \|\phi_n'\|.$$

Similar treatment can be given to the rest of the terms in (2.17).

The proof that $\gamma_n(\lambda)$ is continuous is completely straightforward. □

We now calculate $\mathcal{J}(w_n, 0)$ where $w_n = \exp\{-(x-n\omega)^2/2\}$. It is not difficult to show that

$$\mathcal{J}(w_n, 0) \leq (1-\lambda)\sqrt{\pi} + \int_{-\omega_0}^{\infty} f^2 w_n^2 + C e^{-(\omega_0+n\omega)^2} \leq (1-\lambda)\sqrt{\pi} + C e^{-(\omega_0+n\omega)^2/2}.$$

Hence, there exists $C > 0$ such that when

$$\lambda > 1 + C e^{-(\omega_0+n\omega)^2/2},$$

we have $\mathcal{J}(w_n, 0) < 0$, and therefore, $\gamma_n(\lambda) < 0$. Since, in view of (2.18), for sufficiently large n , $\gamma_n(\lambda) > 0$ whenever $\lambda < \alpha_n$, and since $\gamma_n(\lambda)$ must be continuous, there exists λ_n satisfying (1.15) and $\gamma_n(\lambda_n) = 0$. By lemma 2.3 there exists a minimizer which must satisfy (2.5), which completes the proof of the theorem. □

We note that the above theorem proves, only for sufficiently large n , that bifurcating modes can exist and that $\lambda_n > 1$. Nevertheless, it seems plausible to conjecture that the bifurcation may take place only for $\lambda > 1$. Furthermore, it appears reasonable to believe that λ_n is monotone decreasing, from which the previous conjecture readily follows.

It still remains necessary to find the dimension of the space of solutions of (2.4) for $\lambda = \lambda_n$. Consider then, (2.5), once again. Let $\phi_n^r = \Re \phi_n$, and $\phi_n^i = \Im \phi_n$. Then, the real part satisfies

$$-(\phi_n^r)'' + [(x-n\omega)^2 - \lambda] \phi_n^r + \lambda f^2 (2\phi_n^r + \phi_{-n}^r) = 0, \tag{2.19a}$$

$$-(\phi_{-n}^r)'' + [(x+n\omega)^2 - \lambda] \phi_{-n}^r + \lambda f^2 (2\phi_{-n}^r + \phi_n^r) = 0, \tag{2.19b}$$

$$(\phi_n^r)'(-\omega_0) = (\phi_{-n}^r)'(-\omega_0) = 0, \tag{2.19c}$$

whereas the imaginary part satisfies

$$-(\phi_n^i)'' + [(x-n\omega)^2 - \lambda] \phi_n^i + \lambda f^2 (2\phi_n^i - \phi_{-n}^i) = 0, \tag{2.20a}$$

$$-(\phi_{-n}^i)'' + [(x+n\omega)^2 - \lambda] \phi_{-n}^i + \lambda f^2 (2\phi_{-n}^i - \phi_n^i) = 0, \tag{2.20b}$$

$$(\phi_n^i)'(-\omega_0) = (\phi_{-n}^i)'(-\omega_0) = 0. \tag{2.20c}$$

Consequently, if (ϕ_n^r, ϕ_{-n}^r) is a solution of (2.19), then $(\phi_n^r, -\phi_{-n}^r)$ is a solution of (2.20). By (2.6) we have

$$\phi = \phi_n e^{in\omega x_2} + \phi_{-n} e^{-in\omega x_2}.$$

Substituting in the above a linear combination of the two independent modes (ϕ_n^r, ϕ_{-n}^r) and $(i\phi_n^r, -i\phi_{-n}^r)$ we obtain

$$\phi = C \phi_n^r e^{in\omega x_2} + \bar{C} \phi_{-n}^r e^{-in\omega x_2},$$

where $C \in \mathbb{C}$ is an arbitrary constant. We can now represent ϕ , upon substituting $C = |C|e^{-ix_2^0}$, in the following form:

$$\phi = |C|[\phi_n^r e^{in\omega(x_2-x_2^0)} + \phi_{-n}^r e^{-in\omega(x_2-x_2^0)}].$$

Consequently, the additional mode stands for translations in the x_2 direction and is, therefore, of very limited interest. Furthermore, since λ_n must be of even multiplicity in \mathbb{X} , it is not possible to apply the Crandall–Rabinowitz theorem.²⁴ Thus, it is desirable to confine the discussion to an appropriate *real* subspace of \mathbb{X} . We thus define

$$\mathbb{X}^* = \{u \in \mathbb{X} | \bar{u}(x_1, x_2) = u(x_1, L - x_2)\}.$$

In this space, we have $\phi_n = \phi_n^r$ for all n , and hence we need only to show that the solution space of (2.19) is one-dimensional.

Lemma 2.4: λ_n is a simple eigenvalue of (2.19).

Proof: Let (ϕ_n, ϕ_{-n}) and $(\tilde{\phi}_n, \tilde{\phi}_{-n})$ be two different solutions of (2.19). We show that they must be linearly dependent. To this end we first multiply (2.19a) by $\tilde{\phi}_n$ to obtain

$$\int_{-\omega_0}^{\infty} f^2(\phi_n \tilde{\phi}_{-n} - \tilde{\phi}_n \phi_{-n}) = 0.$$

Hence,

$$\exists x_0 \in (-\omega_0, \infty) \quad [\phi_n \tilde{\phi}_{-n} - \tilde{\phi}_n \phi_{-n}]_{x=x_0} = 0.$$

Consequently,

$$\exists C \in \mathbb{R} \quad \text{such that} \quad \begin{bmatrix} \chi_n \\ \chi_{-n} \end{bmatrix} = \begin{bmatrix} \phi_n \\ \phi_{-n} \end{bmatrix} + C \begin{bmatrix} \tilde{\phi}_n \\ \tilde{\phi}_{-n} \end{bmatrix} \tag{2.21}$$

vanishes at $x = x_0$. Since (χ_n, χ_{-n}) is a solution of (2.19) we must have $\mathcal{J}(\chi_n, \chi_{-n}) = 0$. Let then,

$$\tilde{\chi}_n = \begin{cases} \chi_n & x_0 \leq x \\ -\chi_n & -\omega_0 \leq x < x_0. \end{cases}$$

Clearly, $\mathcal{J}(\tilde{\chi}_n, \tilde{\chi}_{-n}) = 0$, and hence, $(\tilde{\chi}_n, \tilde{\chi}_{-n})$ is a minimizer, which must have a continuous derivative at x_0 . Consequently, $\chi_n'(x_0) = \chi_{-n}'(x_0) = 0$ from which we conclude that $\chi_n \equiv \chi_{-n} \equiv 0$. □

III. WEAKLY NONLINEAR ANALYSIS

In the previous section, we showed that the linearized equation (2.4) has nontrivial solutions for a sequence of eigenvalues satisfying (1.15). However, our goal is to prove that each of these eigenvalues is a bifurcation point for the nonlinear equation

$$F(\lambda, u) = 0, \tag{3.1}$$

where F is defined in (2.3).

In this section we prove the existence of a bifurcating branch at $(0, \lambda_n)$, for sufficiently large n . Furthermore, we prove that the bifurcation is supercritical and prove that the bifurcating branch is energetically lower than $u \equiv 0$, representing the one-dimensional solution (1.8).

A. Existence of the bifurcation

Theorem 3.1: Equation (3.1) has a bifurcation point at $(0, \lambda_n)$ in \mathbb{W}^* .

Proof: We use Theorem 1.7 in Ref. 24 to prove the existence of bifurcation. In view of the results of the previous section it remains to show that

$$F_{\lambda u} \phi \notin R(F_u(0, \lambda_n)),$$

where ϕ spans the solution space of (2.4) in \mathbb{X}^* at $\lambda = \lambda_n$. Alternatively, we can write

$$\Re \left\{ \int_{-\omega_0}^{\infty} \bar{\phi} F_{\lambda u} \phi \right\} \neq 0. \tag{3.2}$$

The above condition may be applied also by applying to (3.1) the Taylor expansion

$$u = \epsilon u^{(0)} + \epsilon^2 \tilde{u}, \tag{3.3a}$$

$$\lambda = \lambda^{(0)} + \epsilon \tilde{\lambda}. \tag{3.3b}$$

In the above $\lambda^{(0)} = \lambda_n$ satisfies (1.15), and $u^{(0)}$ is a solution of (2.4). Theorem 1.18 in Ref. 24 guarantees that

$$\tilde{u} = u^{(1)} + \epsilon u^{(2)} + O(\epsilon^2), \tag{3.4a}$$

$$\tilde{\lambda} = \lambda^{(1)} + \epsilon \lambda^{(2)} + O(\epsilon^2). \tag{3.4b}$$

This Taylor expansion, in powers of ϵ , would be useful while investigating whether the bifurcation is subcritical or supercritical and while estimating the energy of the bifurcating branch near the bifurcation point.

The $O(\epsilon^2)$ equation is given by

$$\begin{aligned} & (i\nabla + x_1 \hat{i}_2)^2 u^{(1)} - \lambda^{(0)} [u^{(1)} - (f(x, \lambda^{(0)}))^2 (2u^{(1)} + \bar{u}^{(1)})] \\ & = \lambda^{(1)} g_\lambda + \lambda^{(0)} f(x, \lambda^{(0)}) [2|u^{(0)}|^2 + (u^{(0)})^2], \end{aligned} \tag{3.5a}$$

where

$$g_\lambda = \frac{\partial}{\partial \lambda} \{ \lambda [u^{(0)} - (f(x, \lambda))^2 (2u^{(0)} + \bar{u}^{(0)})] \} |_{\lambda = \lambda^{(0)}}, \tag{3.5b}$$

which is exactly equation 1.20 in Ref. 24 applied to our particular case. Multiplying (3.5a) by $\bar{u}^{(0)}$ we obtain after some manipulation that

$$\lambda^{(1)} I_\lambda = \lambda^{(0)} \Re \left\{ \int f |u^{(0)}|^2 (u^{(0)} + 2\bar{u}^{(0)}) \right\}, \tag{3.6a}$$

where

$$I_\lambda = \frac{\partial}{\partial \lambda} \left\{ \lambda \int |u^{(0)}|^2 - \frac{1}{2} f^2(x, \lambda^{(0)}) |u^{(0)} + \bar{u}^{(0)}|^2 - f^2 |u^{(0)}|^2 \right\} \Big|_{\lambda = \lambda^{(0)}}. \tag{3.6b}$$

Condition (3.2) is a solvability condition of (3.5). By (3.6) it can be expressed in the form $I_\lambda \neq 0$.

In the previous section we showed that when $\lambda^{(0)} = \lambda_n$ we have

$$u^{(0)} = \phi_n e^{i\omega n x_2} + \phi_{-n} e^{-i\omega n x_2}, \tag{3.7}$$

where (ϕ_n, ϕ_{-n}) is a solution of (3.6). Hence,

$$I_\lambda = \frac{2\pi}{\omega} \int_{-\omega_0}^\infty (|\phi_n'|^2 + (x-n\omega)^2 |\phi_n|^2 + |\phi_{-n}'|^2 + (x+n\omega)^2 |\phi_{-n}|^2 - 2\lambda_n^2 f \frac{\partial f}{\partial \lambda} (|\phi_n|^2 + |\phi_n + \bar{\phi}_{-n}|^2 + |\phi_{-n}|^2)). \tag{3.8}$$

In the following, we prove that $I_\lambda > 0$. To this end we need first the following lemma.

Lemma 3.2: Let

$$\tilde{\beta}_n = \inf_{\phi \in \mathcal{H}} \frac{\int_{-\omega_0}^\infty (|\phi'|^2 + (x-n\omega)^2 |\phi|^2 - 2\lambda_n^2 f \frac{\partial f}{\partial \lambda} |\phi|^2)}{\int_{-\omega_0}^\infty |\phi|^2}. \tag{3.9}$$

Then, $\lim \tilde{\beta}_n = 1$.

Proof: We first prove that

$$\left\| \frac{\partial f}{\partial \lambda} \right\|_\infty \leq C.$$

The equation satisfied by $\partial f / \partial \lambda = f_\lambda$ is

$$-f_\lambda'' + [x^2 - \lambda + 3\lambda f^2] f_\lambda = f(1 - f^2); \quad f_\lambda'(0) = 0.$$

Clearly, there exists x_0 such that

$$x > x_0 \Rightarrow x^2 - \lambda + 3\lambda f^2 > 1.$$

Suppose now, for a contradiction, that at some $x_1 > x_0$, for some $\lambda = \lambda_0$ we have $f_\lambda(x_1, \lambda_0) > 1$ and $f_\lambda'(x_1, \lambda_0) > 0$. Then, since for $x > x_0$ f_λ cannot have a maximum greater than 1 we must have $f_\lambda(x, \lambda_0) > 1$ for all $x > x_1$. Since both f_λ and f_λ' are continuous in λ there must be a neighborhood $(\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ where $f_\lambda(x_1, \lambda) > 1$ and $f_\lambda'(x_1, \lambda) > 0$. Consequently,

$$x \geq x_1; \lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon) \Rightarrow f_\lambda(x, \lambda) > 1,$$

and hence,

$$f(x, \lambda_0 + \epsilon) - f(x, \lambda_0 - \epsilon) \geq 2\epsilon$$

for all $x \geq x_1$, contradicting (1.10). Thus, for $x > x_0$ we have

$$f_\lambda(x, \lambda) > 1 \Rightarrow f_\lambda'(x, \lambda) \leq 0$$

from which we can conclude that, for $x > x_0$,

$$f_\lambda(x, \lambda) \leq \max(f_\lambda(x_0, \lambda), 1) \leq C(\lambda),$$

where C is independent of x . In a similar manner we can obtain a lower bound for f_λ , and hence,

$$\left| f \frac{\partial f}{\partial \lambda} \right| \leq C x^{(\lambda-1)/2} \exp\{-x^2/2\}. \tag{3.10}$$

The lemma now follows from (2.7) with $g = -2\lambda_n^2 f \partial f / \partial \lambda$ and $\xi = -n\omega - \omega_0$. □

We now return to the proof of Theorem 3.1. From (3.8) it easily follows that

$$I_\lambda \geq \frac{4\pi}{\omega} \tilde{\beta}_n,$$

and hence, for sufficiently large n , I_λ must be positive, which proves our theorem.

B. Nature of the bifurcation

In the following we show that in some neighborhood of $(0, \lambda_n)$ in $X^* \times \mathbb{R}$ we must have $\lambda > \lambda_n$ along the bifurcating branch. Alternatively, we can state that the bifurcation is supercritical. From a physical point of view we can say that if we decrease the applied magnetic field (and consequently increase λ) below the critical field which corresponds to λ_n then the bifurcating branch continues to develop, i.e., $\|u\|$ increases.

Consider then, once again, (3.6). Using (3.7) it is not difficult to show that $\lambda^{(1)} = 0$, which is a natural result in as much as we do not expect the sign of $\lambda - \lambda^{(0)}$ to depend on the sign of ϵ . Hence,

$$(i\nabla + x_1 \hat{i}_2)^2 u^{(1)} - \lambda^{(0)} [u^{(1)} - f^2(2u^{(1)} + \overline{u^{(1)}})] = \lambda^{(0)} f [2|u^{(0)}|^2 + (u^{(0)})^2]. \tag{3.11}$$

The next order equation is given by

$$\begin{aligned} & (i\nabla + x_1 \hat{i}_2)^2 u^{(2)} - \lambda^{(0)} [u^{(2)} - f^2(2u^{(2)} + \overline{u^{(2)}})] \\ & = \lambda^{(2)} g_\lambda - \lambda^{(0)} \{ |u^{(0)}|^2 u^{(0)} + 2f[u^{(0)} \overline{u^{(1)}} + u^{(1)} \overline{u^{(0)}}] + u^{(1)} u^{(0)} \}. \end{aligned} \tag{3.12}$$

The above equation no longer follows directly from Theorem 1.18 in Ref. 24. Nevertheless, it can be easily obtained, using the implicit function Theorem, in the same way it is used in the proof of equation 1.20 in Ref. 24.

Multiplying (3.12) by $\overline{u^{(0)}}$ and integrating by parts we obtain

$$\lambda^{(2)} I_\lambda = \lambda^{(0)} \int |u^{(0)}|^4 + 2f|u^{(0)}|^2(u^{(1)} + \overline{u^{(1)}}) + 2f\overline{u^{(0)}}^2 u^{(1)}. \tag{3.13}$$

We now multiply (3.11) by $u^{(1)}$ to obtain

$$\begin{aligned} & \int |(i\nabla + x_1 \hat{i}_2)u^{(1)}|^2 - \lambda^{(0)} [|u^{(1)}|^2 - 2f^2|u^{(1)}|^2 - \frac{1}{2}f^2(\overline{u^{(1)}})^2 + (u^{(1)})^2] \\ & = \lambda^{(0)} \int 2f|u^{(0)}|^2(u^{(1)} + \overline{u^{(1)}}) + \frac{1}{2}f[\overline{u^{(0)}}^2 u^{(1)} + \overline{u^{(1)}}(u^{(0)})^2]. \end{aligned} \tag{3.14}$$

Hence,

$$\lambda^{(2)} I_\lambda = \lambda^{(0)} \int |u^{(0)}|^4 + 2 \int |(i\nabla + x_1 \hat{i}_2)u^{(1)}|^2 - \lambda^{(0)} [|u^{(1)}|^2 - f^2|u^{(1)}|^2 - \frac{1}{2}f^2|\overline{u^{(1)}} + u^{(1)}|^2]. \tag{3.15}$$

By (2.17) and (3.7) we obtain, that if we substitute $u^{(0)}$ instead of $u^{(1)}$ in the second integral on the right hand side of (3.15) it must vanish identically. Furthermore, except for a finite number of n values, (3.7) must span the solution space of (2.4) in X^* when $\lambda^{(0)} = \lambda_n$. Hence, $u^{(0)}$ must serve as the nontrivial global minimizer of the second integral on the right hand side of (3.15). Consequently,

$$\int |(i\nabla + x_1 \hat{i}_2)u^{(1)}|^2 - \lambda^{(0)}[|u^{(1)}|^2 - f^2|u^{(1)}|^2 - \frac{1}{2}f^2|\overline{u^{(1)}} + u^{(1)}|^2] \geq 0 \tag{3.16}$$

and hence,

$$\lambda^{(2)} \geq \frac{\lambda^{(0)}}{I_\lambda} \int |u^{(0)}|^4 > 0. \tag{3.17}$$

This proves our assertion, namely, that $\lambda > \lambda^{(0)}$ along the bifurcating branch, or, that the bifurcation is supercritical.

C. Energy

In this subsection we prove that (1.8) is not a local minimizer of (1.13) for $\lambda > 1$. To this end we show that for every $n \geq n_0$, there exist a right neighborhood of λ_n in \mathbb{R} such that $\mathcal{E}(f+u, \lambda) < \mathcal{E}(f, \lambda)$. Let

$$\mathcal{E}_0(\lambda) = \mathcal{E}(f, \lambda) = \frac{2\pi}{\omega} \int_{-\omega_0}^\infty |f'| + x^2 f^2 + \lambda \left[\frac{1}{2} f^4 - f^2 \right].$$

Let further

$$\begin{aligned} \Delta \mathcal{E}(u, \lambda) &= \mathcal{E}(f+u) - \mathcal{E}_0(\lambda) \\ &= 2\Re \left\{ \int (i\nabla + x_1 \hat{i}_2)f \cdot (-i\nabla + x_1 \hat{i}_2)\bar{u} \right\} + \int |(i\nabla + x_1 \hat{i}_2)u|^2 + 2\lambda \Re \left\{ \int |f|^2 [f\bar{u} + |u|^2] \right. \\ &\quad \left. + \frac{1}{2} f^2 \bar{u}^2 + |u|^2 [f\bar{u} + \frac{1}{4}|u|^2] - f\bar{u} - \frac{1}{2}|u|^2 \right\}. \end{aligned}$$

Using (1.9) and integration by parts yields

$$\Delta \mathcal{E}(u, \lambda) = \int |(i\nabla + x_1 \hat{i}_2)u|^2 + \lambda \int 2|f|^2|u|^2 + 2|u|^2 \Re(f\bar{u}) + \lambda \int \frac{1}{2}|u|^4 - |u|^2 + \frac{1}{2}|f\bar{u} + \bar{\psi}_0 u|^2. \tag{3.18}$$

We now multiply the (3.1) by \bar{u} and integrate to obtain

$$\int |(i\nabla + x_1 \hat{i}_2)u|^2 = -\lambda \int 2|f|^2|u|^2 + |u|^2(2f\bar{u} + u\bar{\psi}_0) + f^2\bar{u}^2 + |u|^4 - |u|^2.$$

Combining the above with (3.18) we obtain

$$\Delta \mathcal{E}(u, \lambda) = -\frac{1}{2}\lambda \int [|f+u|^2 - |f|^2]|u|^2. \tag{3.19}$$

We now expand $\Delta \mathcal{E}$ in powers of ϵ . By (3.4) we have

$$\Delta \mathcal{E} = \epsilon^3 \Delta \mathcal{E}^{(0)} + \epsilon^4 \Delta \mathcal{E}^{(1)} + O(\epsilon^5).$$

Substituting (3.4) into (3.19) yields

$$\Delta \mathcal{E}^{(0)} = -\lambda^{(0)} \int |u^{(0)}|^2 \Re(f\overline{u^{(0)}}) = 0.$$

Once again, this result corresponds to the natural expectation that the sign of ΔE does not depend on the sign of ϵ . The next order term is expressible in the form

$$\Delta \mathcal{E}^{(1)} = -\lambda^{(0)} \int \frac{1}{2} |u^{(0)}|^4 + |u^{(0)}|^2 \Re(fu^{(1)}) + 2\Re(fu^{(0)}) \Re(u^{(0)}\overline{u^{(1)}}). \tag{3.20}$$

Using (3.14) we obtain

$$\Delta \mathcal{E}^{(1)} = -\lambda^{(0)} \int \frac{1}{2} |u^{(0)}|^4 - \left\{ \int |(i\nabla + x_1 \hat{i}_2)u^{(1)}|^2 - \lambda^{(0)}[|u^{(1)}|^2 - f^2|u^{(1)}|^2 - \frac{1}{2}f^2|\overline{u^{(1)}} + u^{(1)}|^2] \right\}$$

and by (3.16) we have

$$\Delta \mathcal{E}^{(1)} \leq -\frac{1}{2} \lambda^{(0)} \int |u^{(0)}|^4 < 0,$$

which proves our assertion, and hence completes the proof of theorem 1.1.

IV. CONCLUDING REMARKS

In Sec. II we proved the existence of a set of critical values $\{\lambda_n\}_{n=n_0}^\infty$ for which nontrivial solutions of (2.4) exist. We also show that $\lambda_n \downarrow 1$ exponentially fast according to (1.15). However, there might exist, finitely many, additional values of λ for which nontrivial solutions of (2.4) can exist. It would be reasonable to conjecture that $\{\lambda_n\}_{n=1}^\infty$ is monotone decreasing, yet, this hypothesis is proved only for large n . In fact, it is not proved yet that $\lambda_n > 1$ for all n .

One can formulate the above conjecture in the following alternative manner: Let $\omega = 1$ and

$$\gamma(\lambda, \alpha) = \inf_{\substack{(\phi_\alpha, \phi_{-\alpha}) \in \mathcal{H} \times \mathcal{H} \\ \|\phi_\alpha\|_{L^2}^2 + \|\phi_{-\alpha}\|_{L^2}^2 = 1}} \mathcal{J}(\phi_\alpha, \phi_{-\alpha}),$$

where \mathcal{J} is defined in (2.17). We look for values of λ and α for which $\gamma = 0$. For sufficiently large α it is proved in Sec. II that there exists $\alpha_0 > 0$ and a function $\lambda(\alpha): [\alpha_0, \infty) \rightarrow \mathbb{R}$ such that $\gamma(\lambda(\alpha), \alpha) = 0$, and such that $\lambda(\alpha) \downarrow 1$ as $\alpha \rightarrow \infty$. If one can show that $\lambda(\alpha)$ can be continued into \mathbb{R}_+ such that $\lambda(\alpha)$ is monotone decreasing, then the above conjecture is proved.

In Sec. III we proved:

- (1) Existence of the bifurcation points;
- (2) super-criticality of the bifurcation;
- (3) that the bifurcating solution is energetically preferable to the one-dimensional surface superconductivity solution.

Statements 1 and 2 were proved only for sufficiently large n . For n which is not large, the existence of nontrivial solutions of (2.4) does not guarantee $I_\lambda > 0$, and hence the bifurcation points do not necessarily exist. In fact, even if the bifurcation from $(\lambda_n, 0)$ exists, it is not clear that it must be supercritical (if $I_\lambda < 0$ then a subcritical bifurcation exists).

In contrast, statement 3 is correct whenever a bifurcating solution exists. It is correct even for n which is not necessarily large, and even in the unlikely situation that the bifurcation takes place at $\lambda < 1$. The surface superconductivity one-dimensional solution becomes therefore locally unstable at each bifurcation point.

Finally, we note that if it was possible to linearly superpose the bifurcating modes then the resulting combination would have the form

$$\psi = f + \sum_{n=-\infty}^\infty C_n \phi_n e^{i\omega n x_2},$$

and since

$$\phi_n \sim e^{-1/2(x_1 - n\omega)^2}$$

as $n \rightarrow \infty$ for $x_1 \sim O(n)$ we have

$$\psi \sim e^{i\omega Px_2} \sum_{n=-\infty}^{\infty} C_{n+P} e^{i\omega n x_2} e^{-1/2[x_1 - (n+P)\omega]^2} \tag{4.1}$$

for $P \gg 1$ and $x \sim O(P)$. The above formula thus approximates ψ far away from the wall at $x_1 = 0$. If

$$\exists N: C_{n+N} = C_n \quad \forall n.$$

Then the right-hand-side of (4.1) is periodic, or an Abrikosov lattice.¹¹⁻¹³

Clearly, it is impossible to linearly superpose modes since the equations are nonlinear and since the bifurcations take place at different values of λ . Nevertheless, if $0 < \lambda - 1 \ll 1$ then $0 < \lambda - \lambda_n < \lambda - 1$ for almost every n . Hence, one might expect that the effect of nonlinearity tends to 0 as $\lambda \rightarrow 1$, and thus, that the solution far away from the wall can be approximated by an Abrikosov lattice.

ACKNOWLEDGMENTS

The author wishes to thank the referee for his many thoughtful suggestions and comments. This research was supported by the New York Metropolitan research fund at the Technion.

APPENDIX A: THE ONE-DIMENSIONAL SOLUTION

Theorem A.1: Let \mathcal{H} and $\beta(z)$ be, respectively, defined by (1.6) and (1.7). Let $\lambda > \beta(z)$. Then, there exists a positive solution in \mathcal{H} to the equation

$$-f'' + x^2 f = \lambda f(1 - f^2); \quad f'(z) = 0. \tag{A1}$$

Moreover,

$$f(x) \sim x^{-1-\lambda/2} e^{-1/2x^2} \quad \text{as } x \rightarrow \infty. \tag{A2}$$

Proof: Let

$$\epsilon_z(\phi) = \int_z^\infty (\phi')^2 + x^2 \phi^2 - \lambda \left[\phi^2 - \frac{1}{2} \phi^4 \right] dx.$$

We first prove the existence of a minimizer \mathcal{H} . Clearly, there exists $x_0 > z$ such that for $x \geq x_0$ we have $x^2 \geq \lambda + 1$. Then, since

$$\phi^2 - \frac{1}{2} \phi^4 \leq \frac{1}{2}$$

we have for all $\phi \in \mathcal{H}$

$$\epsilon_z(\phi) \geq -\frac{\lambda}{2}(x_0 - z).$$

Since ϵ_z is semibounded there is a minimizing sequence $\{\phi_n\}_{n=1}^\infty$ in \mathcal{H} . As

$$\frac{1}{x_0 - z} \left(\int_z^{x_0} \phi_n^2 \right)^2 \leq \int_z^{x_0} \phi_n^4 \leq 2 \int_z^{x_0} \phi_n^2 + C,$$

where C is independent of n , we obtain that $\|\phi_n\|_{L^2[z, x_0]} \leq C$. Recalling that $x^2 - \lambda \geq 1$ in $[x_0, \infty)$ yields $\|\phi_n\|_{L^2(z, \infty)} \leq C$ and hence $\|\phi_n\|_{H^1(z, \infty)} \leq C$. Thus, there is a subsequence which converges weakly in $H^1(z, \infty)$ to a limit, which we denote by f . We skip the proof of lower semicontinuity—some of the details can be found in the proof of lemma 2.4.

Since f is a minimizer of ϵ_z in \mathcal{H} it must satisfy (A1). Suppose for a contradiction that it changes sign at $x = x_1$. Then let

$$g = \begin{cases} f(x) & x \leq x_1 \\ -f(x) & x > x_1 \end{cases}$$

Clearly, $\epsilon_z(f) = \epsilon_z(g)$, and thus, g must be a minimizer and, therefore, a solution of (A1). Thus, either $f \equiv 0$, or f does not change its sign (both f and $-f$ are minimizers).

We now prove that f is nontrivial. Let u_z be the minimizer of the fraction on the right-hand-side of (1.7) such that $\|u_z\|_2 = 1$. Then

$$\epsilon_z(cu_z) = -c^2[\lambda - \beta(z)] + c^4 \int_z^\infty |u_z|^4.$$

Therefore, for sufficiently small c the minimizer must be nontrivial.

It remains necessary, yet, to prove (A2). We first prove that $f \rightarrow 0$ as $x \rightarrow \infty$. Suppose first, for a contradiction, that for some $x_2 > x_0$ we have $f'(x_2) > 0$. Then, since f cannot have a maximum for $x > x_0$, f must be greater than $f(x_2)$, contradicting $f \in \mathcal{H}$. Thus, $f' \leq 0$ for all $x > x_0$ from which we easily conclude that $f \rightarrow 0$ at infinity.

Let $w(x, t)$, where $t > x_0$, denote the decaying solution of

$$-w'' + [x^2 - \lambda]w = 0; \quad x > t \quad w(t, t) = 1.$$

By the maximum principle we must have that $f < f(t)w$ for all $x > x_0$. Let $f(x) = f(t)v(x, t)w(x, t)$. Substituting in (A1) we obtain

$$v' = -\frac{\lambda}{f(t)w^2} \int_x^\infty f^3(s)w(s, t)ds \geq -\lambda f^2(t) \int_x^\infty w^2(s, t)ds.$$

The properties of w have been obtain in Ref. 21 but can be also found in chapter 19 of Ref. 25. From both references we find that as $t \rightarrow \infty$, $x \rightarrow \infty$

$$w \sim \left(\frac{x}{t}\right)^{\lambda-1/2} e^{-1/2(x^2-t^2)} \quad \forall x \geq t$$

and hence, for all $x > t$

$$-2\lambda f^2(t) \frac{x^{\lambda-2}}{t^{\lambda-1}} e^{t^2-x^2} \leq v' < 0.$$

Since v is decreasing it must converge to a limit as $x \rightarrow \infty$. Integrating the above inequality by parts we obtain

$$v_\infty(t) = \lim_{x \rightarrow \infty} v(x, t) \geq 1 - \lambda \frac{f^2(t)}{t}.$$

For sufficiently large t we, therefore, have $v_\infty(t) > 0$, proving (A2).

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