FACTORIZATION ON A RIEMANN SURFACE IN SCATTERING THEORY

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Summary

Motivated by a problem of scattering theory, the authors solve by quadratures a vector Riemann–Hilbert problem with the matrix coefficient of Chebotarev–Khrapkov type. The problem of matrix factorization reduces to a scalar Riemann–Hilbert boundary-value problem on a two-sheeted Riemann surface of genus 3 that is topologically equivalent to a sphere with three handles. The conditions quenching an essential singularity of the solution at infinity lead to the classical Jacobi inversion problem. It is shown that this problem is equivalent to an algebraic equation of degree that coincides with the genus of the Riemann surface. A closed-form solution of this nonlinear problem is found for genus 3. A normal matrix of factorization and the canonical matrix are constructed in explicit form. It is proved that the vector Riemann–Hilbert problem possesses zero partial indices and is, therefore, stable. The proposed technique is illustrated by a problem of scattering of sound waves by a perforated sandwich panel.

1. Introduction

First, the boundary-value problem $\Phi^+(p) = G_\nu \Phi^-(p) + g_\nu,\ \ p \in \Gamma_\nu$ on a Riemann surface $\mathcal{R}$ cut along Γ_ν, was stated by Prym in 1869. Since that time, many papers devoted to analysis of the problem and its generalizations have been published. The best guide to constructive methods for boundary-value problems in the theory of analytic functions on Riemann surfaces is the paper by Zverovich (**1**). This work provides an excellent survey of papers on that problem; see also (**2** to **4**). We mention works by Čibrikova (**5**), Moiseyev and Popov (**6**), Nuller (**7**), Silvestrov (**8**, **9**), Antipov and Moiseyev (**10**), where boundary-value problems on hyperelliptic surfaces were used for solution of problems of elasticity theory. Riemann surfaces have found effective application to analysis of integrable nonlinear systems (see (**11**, **12**)).

Another important application of the theory of boundary-value problems on Riemann surfaces arises in matrix factorization. Moiseyev (**13**) analysed the problem of splitting the Jones matrices

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(**14**) and, in particular, factorization of the Chebotarev–Khrapkov matrix (**15, 16**)

$$\mathbf{G}(t) = \begin{pmatrix} b(t) + c(t)l(t) & c(t)m(t) \\ c(t)n(t) & b(t) - c(t)l(t) \end{pmatrix}, \tag{1.1}$$

where $b(t), c(t)$ are arbitrary Hölder functions, $l(t), m(t), n(t)$ are polynomials of *any* degree. Moiseyev (**13**) showed that the problem of splitting the Chebotarev–Khrapkov matrix is reducible to a homogeneous scalar Riemann–Hilbert boundary-value problem on a two-sheeted surface of the algebraic function $w^2 = f(s)$, where $f(s) = l^2(s) + m(s)n(s)$. If the Khrapkov restriction (**16**) $\deg f(s) \leqslant 2$ holds, then the genus of the corresponding surface is equal to 0. The surface is equivalent to a sphere, and the matrix can be factorized either by the Khrapkov method (**16**), or by solution of the corresponding Riemann–Hilbert problem on a sphere. In this case, both approaches are equivalent. If $\deg f(s) > 2$, then the Khrapkov method fails since it leads to an essential singularity of the matrix factors at infinity. Daniele (**17**) suggested a scheme which eliminates the exponential growth of the factors. To achieve, at worth, algebraic growth of the factors at infinity, it is necessary and sufficient to find a solution to a system of h equations with $h = [(\deg f(s) - 1)/2]$ ($[x]$ is the integer part of the real number x). This system is essentially nonlinear. Daniele managed to find an exact solution of the system in terms of elliptic functions for the case $h = 1$. In addition, the solvability of the system for the general case was analysed. For $h > 1$, neither analytical technique, nor numerical procedure was indicated.

On the contrary, the idea based on the reduction of matrix factorization to a scalar Riemann–Hilbert boundary-value problem on a hyperelliptic surface of genus $h = [(\deg f(s) - 1)/2]$, leads to a nonlinear system that is different from Daniele's system, and, in fact, is a classical problem, namely, Jacobi's inversion problem (Krazer (**18**), Springer (**19**), Zverovich (**1**)). Moiseyev and Popov (**6**) analysed a contact problem of bending of a semi-infinite plate bonded to an elastic half-space. This problem reduces to a vector Riemann–Hilbert boundary-value problem with the Chebotarev–Khrapkov matrix. The degree of the corresponding polynomial $f(s)$ is equal to 6. Because of the symmetry, the authors managed to eliminate the essential singularity of the solution by inversion of the corresponding abelian integral. Motivated by a two-dimensional problem of a composite plane with a crack crossing the interface, Antipov and Moiseyev (**10**) analysed cases when two copies of the complex plane are glued (i) along two finite cuts and (ii) along four semi-infinite straight segments. The positions of the branch points of the function $f^{\frac{1}{2}}(s)$ dictate the choice of the Riemann surface that is of genus 1 in both cases. Because the condition eliminating the essential singularity at infinity can be satisfied by standard inversion of an abelian integral, there was no need to state the Jacobi inversion problem. The unknown parameters were found in terms of elliptic functions.

If the genus of the surface is greater than 1 and there is no special symmetry, the problem is unlikely to be solved in terms of elliptic functions. The main steps of the algorithm in the case $h > 1$ are

(i) to construct canonical A- and B-cross-sections of the Riemann surface $\mathcal{R}$;
(ii) to choose Weierstrass's kernel (an analogue of the Cauchy kernel of the surface $\mathcal{R}$) and write down a meromorphic solution to a Riemann–Hilbert boundary-value problem which possesses h arbitrary points say, q_j $(j = 1, 2, \ldots, h)$, of the surface $\mathcal{R}$, and $2h$ integers;
(iii) to achieve an algebraic behaviour of the solution at infinity by setting the points q_j and the $2h$ integers to provide a solution to Jacobi's inversion problem;
(iv) to compute the A- and B-periods of the abelian differentials of the first kind (Hensel and

Landsberg (**20**), Schiffer and Spencer (**21**), Springer (**19**)) and normalize the basis of the abelian integrals;

(v) to construct a Riemann Θ-function (Krazer (**18**), Farkas and Kra (**22**)) and find its zeros (which, in fact, give rise to a solution to Jacobi's inversion problem) by solving an algebraic equation of degree h.

Thus, this algorithm provides an exact solution of the Riemann–Hilbert problem on the Riemann surface and, therefore, allows to factorize the original Chebotarev–Khrapkov matrix practically for any deg $f(s)$. In addition, it is possible to construct a normal matrix of factorization and the canonical matrix (Vekua (**23**), Gakhov (**24**), Muskhelishvili (**25**)). The canonical matrix defines the partial indices of the corresponding vector Riemann–Hilbert problem. Moiseyev (**26**) computed the partial indices for the case when the genus of the surface is equal to 1 or 2. In the later case it was assumed that the total index of the vector problem was even. The knowledge of the partial indices is crucial for the solvability theory of the vector Riemann–Hilbert boundary-value problem and for constructing an approximate factorization. It is known (Litvinchuk and Spitkovskiĭ (**27**)) that, in general, the partial indices are unstable. This means that in any neighbourhood of the matrix coefficient $G(t)$ there exists a matrix $G_\varepsilon(t)$ such that the corresponding factorization factors $X_\varepsilon^+(t), X_\varepsilon^-(t)$ do not approach $X^+(t), X^-(t)$ as $\varepsilon \to 0$, where $X^\pm(t)$ are the exact factors for the original matrix $G(t)$. The stability of the partial indices is necessary and sufficient for the stability of the solution of the Riemann–Hilbert problem and substantiation of the convergence of an approximate solution to the exact one. It turns out (Gohberg and Krein (**28**), Vekua (**23**)) that the partial indices are stable if and only if the difference between the largest index and the smallest one is less than or equal to 1. Otherwise they are unstable. However, in general, there is no way to find the partial indices a priori without constructing exact factorization. Approximate factors do not give information about the indices.

To the best of the authors' knowledge, the first paper on approximate matrix factorization, is the work by Mandžavidze (**29**). Babeshko (**30**, **31**) proposed to use a rational approximation for the matrix factorization arising in dynamic problems of linear elasticity. Abrahams (**32**) presented a constructive approximate approach for factorizing the Chebotarev–Khrapkov matrix with the polynomial $f(s)$ of any degree. The method hinges on Padé approximation and has been verified by comparing with Daniele's exact results. The Riemann–Hilbert problem in the case considered by Daniele (**17**) possesses zero partial indices and is therefore stable. Abrahams reported numerical results demonstrating a good convergence of an approximate solution to Daniele's exact solution.

Motivation of the present paper is a problem of scattering by a semi-infinite perforated sandwich panel which is reducible to a vector Riemann–Hilbert problem with the matrix coefficient of Chebotarev–Khrapkov type (Leppington (**33**), Jones (**34**)). Using an asymptotic approach, Jones constructed an approximate solution of the problem for a small parameter τ, which accounts for the perforations. However, this parameter does not admit only small values and can also be large. The corresponding polynomial $f(s)$ of the vector Riemann–Hilbert problem is of degree eight and, therefore, the problem of matrix factorization is equivalent to a scalar boundary-value problem on a hyperelliptic surface of genus 3 that is topologically equivalent to a sphere with three handles.

The main objectives of the present paper are

(i) to develop a method of exact factorization of the Chebotarev–Khrapkov matrix for the case deg $f(s) = 8$;

(ii) to solve Jacobi's inversion problem by quadratures for a hyperelliptic surface of genus 3;

(iii) to find a closed-form solution to the problem of scattering of sound waves by a sandwich panel.

The paper is organized as follows. In section 2, the problem of factorization of the Chebotarev–Khrapkov matrix with deg $f(s) = 8$ is stated. A Riemann surface $\mathcal{R}$ of genus 3 of the algebraic function $w^2 = f(s)$ is constructed. The vector Riemann–Hilbert problem on the real axis reduces to a scalar problem on the hyperelliptic surface $\mathcal{R}$.

Section 3 depicts canonical cross-sections of the surface, derives a meromorphic solution of the scalar Riemann–Hilbert problem and states Jacobi's inversion problem for defining three points q_ν on the surface and six integers n_ν, m_ν $(\nu = 1, 2, 3)$.

In section 4.1, a classical algorithm for Jacobi's inversion problem for a surface of genus h is summarized, and it is shown how to express the points q_j $(j = 1, 2, \dots, h)$ in terms of the zeros of Riemann's Θ-function. The next section describes Zverovich's method that replaces the Jacobi problem by a system of nonlinear algebraic equations for the affixes of the zeros. We present an approach reducing this system to a single algebraic equation of degree h and employ this device for the case $h = 2$ and $h = 3$. Implementation of the algorithm for $h = 4$ is reported in Appendix A. In addition, an alternative direct numerical procedure for identification of the zeros is proposed in Appendix B. This technique is based on the argument principle for an analytic function on a Riemann surface.

We construct a solution to Jacobi's inversion problem for the two-sheeted surface defined by the algebraic function $w^2 = s^8 - 2M_1 s^4 + M_2$ $(M_2 > M_1^2)$, in section 5. First, the A- and B-periods of the abelian integrals are evaluated (section 5.1). Next (section 5.2), we normalize the basis of the abelian integrals and verify the symmetry of the matrix $\boldsymbol{\mathcal{B}}$ and the positive definiteness of the matrix $\mathrm{Im}(\boldsymbol{\mathcal{B}})$, where $\boldsymbol{\mathcal{B}}$ is the matrix of the B-periods of the canonical basis of the abelian differentials. In section 5.3, Jacobi's inversion problem reduces to a cubic equation with the coefficients evaluated in section 5.4. Explicit definition of the integers n_ν, m_ν $(\nu = 1, 2, 3)$ completes the solution of the Jacobi problem.

In section 6, the original vector Riemann–Hilbert problem is solved by quadratures.

A normal matrix of factorization and the canonical matrix are obtained in section 7. It is found that both partial indices of the vector Riemann–Hilbert problem arising in the scattering problem of section 8 are equal to 0. This ensures their stability.

As an illustration of the proposed technique, the problem of scattering of sound waves by a perforated sandwich panel is solved by quadratures in section 8. Concluding remarks are given in the final section.

2. Vector Riemann–Hilbert problem

Let L be the real axis $\mathbb{R}$ and $\mathbb{C}^+, \mathbb{C}^-$ be the upper $(\mathrm{Im}(s) > 0)$ and lower $(\mathrm{Im}(s) < 0)$ half-planes, respectively. Consider the following Riemann–Hilbert problem.

Given a 2×2 matrix $\mathbf{G}(t)$ and a vector $\mathbf{g}(t)$ find two vector functions $\boldsymbol{\Phi}^+(s)$, $\boldsymbol{\Phi}^-(s)$, analytic in the domains $\mathbb{C}^+, \mathbb{C}^-$, respectively, vanishing at infinity and satisfying the boundary condition

$$\mathbf{G}(t)\boldsymbol{\Phi}^+(t) + \boldsymbol{\Phi}^-(t) = \mathbf{g}(t), \quad t \in L. \tag{2.1}$$

The matrix $\mathbf{G}(t)$ and the vector $\mathbf{g}(t)$ satisfy the Hölder condition on every finite segment of L and at infinity $\det \mathbf{G}(t) = 1 + O(t^{-\delta})$ and $\mathbf{g}(t) = O(t^{-\delta})$ $(\delta > 0)$. The matrix $\mathbf{G}(t)$ is also non-singular on L.

Let the matrix $\mathbf{G}(t)$ have the Chebotarev–Khrapkov structure (**15, 16**):

$$\mathbf{G}(t) = b(t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c(t)\begin{pmatrix} l(t) & m(t) \\ n(t) & -l(t) \end{pmatrix}, \tag{2.2}$$

where $b(t), c(t)$ are Hölder functions, $l(t), m(t), n(t)$ are polynomials. Following Khrapkov (**16**) we introduce the notation for the polynomial $f(t) = l^2(t)+m(t)n(t)$ and the characteristic functions (eigenvalues of the matrix $\mathbf{G}(t)$):

$$\lambda_1(t) = b(t) + c(t) f^{\frac{1}{2}}(t), \quad \lambda_2(t) = b(t) - c(t) f^{\frac{1}{2}}(t). \tag{2.3}$$

Note that then $\det \mathbf{G}(t) = \lambda_1(t)\lambda_2(t)$. Our main assumptions are

(i) the polynomial $f(t)$ is of eighth degree:

$$f(t) = t^8 - 2M_1 t^4 + M_2, \quad M_1 \in \mathbb{R}, \quad M_2 > M_1^2; \tag{2.4}$$

(ii) the increment of the arguments of the characteristic functions $\lambda_1(t)$ and $\lambda_2(t)$ when t traverses the contour L in the positive direction, equals zero:

$$\operatorname{ind}_L \lambda_1(t) = \operatorname{ind}_L \lambda_2(t) = 0. \tag{2.5}$$

REMARK The degrees of the polynomials $l(t), m(t), n(t)$ are supposed to be not higher than 4. Among the above restrictions (i), (ii) there is only one essential condition that $\deg f(t) = 8$. The others have been assumed for simplicity and can be always overcome.

To fix a branch of the function $f^{\frac{1}{2}}(s)$, first, we find the eight zeros of the polynomial $f(s)$ which are

$$s_j = \rho_0 e^{i\theta_j}, \quad j = 1, 2, \dots, 8,$$

$$\theta_{2m-1} = \theta_0 + \frac{m-1}{2}\pi, \quad \theta_{2m} = -\theta_0 + \frac{m}{2}\pi, \quad m = 1, 2, 3, 4,$$

$$\rho_0 = M_2^{\frac{1}{8}}, \quad \theta_0 = \frac{1}{4} \arccos \frac{M_1}{M_2^{\frac{1}{2}}}. \tag{2.6}$$

The points s_j $(j = 1, 2, \dots)$ are the branch points of the function $f^{\frac{1}{2}}(s)$. Cut the s-plane along the arcs joining the branch points s_1 with s_2, s_3 with s_4, s_5 with s_6 and s_7 with s_8 (Fig. 1). Put $f^{\frac{1}{2}}(0) > 0$, that is $f^{\frac{1}{2}}(0) = \sqrt{M_2}$. The chosen branch is positive everywhere on the real and imaginary axes:

$$f^{\frac{1}{2}}(t) > 0, \; f^{\frac{1}{2}}(it) > 0 \quad (-\infty < t < \infty).$$

The key step of solution of the vector problem (2.1) is splitting the matrix $\mathbf{G}(t)$ into a product of two matrices

$$\mathbf{G}(t) = \mathbf{X}^+(t)[\mathbf{X}^-(t)]^{-1}, \quad t \in \mathbb{R}, \tag{2.7}$$

which are analytically continuable into the upper and lower half-planes respectively, with the exception maybe of a finite number of poles and points at which the matrices $[\mathbf{X}^+(s)]^{-1}, \mathbf{X}^-(s)$

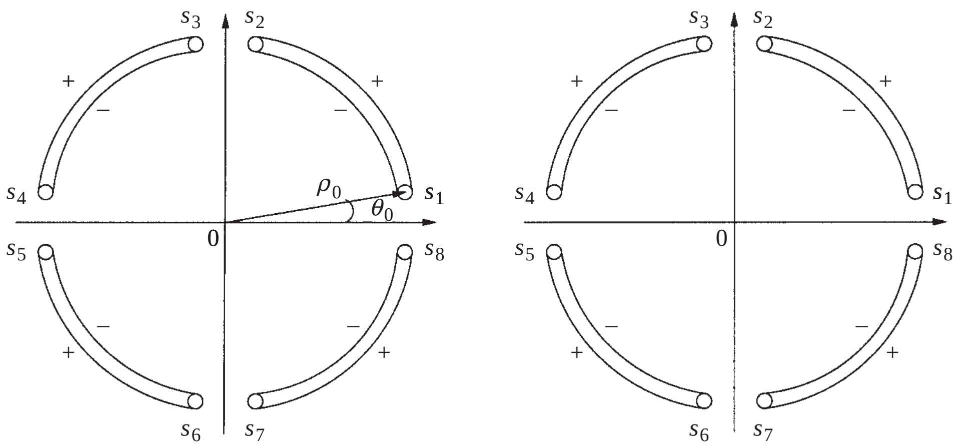

Fig. 1 Two copies of the cut complex planes generating the surface $\mathcal{R}$.

have poles. To do this we reduce the matrix factorization problem (2.1) to a scalar Riemann–Hilbert problem on a Riemann surface.

Let $\mathcal{R}$ be the hyperelliptic surface given by the algebraic equation

$$w^2 = f(s). \tag{2.8}$$

The surface $\mathcal{R}$ is formed by gluing two copies $\mathbb{C}_1$ and $\mathbb{C}_2$ of the extended complex plane $\mathbb{C} \cup \{\infty\}$ cut along the system of the arcs s_1s_2, s_3s_4, s_5s_6, s_7s_8. The positive sides of the cuts in $\mathbb{C}_1$ are glued with the negative sides of the corresponding cuts on the second sheet $\mathbb{C}_2$, and the negative banks of the cuts in $\mathbb{C}_1$ are glued with the positive sides of the corresponding cuts on the sheet $\mathbb{C}_2$. In such a way, we obtain a two-sheeted Riemann surface of genus 3. Now the function $w(s)$ defined by (2.8) is single-valued on the surface $\mathcal{R}$:

$$w = \begin{cases} f^{\frac{1}{2}}(s), & s \in \mathbb{C}_1, \\ -f^{\frac{1}{2}}(s), & s \in \mathbb{C}_2, \end{cases} \tag{2.9}$$

where $f^{\frac{1}{2}}(s)$ is the branch chosen earlier. Denote the point of the surface $\mathcal{R}$ with affix s on $\mathbb{C}_1$ by the pair $(s, f^{\frac{1}{2}}(s))$ and that on the sheet $\mathbb{C}_2$ by the pair $(s, -f^{\frac{1}{2}}(s))$. Define a contour Γ on the surface $\mathcal{R}$ by $\Gamma = L_1 \cup L_2$, where $L_1 = \mathbb{R} \subset \mathbb{C}_1$ and $L_2 = \mathbb{R} \subset \mathbb{C}_2$. The factorization of the matrix $\mathbf{G}(t)$

$$\mathbf{G}(t) = \mathbf{X}^+(t)[\mathbf{X}^-(t)]^{-1} = [\mathbf{X}^-(t)]^{-1}\mathbf{X}^+(t), \quad t \in L, \tag{2.10}$$

in terms of a solution of the Riemann–Hilbert problem on the Riemann surface is given by (**10, 13**)

$$\mathbf{X}(s) = F(s, w)\mathbf{Y}(s, w) + F(s, -w)\mathbf{Y}(s, -w),$$

$$[\mathbf{X}(s)]^{-1} = \frac{\mathbf{Y}(s, w)}{F(s, w)} + \frac{\mathbf{Y}(s, -w)}{F(s, -w)}, \tag{2.11}$$

where

$$\mathbf{Y}(s, w) = \frac{1}{2}\left(\mathbf{I} + \frac{1}{w}\mathbf{Q}(s)\right),$$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} l(s) & m(s) \\ n(s) & -l(s) \end{pmatrix}, \tag{2.12}$$

$F(s, w)$ is a function on the surface $\mathcal{R}$ to be determined. Derivation of formulae (2.11) is rather straightforward. Indeed, by using the identities

$$\mathbf{Q}^2(s) = f(s)\mathbf{I},$$

$$\mathbf{Y}^2(s, w) = \mathbf{Y}(s, w), \quad \mathbf{Y}(s, w)\mathbf{Y}(s, -w) = \mathbf{0} \tag{2.13}$$

from relations (2.11) we get

$$\mathbf{X}^+(t)[\mathbf{X}^-(t)]^{-1} = \frac{1}{2}\left(\frac{F^+(t, \xi)}{F^-(t, \xi)} + \frac{F^+(t, -\xi)}{F^-(t, -\xi)}\right)\mathbf{I} + \frac{1}{2\xi}\left(\frac{F^+(t, \xi)}{F^-(t, \xi)} - \frac{F^+(t, -\xi)}{F^-(t, -\xi)}\right)\mathbf{Q}(t), \quad t \in L, \tag{2.14}$$

where $\xi = f^{\frac{1}{2}}(t)$. Next, by comparing the last relation with formula (2.2) we obtain that in order for the boundary condition (2.10) to hold, we have to put

$$\frac{F^+(t, \xi)}{F^-(t, \xi)} \pm \frac{F^+(t, -\xi)}{F^-(t, -\xi)} = [b(t) + \xi c(t)] \pm [b(t) - \xi c(t)], \quad t \in L. \tag{2.15}$$

By introducing the new function

$$\lambda(t, \xi) = b(t) + \xi c(t), \quad (t, \xi) \in \Gamma, \quad \xi = w(t), \tag{2.16}$$

defined on the contour Γ of the Riemann surface $\mathcal{R}$, we realize that the conditions (2.15) are equivalent to the following Riemann–Hilbert problem on the surface $\mathcal{R}$:

$$F^+(t, \xi) = \lambda(t, \xi)F^-(t, \xi), \quad (t, \xi) \in \Gamma. \tag{2.17}$$

This problem is the subject of the next section.

3. Homogeneous Riemann–Hilbert boundary-value problem on a hyper elliptic surface of genus 3

In this section we aim to analyse the boundary-value problem (2.17) on the surface $\mathcal{R}$ and reduce it to Jacobi's inversion problem. Consider the following problem.

Find a function $F(s, w)$ meromorphic on $\mathcal{R} \setminus \Gamma$ which admits an H-continuous extension to Γ ($F^{\pm}(t, \xi)$ are Hölder functions on Γ) and satisfies the boundary condition

$$F^+(t, \xi) = \lambda(t, \xi)F^-(t, \xi), \quad (t, \xi) \in \Gamma. \tag{3.1}$$

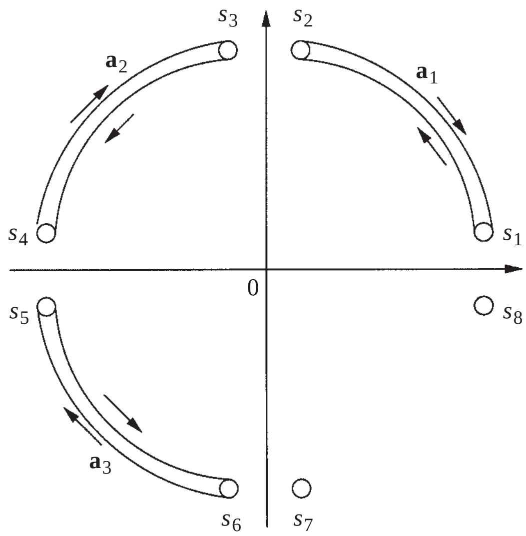

Fig. 2 Canonical cross-sections $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$

The last condition on the contour Γ of the Riemann surface can be rewritten on the contours L_1, L_2 of the sheets $\mathbb{C}_1, \mathbb{C}_2$ as follows:

$$\Xi^+(t) = \lambda_1(t)\Xi^-(t), \quad t \in L_1,$$

$$\Xi^+(t) = \lambda_2(t)\Xi^-(t), \quad t \in L_2 \tag{3.2}$$

with

$$F^\pm(t,\xi) = \begin{cases} \Xi^\pm(t), & \xi = +\sqrt{f(t)}, \quad t \in L_1, \\ \Xi^\pm(t), & \xi = -\sqrt{f(t)}, \quad t \in L_2. \end{cases} \tag{3.3}$$

Let us find a canonical function of the problem (3.1) that is a solution of the problem bounded at infinity and admitting a finite number of poles and zeros.

First, we construct a system of canonical cross-sections of the surface $\mathcal{R}$: $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$. The curves $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are smooth closed contours which coincide with the banks of the cuts s_1s_2, s_3s_4, s_5s_6, respectively. They lie on both sheets $\mathbb{C}_1, \mathbb{C}_2$. These cross-sections $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are indicated on Fig. 2. The positive direction is chosen in such a way that the first sheet $\mathbb{C}_1$ is on the left when a point traverses the contour.

The cross-section $\mathbf{b}_1$ lies on both sheets $\mathbb{C}_1, \mathbb{C}_2$ and consists of the arcs $s_2s_7 \subset \mathbb{C}_1$ and $s_7s_2 \subset \mathbb{C}_2$. The starting point is $s_2 \in \mathbb{C}_1$. Then a point traverses the part of contour $\mathbf{b}_1 \subset \mathbb{C}_1$ to the point $s_7 \in \mathbb{C}_1$, passes to the second sheet $\mathbb{C}_2$ and returns from $s_7 \in \mathbb{C}_2$ to the terminal point $s_2 \in \mathbb{C}_2$ that coincides with the starting point $s_2 \in \mathbb{C}_1$. In Fig. 3, the part of $\mathbf{b}_1$ on $\mathbb{C}_1$ is indicated by the solid line and the rest, lying on $\mathbb{C}_2$, is shown as the broken line.

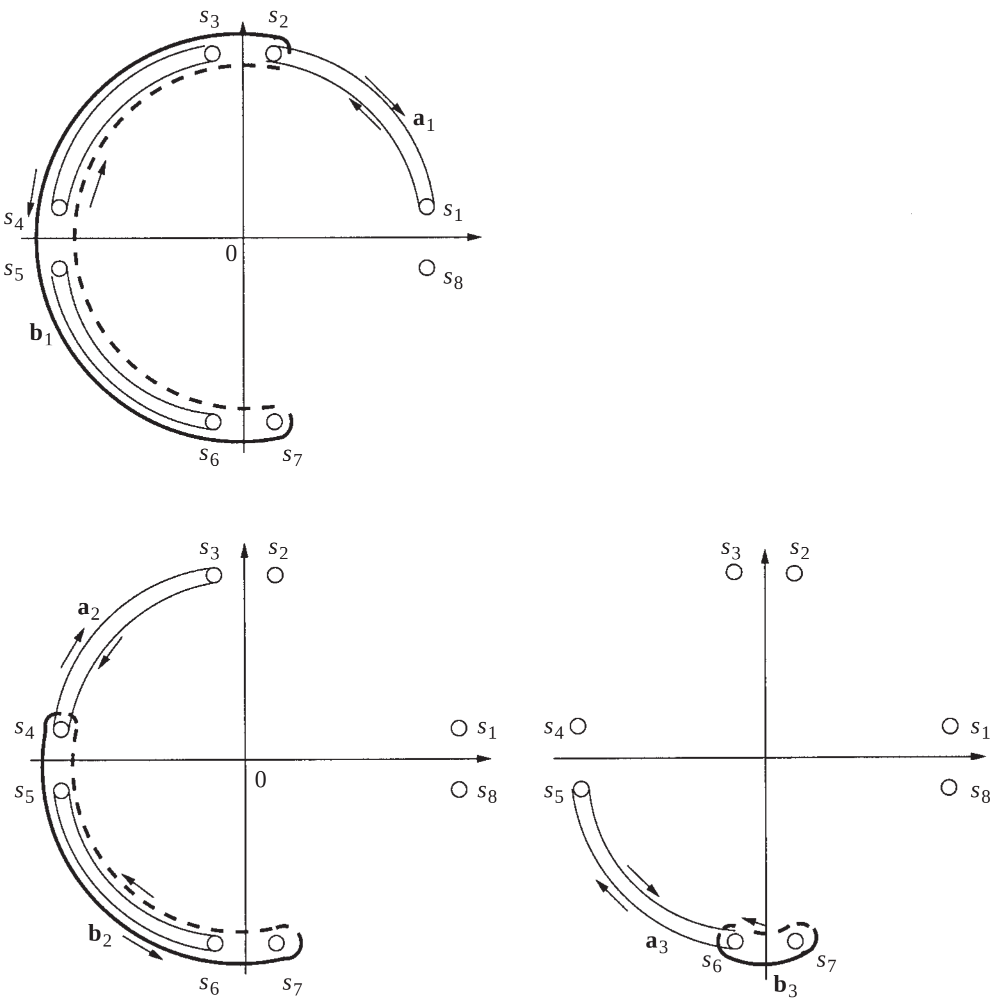

Fig. 3 Canonical cross-sections $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$

The cross-sections $\mathbf{b}_2$ and $\mathbf{b}_3$ are illustrated in Fig. 3 and defined as follows. The contour $\mathbf{b}_2$ consists of the arc s_4s_7 of the first sheet $\mathbb{C}_1$ and the arc s_7s_4 of the second sheet $\mathbb{C}_2$. The positive direction is from the point $s_4 \in \mathbb{C}_1$ to the point $s_7 \in \mathbb{C}_1$ and then from $s_7 \in \mathbb{C}_2$ back to the point $s_4 \in \mathbb{C}_2$. The last cross-section, $\mathbf{b}_3$, is a union of the arcs $s_6s_7 \subset \mathbb{C}_1$ and $s_7s_6 \subset \mathbb{C}_2$. A point traverses the contour in the positive direction with the starting point $s_6 \in \mathbb{C}_1$ and the terminal point $s_6 \in \mathbb{C}_2$. The contour $\mathbf{a}_j$ $(j = 1, 2, 3)$ intersects the curve $\mathbf{b}_j$ from left to the right and there is one point of intersection only. We note that the cross-sections $\mathbf{a}_j$ and $\mathbf{b}_k$ do not intersect if $j \neq k$.

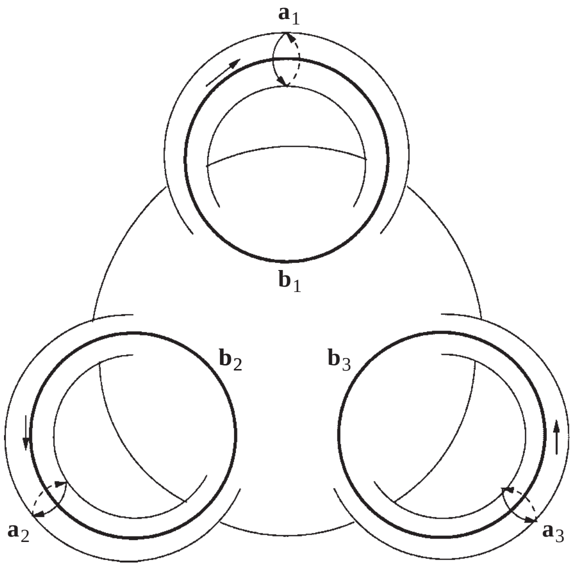

Fig. 4 A sphere with three handles and the canonical cross-sections

Topologically, the hyperelliptic surface $\mathcal{R}$ is equivalent to a sphere with three handles. The loop-cuts $\mathbf{a}_j$ and $\mathbf{b}_j$ $(j = 1, 2, 3)$ are shown schematically in Fig. 4.

The second step of the algorithm of the solution of a Riemann–Hilbert problem on Riemann surfaces is a choice of an analogue of the Cauchy kernel on the surface that is one of Weierstrass's kernels. We shall use the following kernel (**1**):

$$dW = \frac{w+\xi}{2\xi}\frac{dt}{t-s}, \tag{3.4}$$

with $w = (-1)^{j-1} f^{\frac{1}{2}}(s),\ s \in \mathbb{C}_j$ and $\xi = (-1)^{j-1} f^{\frac{1}{2}}(t),\ t \in L_j \subset \mathbb{C}_j\ (j = 1, 2)$. If a point (s, w) is fixed then the differential dW decays for $(t, \xi) \to (\infty, \infty_j)$ as $\frac{1}{2}dt/t$. Here and later we denote by (∞, ∞_1) and (∞, ∞_2) the infinite points of the first and the second sheets $\mathbb{C}_1, \mathbb{C}_2$. On the other hand, if a point (t, ξ) is fixed, then the kernel dW has a pole of the third order at infinity. Therefore we choose the solution to the homogeneous Riemann–Hilbert problem (3.1) in the form which enables us to remove the pole of the solution at infinity (**1**)

$$F(s, w) = \exp\{\varphi(s, w)\}, \quad (s, w) \in \mathcal{R}, \tag{3.5}$$

$$\varphi(s, w) = \frac{1}{2\pi i}\int_\Gamma \log\lambda(t, \xi)dW + \sum_{j=1}^{3}\left[\int_{(\delta_j, v_j)}^{(\sigma_j, w_j)} dW + m_j \oint_{\mathbf{a}_j} dW + n_j \oint_{\mathbf{b}_j} dW\right]. \tag{3.6}$$

The exponent of the first integral,

$$\varphi_0(s, w) = \frac{1}{2\pi i}\int_\Gamma \log\lambda(t, \xi)dW, \tag{3.7}$$

satisfies the boundary condition of the problem (3.1)

$$\exp\{\varphi_0^+(t,\xi)\} = \lambda(t,\xi)\exp\{\varphi_0^-(t,\xi)\}, \quad (t,\xi)\in\Gamma. \tag{3.8}$$

However, since $\varphi_0(s,w)$ grows at infinity as s^3, the function $\exp\{\varphi_0(s,w)\}$ has an essential singularity at infinity. To remove it, we need the second term in the formula (3.6):

$$\varphi_1(s,w) = \sum_{j=1}^{3}\left[\int_{(\delta_j,v_j)}^{(\sigma_j,w_j)} dW + m_j\oint_{\mathbf{a}_j} dW + n_j\oint_{\mathbf{b}_j} dW\right], \tag{3.9}$$

where m_j, n_j are unknown integers. The integral

$$\int_{(\delta_j,v_j)}^{(\sigma_j,w_j)} dW \quad (j=1,2,3) \tag{3.10}$$

is taken over any contour γ_j with starting point (δ_j,v_j) and terminal point (σ_j,w_j). The contours γ_1,γ_2 and γ_3 lie on the surface $\mathcal{R}$ and do not intersect the cross-sections $\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3$ and $\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3$. The points $(\delta_j,v_j)\in\mathbb{C}_1$ $(j=1,2,3)$ are any fixed points such that $(\delta_j,v_j)\notin L_1$ and $v_j = f^{\frac{1}{2}}(\delta_j)$. The points (σ_j,w_j), $(w_j = f^{\frac{1}{2}}(\sigma_j)$ or $w_j = -f^{\frac{1}{2}}(\sigma_j)$ $(j=1,2,3))$ may lie on both sheets of the surface $\mathcal{R}$. The points (σ_j,w_j) $(j=1,2,3)$ will be specified later. The integral (3.10) satisfies the Sokhotski–Plemelj formulae on the surface $\mathcal{R}$ and is therefore discontinuous through the contour γ_j with a jump of $2\pi i$. In a neighbourhood of the end points (δ_j,v_j) and (σ_j,w_j) $(j=1,2,3)$ the integral (3.10) possesses the logarithmic singularities

$$\int_{(\delta_j,v_j)}^{(\sigma_j,w_j)} dW = \begin{cases}\log(s-\sigma_j)+O(1), & (s,w)\to(\sigma_j,w_j),\\ -\log(s-\delta_j)+O(1), & (s,w)\to(\delta_j,v_j).\end{cases} \tag{3.11}$$

At infinity, the integral (3.10) has a pole of the third order. The other integrals

$$\oint_{\mathbf{a}_j} dW \quad \text{and} \quad \oint_{\mathbf{b}_j} dW \tag{3.12}$$

are analytic on the surface $\mathcal{R}$ cut along the contours $\mathbf{a}_j,\mathbf{b}_j$ $(j=1,2,3)$, respectively and discontinuous through the corresponding contours $\mathbf{a}_j,\mathbf{b}_j$ with jumps of $2\pi i$. It is needless to say that both integrals (3.12) have poles of the third order at infinity.

Because the numbers m_j, n_j are integers, the function $F(s,w)$ is continuous through all the contours $\gamma_j,\mathbf{a}_j,\mathbf{b}_j$ $(j=1,2,3)$. At the points (δ_j,v_j) $(j=1,2,3)$ this function has simple poles. (We note that all the points (δ_j,v_j) are distinct.) The points (σ_j,w_j) are zeros of the function $F(s,w)$. Their order depends on whether the points (σ_1,w_1), (σ_2,w_2) and (σ_3,w_3) are different or some of them coincide. It is always possible by choosing the initial points (δ_j,v_j) $(j=1,2,3)$ to avoid the multiplicity of the zeros (σ_j,w_j). In general, the function $\varphi(s,w)$ grows at infinity as s^3. To define the conditions of boundedness of the function $\varphi(s,w)$ at infinity, we rewrite formula (3.6) as follows:

$$\begin{aligned}\varphi(s,w) = {}& \frac{1}{4\pi i}\int_L \log\{\lambda_1(t)\lambda_2(t)\}\frac{dt}{t-s} + \frac{w(s)}{4\pi i}\int_L \log\frac{\lambda_1(t)}{\lambda_2(t)}\frac{dt}{f^{\frac{1}{2}}(t)(t-s)}\\ &+\frac{1}{2}\sum_{j=1}^{3}\left[\int_{\delta_j}^{\sigma_j}\frac{dt}{t-s} + w(s)\left(\int_{(\delta_j,v_j)}^{(\sigma_j,w_j)} + m_j\oint_{\mathbf{a}_j} + n_j\oint_{\mathbf{b}_j}\right)\frac{dt}{\xi(t)(t-s)}\right],\end{aligned} \tag{3.13}$$

and use the identity

$$\frac{1}{t-s} = -\frac{1}{s} - \frac{t}{s^2} - \frac{t^2}{s^3} + \frac{t^3}{s^3(t-s)}. \tag{3.14}$$

The analysis of the behaviour of the function $\varphi(s, w)$ at infinity shows that

$$\varphi(s, w) = -\frac{1}{2}\sum_{\nu=1}^{3}\left[\frac{1}{2\pi i}\int_L \log\frac{\lambda_1(t)}{\lambda_2(t)}\frac{t^{\nu-1}dt}{f^{\frac{1}{2}}(t)} + \sum_{j=1}^{3}\left(\int_{(\delta_j,v_j)}^{(\sigma_j,w_j)} + m_j\oint_{\mathbf{a}_j} + n_j\oint_{\mathbf{b}_j}\right)\frac{t^{\nu-1}dt}{\xi(t)}\right]\frac{w(s)}{s^\nu} + O(1), \quad s\to\infty. \tag{3.15}$$

In order for the function $\varphi(s, w)$ to be bounded at infinity, it is necessary and sufficient that the following three conditions be satisfied:

$$\sum_{j=1}^{3}\left(\int_{(\delta_j,v_j)}^{(\sigma_j,w_j)} d\omega_\nu + m_j\oint_{\mathbf{a}_j} d\omega_\nu + n_j\oint_{\mathbf{b}_j} d\omega_\nu\right) = d_\nu^\circ, \quad \nu = 1, 2, 3, \tag{3.16}$$

where

$$d_\nu^\circ = -\frac{1}{2\pi i}\int_L \log\frac{\lambda_1(t)}{\lambda_2(t)}\frac{t^{\nu-1}}{f^{\frac{1}{2}}(t)}dt,$$

$$d\omega_\nu = \frac{t^{\nu-1}dt}{\xi(t)}. \tag{3.17}$$

The differentials $d\omega_1, d\omega_2, d\omega_3$ form the basis of abelian differentials of the first kind on the surface $\mathcal{R}$. The integrals

$$A_{\nu j} = \oint_{\mathbf{a}_j}\frac{t^{\nu-1}dt}{\xi(t)} \quad \text{and} \quad B_{\nu j} = \oint_{\mathbf{b}_j}\frac{t^{\nu-1}dt}{\xi(t)} \tag{3.18}$$

are the A-periods and the B-periods of the abelian integrals

$$\omega_\nu = \omega_\nu(s, w) = \int_{(s_8,0)}^{(s,w)}\frac{t^{\nu-1}dt}{\xi(t)}, \quad \nu = 1, 2, 3. \tag{3.19}$$

By using the notation (3.17) to (3.19) we rewrite the equations (3.16) as follows:

$$\sum_{j=1}^{3}\big(\omega_\nu(\sigma_j, w_j) + m_j A_{\nu j} + n_j B_{\nu j}\big) = d_\nu^*, \quad \nu = 1, 2, 3, \tag{3.20}$$

where

$$d_\nu^* = d_\nu^\circ + \omega_\nu(\delta_1, v_1) + \omega_\nu(\delta_2, v_2) + \omega_\nu(\delta_3, v_3). \tag{3.21}$$

The nonlinear system (3.20) with respect to the points $(\sigma_j, w_j) \in \mathcal{R}$ and the integers m_j, n_j $(j = 1, 2, 3)$ is Jacobi's inversion problem (**1, 18**).

4. Jacobi's inversion problem

4.1 *Solution in terms of the zeros of Riemann's Θ-function*

This section will summarize the algorithm (**1**) that reduces Jacobi's inversion problem to the finding of the zeros of Riemann's Θ-function. Let $\mathcal{R}$ be a hyperelliptic Riemann surface of genus h defined by the algebraic function

$$w^2 = (s - s_1)(s - s_2)\dots(s - s_{2h+2}), \tag{4.1}$$

where $s_1, s_2, \dots, s_{2h+2}$ are distinct fixed complex numbers. The canonical cross-sections $\mathbf{a}_j, \mathbf{b}_j$ $(j = 1, 2, \dots, h)$ are chosen in the same manner as in section 3. Assume that $\{u_1(q), u_2(q), \dots, u_h(q)\}$ is the normalized basis of the abelian integrals of the first kind. Here and later $q = (s, w) \in \mathcal{R}$. The basis $\{u_\nu(q)\}$ $(\nu = 1, 2, \dots, h)$ has the following A- and B-periods:

$$\mathcal{A}_{\nu j} = \oint_{\mathbf{a}_j} du_\nu(p) = \delta_{j\nu}, \quad \mathcal{B}_{\nu j} = \oint_{\mathbf{b}_j} du_\nu(p) \quad (j, \nu = 1, 2, \dots, h), \tag{4.2}$$

where $p = (t, \xi) \in \mathcal{R}$ and $\delta_{j\nu}$ is Kronecker's symbol. The matrix $\boldsymbol{\mathcal{B}} = (\mathcal{B}_{\nu j})$ is symmetric and $\mathrm{Im}(\boldsymbol{\mathcal{B}})$ is positive definite.

Consider the classical Jacobi inversion problem on the surface $\mathcal{R}$.

Given h constants $d_1, d_2, \dots, d_h$ find h points $q_1, q_2, \dots, q_h$ on the surface $\mathcal{R}$ and $2h$ integers $m_1, m_2, \dots, m_h$ and $n_1, n_2, \dots, n_h$ such that

$$\sum_{j=1}^{h} [u_\nu(q_j) + n_j \mathcal{B}_{\nu j}] + m_\nu = d_\nu, \quad \nu = 1, 2, \dots, h. \tag{4.3}$$

Here $q_j = (\sigma_j, w_j)$, $w_j = w(\sigma_j)$. It was proved (**1, 18**) that the solution of this problem always exists. First, the problem (4.3) can be rewritten in the following form more convenient for further analysis:

$$\sum_{j=1}^{h} [u_\nu(q_j) + n_j \mathcal{B}_{\nu j}] + m_\nu = e_\nu - k_\nu, \quad \nu = 1, 2, \dots, h, \tag{4.4}$$

where $e_\nu = d_\nu + k_\nu$ and k_ν are Riemann's constants of the surface $\mathcal{R}$:

$$k_\nu = -\tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}_{\nu\nu} - \sum_{j=1,\, j\neq\nu}^{h} \oint_{\mathbf{a}_j} u_\nu^-(p)\, du_j(p), \tag{4.5}$$

and $u_\nu^-(p)$ is the limit value of the function $u_\nu(q)$ on the cross-section $\mathbf{a}_j$ from the side of the second sheet $\mathbb{C}_2$:

$$u_\nu^-(p) = \lim_{q \to p,\, q \in \mathbb{C}_2} u_\nu(q). \tag{4.6}$$

Next, we take the Riemann Θ-function

$$\mathfrak{F}(q) = \Theta(u_1(q) - e_1, u_2(q) - e_2, \ldots, u_h(q) - e_h)$$
$$= \sum_{l_1, l_2, \ldots l_h = -\infty}^{\infty} \exp\left\{\pi i \sum_{\mu=1}^{h} \sum_{\nu=1}^{h} \mathcal{B}_{\mu\nu} l_\mu l_\nu + 2\pi i \sum_{\nu=1}^{h} l_\nu [u_\nu(q) - e_\nu]\right\} \tag{4.7}$$

and show that its zeros provide the solution of the inversion problem. Since

$$\sum_{\mu=1}^{h} \sum_{\nu=1}^{h} \operatorname{Im}(\mathcal{B}_{\mu\nu}) l_\mu l_\nu \tag{4.8}$$

is a positive definite form, the series (4.7) converges exponentially everywhere on $\mathcal{R}$ and therefore $\mathfrak{F}(q)$ is an entire function on the surface $\hat{\mathcal{R}}$ formed from the original surface $\mathcal{R}$ by cutting it along the cuts $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_h$. Moreover, this function is single-valued on $\hat{\mathcal{R}}$, is discontinuous through the cuts $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_h$, and satisfies the boundary condition

$$\mathfrak{F}^+(p) = \mathfrak{F}^-(p) \exp\{\pi i \mathcal{B}_{jj} - 2\pi i e_j + 2\pi i u_j^+(p)\}, \quad p \in \mathbf{a}_j, \quad j = 1, 2, \ldots, h. \tag{4.9}$$

Here

$$u_j^+(p) = \lim_{q \to p, q \in \mathbb{C}_1} u_j(q) \tag{4.10}$$

is the limit value of the function $u_j(q)$ on the cross-section $\mathbf{a}_j$ from the side of the sheet $\mathbb{C}_1$. To establish the boundary condition (4.9) it is necessary to use the property of quasi-periodicity of the Θ-function:

$$\Theta(u_1 + \mathcal{B}_{1\nu}, u_2 + \mathcal{B}_{2\nu}, \ldots, u_h + \mathcal{B}_{h\nu}) = \exp(-\pi i \mathcal{B}_{\nu\nu} - 2\pi i u_\nu) \Theta(u_1, u_2, \ldots, u_h). \tag{4.11}$$

Note that the function $\mathfrak{F}(p)$ is continuous through the cross-sections $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_h$ since it is periodic with the unit period in each of its arguments:

$$\Theta(u_1, \ldots, u_\nu + 1, \ldots, u_h) = \Theta(u_1, \ldots, u_\nu, \ldots, u_h). \tag{4.12}$$

The increment of the argument of the function $\exp\{2\pi i u_j^+(p)\}$ along the contour $\mathbf{a}_j$ is 2π. This is because the basis $\{u_\nu\}$ is assumed to be normalized. Therefore the argument principle applied for the cut surface $\hat{\mathcal{R}}$ enables us to find the number of zeros of the function $\mathfrak{F}(q)$ on the surface $\mathcal{R}$. The Θ-function $\mathfrak{F}(q)$ has exactly h zeros (the zeros are counted according to the multiplicity). Let $q_1, q_2, \ldots, q_h$ be the zeros of the function $\mathfrak{F}(q)$ on $\mathcal{R}$. It might turn out that the function $\mathfrak{F}(q)$ is identically equal to zero. The simple criterion whether the Riemann Θ-function is trivial or not was proposed by Zverovich (**1**). Choose any $h + 1$ distinct points on $\hat{\mathcal{R}}$ and observe whether or not all the $h + 1$ values of the function $\mathfrak{F}(q)$ vanish. It is clear that in the first case $\mathfrak{F}(q) \equiv 0$. If the Θ-function is trivial we analyse $h + 1$ values of its partial derivative of the first order with respect to u_j:

$$\frac{\partial \Theta}{\partial u_j}(u_1 - e_1, u_2 - e_2, \ldots, u_h - e_h), \quad j = 1, 2, \ldots, h. \tag{4.13}$$

Each non-trivial function (4.13) has on $\hat{\mathcal{R}}$ precisely h zeros providing the solution of Jacobi's inversion problem. In the case of the problem (3.20) on the surface $\mathcal{R}$ of genus $h = 3$, there is another possibility. The coefficients d_ν^* corresponding to the problem (3.20) depend on the choice of the initial points (δ_j, v_j), $j = 1, 2, 3$. Therefore by changing the position of these points, it is possible to get a new Θ-function to be non-trivial and to avoid analysing the derivatives of the Θ-function.

Next, consider the integral

$$\mathcal{J}_\nu = \frac{1}{2\pi i}\int_{\partial\tilde{\mathcal{R}}} u_\nu(p)\frac{\mathfrak{F}'(p)}{\mathfrak{F}(p)}dp, \tag{4.14}$$

where $\tilde{\mathcal{R}}$ is the surface $\mathcal{R}$ cut along the union of the cross-sections $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_h$, and $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_h$ and $\partial\tilde{\mathcal{R}}$ is the boundary of the surface $\tilde{\mathcal{R}}$. On one hand, by the logarithmic residue theorem, the integral $\mathcal{J}_\nu$ can be expressed through the zeros of the function $\mathfrak{F}(q)$, namely

$$\mathcal{J}_\nu = \sum_{j=1}^{h} u_\nu(q_j). \tag{4.15}$$

On the other hand, the above integral can be written as a sum of the integrals over the contours $\mathbf{a}_j^+, \mathbf{a}_j^-, \mathbf{b}_j^+, \mathbf{b}_j^-$ $(j = 1, 2, \dots, h)$. All the contours $\mathbf{a}_j^+, \mathbf{a}_j^-, \mathbf{b}_j^+, \mathbf{b}_j^-$ are closed curves. The loops $\mathbf{a}_j^+, \mathbf{b}_j^+$ are the left sides of the cross-sections $\mathbf{a}_j, \mathbf{b}_j$, and the other curves $\mathbf{a}_j^-, \mathbf{b}_j^-$ are the right sides of $\mathbf{a}_j, \mathbf{b}_j$, respectively. We mention that $\mathbf{a}_j^+ \subset \mathbb{C}_1$ and $\mathbf{a}_j^- \subset \mathbb{C}_2$ $(j = 1, 2, \dots, h)$. Then we take into account that the abelian integrals of the first kind $u_\nu(p)$ are multiple-valued. Using the boundary condition (4.9) the following expression for the integral (4.14) is found:

$$\mathcal{J}_\nu = -\sum_{j=1}^{h} n_j \mathcal{B}_{\nu j} - m_\nu + e_\nu - k_\nu. \tag{4.16}$$

By comparing (4.15) and (4.16) we arrive at the system of nonlinear equations that is Jacobi's inversion problem (4.4). This means that to solve the problem (4.3), we have to find the zeros of the Riemann Θ-function $\mathfrak{F}(q)$. The final step of the algorithm is to find the integers m_ν and n_ν $(\nu = 1, 2, \dots, h)$ from the system (4.4) which is a linear algebraic system with respect to these integers. Separate the real and imaginary parts of the left- and right-hand-sides of the equations to obtain the system for the integers n_ν:

$$\sum_{j=1}^{h} n_j \operatorname{Im}(\mathcal{B}_{\nu j}) = \operatorname{Im}(\eta_\nu), \quad \nu = 1, 2, \dots, h, \tag{4.17}$$

where

$$\eta_\nu = e_\nu - k_\nu - \sum_{j=1}^{h} u_\nu(q_j). \tag{4.18}$$

We mention that the matrix $\operatorname{Im}(\boldsymbol{\mathcal{B}}) = (\operatorname{Im}\{\mathcal{B}_{\nu j}\})$ $(\nu, j = 1, 2, \dots, h)$ is symmetric and positive definite. The other integers m_ν are defined explicitly:

$$m_\nu = \operatorname{Re}(\eta_\nu) - \sum_{j=1}^{h} n_j \operatorname{Re}(\mathcal{B}_{\nu j}). \tag{4.19}$$

Thus, the key point in the solution of Jacobi's inversion problem is defining the zeros of the Riemann Θ-function. To find them one can use an analytical technique proposed by Zverovich (**1**). It is also possible to apply a direct numerical method based on the argument principle (see Appendix B).

4.2 *Zverovich's procedure*

This method reduces the problem of identification of the zeros to a system of h algebraic equations. Let us describe the procedure. First, consider the auxiliary integral

$$\mathcal{I}_\nu = \frac{1}{2\pi i} \int_{\partial \hat{\mathcal{R}}} t^\nu d \log \mathfrak{F}(p) \tag{4.20}$$

over the boundary of the surface $\mathcal{R}$ cut along the cross-sections $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_h$. By the logarithmic residue theorem

$$\mathcal{I}_\nu = \sum_{j=1}^{h} \sigma_j^\nu + \operatorname*{res}_{q=\infty\in\mathbb{C}_1} \frac{s^\nu \mathfrak{F}'(q)}{\mathfrak{F}(q)} + \operatorname*{res}_{q=\infty\in\mathbb{C}_2} \frac{s^\nu \mathfrak{F}'(q)}{\mathfrak{F}(q)}, \tag{4.21}$$

where $q = (s, w) \in \mathcal{R}$, $w = w(s)$, and $q_j = (\sigma_j, w_j)$ are the zeros of the θ-function $\mathfrak{F}(q)$. On the other hand, the integral $\mathcal{I}_\nu$ can be represented as a sum of the integrals over $\mathbf{a}_j^+$ and $\mathbf{a}_j^-$ $(j = 1, 2, \ldots, h)$:

$$\mathcal{I}_\nu = \frac{1}{2\pi i} \sum_{j=1}^{h} \left(\oint_{\mathbf{a}_j^+} + \oint_{\mathbf{a}_j^-} \right) t^\nu d \log \mathfrak{F}(p). \tag{4.22}$$

The directions of the loops $\mathbf{a}_j^+$ and $\mathbf{a}_j^-$ are opposite (the positive directions on the contours are chosen so that the surface $\hat{\mathcal{R}}$ is on the left when a point traverses the curves). Therefore

$$\mathcal{I}_\nu = \frac{1}{2\pi i} \sum_{j=1}^{h} \oint_{\mathbf{a}_j} t^\nu [d \log \mathfrak{F}^+(p) - d \log \mathfrak{F}^-(p)]. \tag{4.23}$$

By using the boundary condition (4.9) we get

$$\mathcal{I}_\nu = \frac{1}{2\pi i} \sum_{j=1}^{h} \oint_{\mathbf{a}_j} t^\nu d[\pi i \mathcal{B}_{jj} - 2\pi i e_j + 2\pi i u_j^+(p)] = \sum_{j=1}^{h} \oint_{\mathbf{a}_j} t^\nu du_j^+(p). \tag{4.24}$$

Taking into account formula (4.21) we obtain the following nonlinear system of algebraic equations for the affixes $\sigma_1, \sigma_2, \ldots, \sigma_h$ of the points $q_1, q_2, \ldots, q_h$ on the Riemann surface $\mathcal{R}$:

$$\sum_{j=1}^{h} \sigma_j^\nu = \varepsilon_\nu, \quad \nu = 1, 2, \ldots, h, \tag{4.25}$$

where ε_ν are determined by quadratures

$$\varepsilon_\nu = \sum_{j=1}^{h} \oint_{\mathbf{a}_j} t^\nu du_j^+(p) - \operatorname*{res}_{q=\infty\in\mathbb{C}_1} \frac{s^\nu \mathfrak{F}'(q)}{\mathfrak{F}(q)} - \operatorname*{res}_{q=\infty\in\mathbb{C}_2} \frac{s^\nu \mathfrak{F}'(q)}{\mathfrak{F}(q)}. \tag{4.26}$$

The solution of the system (4.25) provides the affixes of the points $q_1 = (\sigma_1, w_1), q_2 = (\sigma_2, w_2), \dots, q_h = (\sigma_h, w_h)$ on the Riemann surface $\mathcal{R}$. Each point q_ν $(\nu = 1, 2, \dots, h)$ may lie on either sheet of the surface $\mathcal{R}$.

The final step of the algorithm is to find the numbers n_ν $(\nu = 1, 2, \dots, h)$ from the linear algebraic system system (4.17); the numbers m_ν are given explicitly by (4.19). Those sets of the points q_j $(j = 1, 2, \dots)$ are a genuine solution for which all the numbers m_j, n_j are integers.

4.3 *Equivalence of Jacobi's inversion problem to an algebraic equation*

In sections 4.1 and 4.2 it was revealed that the classical approach for Jacobi's inversion problem leads, first, to identification of the zeros of Riemann's θ-function and secondly, to solution of the system of h algebraic equations (4.25). Let us show that the latter system is equivalent to an algebraic equation of degree h.

1. Obviously, if $h = 2$, then the system (4.25) reduces to the quadratic equation

$$2\sigma^2 - 2\varepsilon_1\sigma + \varepsilon_1^2 - \varepsilon_2 = 0.$$

2. For $h = 3$, we notice that from the first and second equations in (4.25) the unknowns σ_2 and σ_3 can be expressed in terms of σ_1 only:

$$\sigma_2 = \frac{1}{2}\left(\varepsilon_1 - \sigma_1 + \sqrt{\Delta(\sigma_1)}\right), \quad \sigma_3 = \frac{1}{2}\left(\varepsilon_1 - \sigma_1 - \sqrt{\Delta(\sigma_1)}\right), \tag{4.27}$$

where $\sqrt{\Delta(s)}$ is any fixed branch of the function $\Delta^{\frac{1}{2}}(s)$ and

$$\Delta(\sigma) = -3\sigma^2 + 2\varepsilon_1\sigma - \varepsilon_1^2 + 2\varepsilon_2. \tag{4.28}$$

By substituting formulae (4.27) into the last equation (4.25), we arrive at the cubic equation with respect to σ_1:

$$6\sigma^3 - 6\varepsilon_1\sigma^2 + 3(\varepsilon_1^2 - \varepsilon_2)\sigma - \varepsilon_1^3 + 3\varepsilon_1\varepsilon_2 - 2\varepsilon_3 = 0. \tag{4.29}$$

For each root $\sigma_1^{(\nu)}(\nu = 1, 2, 3)$ of this equation, there is a definite pair $(\sigma_2^{(\nu)}, \sigma_3^{(\nu)})$ with $\sigma_2^{(\nu)}, \sigma_3^{(\nu)}$ determined by formulae (4.27). However, due to the symmetry of the system (4.25), all the triples $(\sigma_1^{(\nu)}, \sigma_2^{(\nu)}, \sigma_3^{(\nu)})$ $(\nu = 1, 2, 3)$ coincide. Thus, the three roots of the cubic equation (4.29) provide the solution of the system (4.25): $\sigma_1 = \sigma_1^{(1)}, \sigma_2 = \sigma_1^{(2)}, \sigma_3 = \sigma_1^{(3)}$.

3. In general, applying the Viéte theorem yields an algebraic equation of degree h. Realization of this idea for the case $h = 4$ is reported in Appendix A.

5. Solution of Jacobi's inversion problem for the hyperelliptic surface of genus 3 by quadratures

The main aim of this section is to construct an efficient solution of Jacobi's inversion problem (3.20) that the Riemann–Hilbert boundary-value problem (3.1) was reduced to.

5.1 *Determination of the A- and B-periods of the abelian integrals*

To apply the procedure of the previous section, we have to evaluate the A- and B-periods of the abelian integrals defined in (3.19) and normalize the basis $\omega_1, \omega_2, \omega_3$. Let us start with the integrals

$$A_{\nu j} = \oint_{\mathbf{a}_j} \frac{t^{\nu-1} dt}{\xi(t)} \quad (\nu, j = 1, 2, 3), \tag{5.1}$$

where $\mathbf{a}_j$ is the closed contour on the sheet $\mathbb{C}_1$. It is convenient to transform the expression of the branch of the function $\xi(t) = f^{\frac{1}{2}}(t)$ that was chosen in section 2. Let us write $t = \rho_0 \exp(i\theta)$. Then it is easy to verify that

$$t - s_j = 2i\rho_0 \sin \frac{\theta - \theta_j}{2} e^{i(\theta+\theta_j)/2} \quad (j = 1, 2, \dots, 8), \tag{5.2}$$

where $s_j = \rho_0 \exp(i\theta_j)$ are branch points of the function $f^{\frac{1}{2}}(s)$. By direct multiplication, the function $\xi(t)$ becomes

$$\xi(t) = \left[\prod_{j=1}^{8} (t - s_j)\right]^{\frac{1}{2}} = 16\rho_0^4 e^{2i\theta} \xi_0(\theta), \tag{5.3}$$

where

$$\xi_0(\theta) = \left(\prod_{j=1}^{8} \sin \frac{\theta - \theta_j}{2}\right)^{\frac{1}{2}}. \tag{5.4}$$

We note that due to the symmetry

$$e^{i(\theta_1+\theta_2+\cdots+\theta_8)/2} = 1. \tag{5.5}$$

The branch of the new function $\xi_0(\theta)$ must be consistent with the chosen branch $\xi(t)$ of the function $f^{\frac{1}{2}}(t)$. Since $\xi(s) \sim s^4$ as $s \to \infty$, we fix the branch of the function $\xi_0(\theta)$ so that $\xi_0(0) > 0$. Formula (5.3) enables us to write down the limit values of the function $\xi(t)$ on the contour $\mathbf{a}_j$ $(j = 1, 2, 3)$:

$$\xi(t) = \pm 16i\rho_0^4 e^{2i\theta} |\xi_0(\theta)| \quad \text{as} \quad |t| = \rho_0 \pm 0, \quad t \in \mathbf{a}_1 \cup \mathbf{a}_3,$$

$$\xi(t) = \mp 16i\rho_0^4 e^{2i\theta} |\xi_0(\theta)| \quad \text{as} \quad |t| = \rho_0 \pm 0, \quad t \in \mathbf{a}_2. \tag{5.6}$$

Substituting the last relations into the definition (5.1) of the A-periods gives

$$A_{\nu 1} = -\frac{\rho_0^{\nu-4}}{8} \int_{\theta_0}^{\frac{1}{2}\pi-\theta_0} \frac{e^{(\nu-2)i\theta}}{|\xi_0(\theta)|} d\theta \quad (\nu = 1, 2, 3). \tag{5.7}$$

Because of the symmetry of the location of the points $s_1, s_2, \dots, s_8$, we obtain for the other A-periods

$$A_{\nu 2} = i^{\nu} A_{\nu 1}, \quad A_{\nu 3} = (-1)^{\nu} A_{\nu 1} \quad (\nu = 1, 2, 3). \tag{5.8}$$

Here we have used the quasi-periodicity of the function $\xi_0(\theta)$:

$$\xi_0\left(\frac{1}{2}\pi+\theta\right)=\left(\prod_{j=1}^{8}\sin\frac{\frac{1}{2}\pi+\theta-\theta_j}{2}\right)^{\frac{1}{2}}=-\xi_0(\theta). \tag{5.9}$$

Next, compute the B-periods of the abelian integrals

$$B_{\nu j}=\oint_{\mathbf{b}_j}\frac{t^{\nu-1}dt}{\xi(t)}\quad (j,\nu=1,2,3). \tag{5.10}$$

Let us start with $B_{\nu 1}$ $(\nu=1,2,3)$. It is worthwhile to recall the definition of the closed contour $\mathbf{b}_1$. It consists of the arcs $s_2s_3, s_3s_4, \dots, s_6s_7$ on the first sheet $\mathbb{C}_1$ and the arcs $s_7s_6, s_6s_5, \dots, s_3s_2$ on the second sheet $\mathbb{C}_2$ (Fig. 3). Therefore, from (5.10)

$$B_{\nu 1}=\sum_{r=2}^{6}\int_{s_rs_{r+1}\subset\mathbb{C}_1}\frac{t^{\nu-1}dt}{\xi^+(t)}+\sum_{r=2}^{6}\int_{s_{9-r}s_{8-r}\subset\mathbb{C}_2}\frac{t^{\nu-1}dt}{\xi^-(t)}. \tag{5.11}$$

The new functions $\xi^+(t)$ and $\xi^-(t)$ are the limit values of the function $w(s)$ defined by

$$\xi^+(t)=\lim_{s\to t\in\mathbf{b}_1, s\in\mathbb{C}_1}w(s)=f^{\frac{1}{2}}(t)\Big|_{|t|=\rho_0+0},$$

$$\xi^-(t)=\lim_{s\to t\in\mathbf{b}_1, s\in\mathbb{C}_2}w(s)=-f^{\frac{1}{2}}(t)\Big|_{|t|=\rho_0-0}. \tag{5.12}$$

The continuity of the function $f^{\frac{1}{2}}(t)$ through the arcs s_2s_3, s_4s_5 and s_6s_7 guarantees the discontinuity of the function $w(s)$ through these portions of the cross-section $\mathbf{b}_1$. On the contrary, because of the condition (5.6), the function $w(s)$ is continuous through the arcs s_3s_4 and s_5s_6. On the arcs of the discontinuity, by using the alternative representation (5.3) of the function $\xi(t)$, we have the following formulae for the limit values of the function $w(s)$:

$$\xi^{\pm}(t)=\mp 16\rho_0^4e^{2i\theta}|\xi_0(\theta)|,\quad t\in(s_2s_3)\cup(s_6s_7),$$

$$\xi^{\pm}(t)=\pm 16\rho_0^4e^{2i\theta}|\xi_0(\theta)|,\quad t\in(s_4s_5)\cup(s_8s_1). \tag{5.13}$$

The direction of the arcs $s_2s_3,\dots,s_6s_7$ on $\mathbb{C}_1$ is opposite to that for the arcs $s_7s_6,\dots,s_3s_2$ on $\mathbb{C}_2$. By cancelling the integrals over the arcs s_3s_4, s_5s_6 on the first sheet $\mathbb{C}_1$ and s_6s_5, s_4s_3 on $\mathbb{C}_2$, in virtue of (5.11), (5.13), (5.3) and (5.4), we get

$$B_{\nu 1}=-\frac{i\rho_0^{\nu-4}}{8}\left(\int_{\frac{1}{2}\pi-\theta_0}^{\frac{1}{2}\pi+\theta_0}-\int_{\pi-\theta_0}^{\pi+\theta_0}+\int_{\frac{3}{2}\pi-\theta_0}^{\frac{3}{2}\pi+\theta_0}\right)\frac{e^{(\nu-2)i\theta}}{|\xi_0(\theta)|}d\theta\quad (\nu=1,2,3). \tag{5.14}$$

By the property of quasi-periodicity (5.9), the last integral is transformed into

$$B_{\nu 1}=-\frac{i\rho_0^{\nu-4}}{8}\int_{-\theta_0}^{\theta_0}\frac{e^{(\nu-2)i\theta}}{|\xi_0(\theta)|}d\theta\quad (\nu=1,2,3). \tag{5.15}$$

To compute the other periods $B_{\nu 2}$ and $B_{\nu 3}$, we notice that the closed contour $\mathbf{b}_2$ consists of the arcs s_4s_5, s_5s_6, s_6s_7 on $\mathbb{C}_1$ and the arcs s_7s_6, s_5s_4, s_4s_3 on $\mathbb{C}_2$.The contour $\mathbf{b}_3$ consists of the arcs $s_6s_7 \subset \mathbb{C}_1$ and $s_7s_6 \subset \mathbb{C}_2$. Similarly to computing the periods $B_{\nu 1}$ ($\nu = 1, 2, 3$), we get

$$B_{\nu 2} = -\frac{i\rho_0^{\nu-4}}{8}\left(-\int_{\pi-\theta_0}^{\pi+\theta_0} + \int_{\frac{3}{2}\pi-\theta_0}^{\frac{3}{2}\pi+\theta_0}\right)\frac{e^{(\nu-2)i\theta}}{|\xi_0(\theta)|}d\theta = -[(-1)^\nu + (-i)^\nu]B_{\nu 1},$$

$$B_{\nu 3} = -\frac{i\rho_0^{\nu-4}}{8}\int_{\frac{3}{2}\pi-\theta_0}^{\frac{3}{2}\pi+\theta_0}\frac{e^{(\nu-2)i\theta}}{|\xi_0(\theta)|}d\theta = -(-i)^\nu B_{\nu 1} \quad (\nu = 1, 2, 3). \tag{5.16}$$

Thus, the desirable matrices of the A- and B-periods of the abelian integrals $\mathbf{A} = (A_{\nu j})$, $\mathbf{B} = (B_{\nu j})$ $(\nu, j = 1, 2, 3)$ have been found:

$$\mathbf{A} = \begin{pmatrix} A_{11} & iA_{11} & -A_{11} \\ A_{21} & -A_{21} & A_{21} \\ A_{31} & -iA_{31} & -A_{31} \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} B_{11} & (1+i)B_{11} & iB_{11} \\ B_{21} & 0 & B_{21} \\ B_{31} & (1-i)B_{31} & -iB_{31} \end{pmatrix}. \tag{5.17}$$

We carried out the calculation of the periods $A_{\nu 1}$ and $B_{\nu 1}$ ($\nu = 1, 2, 3$) and they are given by formulae (5.7) and (5.15).

5.2 *Normalization of the basis of the abelian integrals*

The canonical basis for the abelian integrals of the first kind is formed by

$$u_\nu = \mu_{\nu 1}\omega_1 + \mu_{\nu 2}\omega_2 + \mu_{\nu 3}\omega_3, \quad \nu = 1, 2, 3, \tag{5.18}$$

where the coefficients $\mu_{\nu r}$ ($r = 1, 2, 3$) provide the solution of the three separate systems of linear algebraic equations

$$\sum_{\nu=1}^{3} A_{\nu j}\mu_{r\nu} = \delta_{jr} \quad (j, r = 1, 2, 3) \tag{5.19}$$

(δ_{jr} is Kronecker's symbol). By substituting the elements of the matrix $\mathbf{A}$ from (5.17) into (5.19) and solving the last systems with respect to $\mu_{\nu r}$ we find the matrix of transformation from the old non-normalized basis $\omega_1, \omega_2, \omega_3$ to the new canonical basis u_1, u_2, u_3:

$$\mathbf{u} = \mathbf{M}\boldsymbol{\omega}, \tag{5.20}$$

where

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix},$$

$$\mathbf{M} = \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix} = \begin{pmatrix} \dfrac{1-i}{4A_{11}} & \dfrac{1}{2A_{21}} & \dfrac{1+i}{4A_{31}} \\ -\dfrac{i}{2A_{11}} & 0 & \dfrac{i}{2A_{31}} \\ -\dfrac{1+i}{4A_{11}} & \dfrac{1}{2A_{21}} & -\dfrac{1-i}{4A_{31}} \end{pmatrix}. \tag{5.21}$$

The definition of the A-periods and the representation (5.18) reveal the links between the A-periods of the old and the new basis of the abelian integrals:

$$\begin{aligned} \mathcal{A}_{\nu j} &= \oint_{\mathbf{a}_j} du_\nu = \oint_{\mathbf{a}_j} (\mu_{\nu 1} d\omega_1 + \mu_{\nu 2} d\omega_2 + \mu_{\nu 3} d\omega_3) \\ &= \mu_{\nu 1} A_{1j} + \mu_{\nu 2} A_{2j} + \mu_{\nu 3} A_{3j}, \end{aligned} \tag{5.22}$$

that is, $\mathcal{A}_{\nu j}$ $(\nu, j = 1, 2, 3)$ are the elements of the product of the matrices **M** and **A**: $\boldsymbol{\mathcal{A}} = \mathbf{MA}$. It is easy to verify that $\mathcal{A}_{\nu j} = \delta_{\nu j}$. Similarly, the B-periods of the basis u_1, u_2, u_3

$$\mathcal{B}_{\nu j} = \oint_{\mathbf{b}_j} du_\nu \tag{5.23}$$

are the elements of the matrix **MB**. By direct calculation we find that

$$\boldsymbol{\mathcal{B}} = \frac{1}{2} \begin{pmatrix} \dfrac{1-i}{2}\Pi_1 + \Pi_2 + \dfrac{1+i}{2}\Pi_3 & \Pi_1 + \Pi_3 & \dfrac{1+i}{2}\Pi_1 + \Pi_2 + \dfrac{1-i}{2}\Pi_3 \\ -i\Pi_1 + i\Pi_3 & (1-i)\Pi_1 + (1+i)\Pi_3 & \Pi_1 + \Pi_3 \\ -\dfrac{1+i}{2}\Pi_1 + \Pi_2 - \dfrac{1-i}{2}\Pi_3 & -i\Pi_1 + i\Pi_3 & \dfrac{1-i}{2}\Pi_1 + \Pi_2 + \dfrac{1+i}{2}\Pi_3 \end{pmatrix}, \tag{5.24}$$

where

$$\Pi_\nu = \frac{B_{\nu 1}}{A_{\nu 1}}, \quad \nu = 1, 2, 3. \tag{5.25}$$

Since the matrix of the B-periods corresponds to the canonical basis it must, by the theory of abelian integrals, first, be symmetric and second, satisfy the condition that $\mathrm{Im}(\boldsymbol{\mathcal{B}})$ is a positive definite matrix. At first glance, the matrix $\boldsymbol{\mathcal{B}}$ is not symmetric. Let us prove that, in fact, the matrix (5.24) enjoys the property that $\mathcal{B}_{\nu j} = \mathcal{B}_{j\nu}$. To do this we notice that the following identities

$$A_{31} = i\rho_0^2 A_{11}, \quad B_{31} = \rho_0^2 B_{11} \tag{5.26}$$

hold. Indeed, formula (5.7) yields

$$A_{31} = -\frac{1}{8\rho_0} \int_{\theta_0}^{\frac{1}{2}\pi - \theta_0} \frac{e^{i\theta} d\theta}{|\xi_0(\theta)|}. \tag{5.27}$$

By changing the variables $\theta = \frac{1}{2}\pi - \theta_*$ and using the relation

$$\left| \xi_0\left(\frac{\pi}{2} - \theta_*\right) \right| = |\xi_0(\theta_*)| \tag{5.28}$$

we obtain another expression for the same integral A_{31}

$$A_{31} = -\frac{i}{8\rho_0}\int_{\theta_0}^{\frac{1}{2}\pi-\theta_0} \frac{e^{-i\theta}d\theta}{|\xi_0(\theta)|}. \tag{5.29}$$

Comparing formula (5.29) with (5.7) shows that the identity $A_{31} = i\rho_0^2 A_{11}$ is valid. The second relation in (5.26) is proved similarly. Identities (5.26) enable us to simplify the matrix of the B-periods

$$\boldsymbol{\mathcal{B}} = i\begin{pmatrix} \beta_1+\beta_2 & \beta_1 & \beta_2 \\ \beta_1 & 2\beta_1 & \beta_1 \\ \beta_2 & \beta_1 & \beta_1+\beta_2 \end{pmatrix}, \tag{5.30}$$

where

$$\beta_1 = -\frac{1+i}{2}\frac{B_{11}}{A_{11}}, \quad \beta_2 = -\frac{i}{2}\frac{B_{21}}{A_{21}}. \tag{5.31}$$

It is now seen that the matrix $\boldsymbol{\mathcal{B}}$ is symmetric. To verify the second property of the matrix $\boldsymbol{\mathcal{B}}$, namely, the positive definiteness of the matrix $\mathrm{Im}(\boldsymbol{\mathcal{B}})$, we analyse the coefficients β_1 and β_2. Because of the quasi-periodicity (5.28) of the function $\xi_0(\theta)$ we can reduce the expressions for the periods A_{11}, A_{21}, B_{11} and B_{21} to the form

$$\begin{gathered} A_{11} = \frac{i-1}{4\rho_0^3}\alpha_1, \quad A_{21} = -\frac{\alpha_2}{4\rho_0^2}, \\ B_{11} = -\frac{i\alpha_3}{4\rho_0^3}, \quad B_{21} = -\frac{i\alpha_4}{4\rho_0^2}, \end{gathered} \tag{5.32}$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are positive and are defined by the integrals

$$\alpha_1 = \frac{1}{2}\int_{\theta_0}^{\frac{1}{4}\pi} \frac{\cos\theta+\sin\theta}{|\xi_0(\theta)|}d\theta, \quad \alpha_2 = \int_{\theta_0}^{\frac{1}{4}\pi}\frac{d\theta}{|\xi_0(\theta)|},$$

$$\alpha_3 = \int_0^{\theta_0}\frac{\cos\theta}{|\xi_0(\theta)|}d\theta, \quad \alpha_4 = \int_0^{\theta_0}\frac{d\theta}{|\xi_0(\theta)|}. \tag{5.33}$$

By the definition (5.31) and from formulae (5.32) and (5.33) the coefficients β_1 and β_2 are positive:

$$\beta_1 = \frac{\alpha_3}{2\alpha_1}, \quad \beta_2 = \frac{\alpha_4}{2\alpha_2}. \tag{5.34}$$

Consider the quadratic form

$$\sum_{\nu=1}^{3}\sum_{j=1}^{3}\mathcal{B}_{\nu j}x_\nu x_j = i[\beta_1(x_1+x_2)^2 + \beta_2(x_1+x_3)^2 + \beta_1(x_2+x_3)^2], \tag{5.35}$$

where x_1, x_2, x_3 are real. Since $\beta_1 > 0$, $\beta_2 > 0$ the form $\mathrm{Im}(\boldsymbol{\mathcal{B}})$ is positive definite and the real part of the matrix $\boldsymbol{\mathcal{B}}$ vanishes: $\mathrm{Re}(\boldsymbol{\mathcal{B}}) = 0$.

Identities (5.26) allow us to simplify the expressions for the elements of the canonical basis

$$u_1 = \frac{1-i}{4A_{11}}\left(\omega_1 + \frac{\omega_3}{\rho_0^2}\right) + \frac{\omega_2}{2A_{21}},$$

$$u_2 = \frac{1}{2A_{11}}\left(-i\omega_1 + \frac{\omega_3}{\rho_0^2}\right),$$

$$u_3 = -\frac{1+i}{4A_{11}}\left(\omega_1 - \frac{\omega_3}{\rho_0^2}\right) + \frac{\omega_2}{2A_{21}}. \tag{5.36}$$

5.3 *Reduction of the inversion problem* (3.20) *to a cubic equation*

We are now able to state the Jacobi inversion problem (3.20) in terms of the canonical basis u_1, u_2, u_3. To do this we rewrite the original problem (3.20) as follows:

$$\sum_{j=1}^{3}\left[\boldsymbol{\omega}(\sigma_j, w_j) + m_j\mathbf{A}_j + n_j\mathbf{B}_j\right] = \mathbf{d}^*, \tag{5.37}$$

where $\mathbf{A}_j, \mathbf{B}_j$ $(j = 1, 2, 3)$ and $\mathbf{d}^*$ are the vector-columns

$$\mathbf{A}_j = \begin{pmatrix} A_{1j} \\ A_{2j} \\ A_{3j} \end{pmatrix}, \quad \mathbf{B}_j = \begin{pmatrix} B_{1j} \\ B_{2j} \\ B_{3j} \end{pmatrix}, \quad \mathbf{d}^* = \begin{pmatrix} d_1^* \\ d_2^* \\ d_3^* \end{pmatrix}. \tag{5.38}$$

Multiply the vector equation (5.37) from the left by the transformation matrix $\mathbf{M}$. Then

$$\sum_{j=1}^{3}\left[\mathbf{u}(\sigma_j, w_j) + m_j\boldsymbol{\mathcal{A}}_j + n_j\boldsymbol{\mathcal{B}}_j\right] = \mathbf{d}, \tag{5.39}$$

where $\boldsymbol{\mathcal{B}}_j$ is the jth column of the matrix $\boldsymbol{\mathcal{B}}$ defined in (5.30), $\boldsymbol{\mathcal{A}}_j$ is the jth column of the unit matrix and $\mathbf{d}$ is the vector-column with the elements

$$d_1 = \frac{1-i}{4A_{11}}\left(d_1^* + \frac{d_3^*}{\rho_0^2}\right) + \frac{d_2^*}{2A_{21}},$$

$$d_2 = \frac{1}{2A_{11}}\left(-id_1^* + \frac{d_3^*}{\rho_0^2}\right),$$

$$d_3 = -\frac{1+i}{4A_{11}}\left(d_1^* - \frac{d_3^*}{\rho_0^2}\right) + \frac{d_2^*}{2A_{21}}. \tag{5.40}$$

Therefore the Jacobi inversion problem in the canonical basis is given by

$$\sum_{j=1}^{3}[u_\nu(\sigma_j, w_j) + n_j\mathcal{B}_{\nu j}] + m_\nu = d_\nu, \quad \nu = 1, 2, 3; \tag{5.41}$$

that is a particular case $h = 3$ of the problem (4.3) considered in section 4. In that section, this problem was reduced to the algebraic system (4.25) which in the case under consideration becomes

$$\begin{aligned}\sigma_1 + \sigma_2 + \sigma_3 &= \varepsilon_1,\\ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 &= \varepsilon_2,\\ \sigma_1^3 + \sigma_2^3 + \sigma_3^3 &= \varepsilon_3\end{aligned} \tag{5.42}$$

with the coefficients $\varepsilon_1, \varepsilon_2, \varepsilon_3$ defined from (4.26) for $h = 3$:

$$\varepsilon_\nu = \sum_{j=1}^{3}\sum_{r=1}^{3}\mu_{jr}A_{\nu+r,j} - R_{\nu 1} - R_{\nu 2} \quad (\nu = 1, 2, 3), \tag{5.43}$$

where

$$R_{\nu r} = \operatorname*{res}_{q=\infty\in\mathbb{C}_r} \frac{s^\nu \mathfrak{F}'(q)}{\mathfrak{F}(q)} \quad (r = 1, 2). \tag{5.44}$$

We have already established (section 4.3) that the system (5.42) is equivalent to the cubic equation

$$\sigma^3 - \varepsilon_1\sigma^2 + \tfrac{1}{2}(\varepsilon_1^2 - \varepsilon_2)\sigma - \tfrac{1}{6}(\varepsilon_1^3 - 3\varepsilon_1\varepsilon_2 + 2\varepsilon_3) = 0. \tag{5.45}$$

5.4 *Evaluation of the coefficients of the cubic equation*

Thus, to find the points $(\sigma_1, w_1), (\sigma_2, w_2), (\sigma_3, w_3)$ or, equivalently, to solve the cubic equation (5.45) it is necessary to know the coefficients $\varepsilon_1, \varepsilon_2$ and ε_3. Formula (5.43) gives ε_j $(j = 1, 2, 3)$ in terms of $A_{\nu+r,j}$ and the residues $R_{\nu r}$. Obviously, the first and second sets of the coefficients A_{2j} and A_{3j} are the corresponding A-periods of the basis $\omega_1, \omega_2, \omega_3$ defined in (5.7), (5.8). The other coefficients can be expressed through the basic vector-column $A_{\nu 1}$ $(\nu = 1, 2, 3)$:

$$\begin{aligned}A_{\nu+r,1} &= -\frac{\rho_0^{\nu+r-4}}{8}\int_{\theta_0}^{\frac{1}{2}\pi-\theta_0} \frac{e^{(\nu+r-2)i\theta}}{|\xi_0(\theta)|}d\theta,\\ A_{\nu+r,2} &= i^{\nu+r}A_{\nu+r,1}, \quad A_{\nu+r,3} = (-1)^{\nu+r}A_{\nu+r,1} \quad (\nu + r = 4, 5, 6).\end{aligned} \tag{5.46}$$

To evaluate the residues $R_{\nu 1}$, $R_{\nu 2}$, we find the coefficient c_{-1} in the Laurent expansion of the meromorphic function $s^\nu\mathfrak{F}'(q)[\mathfrak{F}(q)]^{-1}$ in a neighbourhood of the infinite points on both sheets $\mathbb{C}_1$, $\mathbb{C}_2$: (∞, ∞_1) and (∞, ∞_2). First, by substituting (5.30) into (4.7), we rewrite the representation for the Riemann Θ-function $\mathfrak{F}(q)$ in the form

$$\begin{aligned}\mathfrak{F}(q) = \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}\sum_{l_3=-\infty}^{\infty} &\exp\{-\pi[\beta_1(l_1+l_2)^2 + \beta_2(l_1+l_3)^2 + \beta_1(l_2+l_3)^2]\cdot\\ &+2\pi i[l_1(u_1(q)-e_1) + l_2(u_2(q)-e_2) + l_3(u_3(q)-e_3)]\},\end{aligned} \tag{5.47}$$

where $e_\nu = d_\nu + k_\nu$, the coefficients d_ν and Riemann's constants k_ν are defined by (5.40) and (4.5), respectively.

5.4.1 *Riemann's constants.* To simplify the expressions for Riemann's constants k_ν of the surface $\mathcal{R}$, we evaluate the integral in (4.5). Based on the relation

$$u_\nu^-(t) - u_\nu^+(t) = \oint_{\mathbf{b}_j} du_\nu(p), \quad t \in \mathbf{a}_j \quad (\nu, j = 1, 2, 3) \tag{5.48}$$

and allowing for the value $\mathcal{B}_{\nu j}$ of the above integral, we may now rewrite the Riemann constants as follows:

$$k_\nu = -\frac{1}{2} + \frac{1}{2}\mathcal{B}_{\nu\nu} - \sum_{j=1,\, j\neq\nu}^{3} \left[\oint_{\mathbf{a}_j} u_\nu^+(t) du_j(t) + \mathcal{B}_{\nu j} \oint_{\mathbf{a}_j} du_j(t) \right]. \tag{5.49}$$

Clearly, the last integral in (5.49) is equal to 1. Recall that $u_\nu^+(t)$ is the limit value of the function $u_\nu(q)$ on both banks of the slit $s_{2j-1}s_{2j} \subset \mathbb{C}_1$. Let $t = t^+$ be a point of the upper side of the cut ($|s| = \rho_0 + 0$) on $\mathbb{C}_1$. Then

$$u_\nu^+(t^+) = \int_{s_8}^{s_{2j-1}} du_\nu(s) + \int_{s_{2j-1}}^{t} du_\nu(\tau^+) = u_\nu^+(s_{2j-1}) + \tilde{u}_{\nu j}(t), \tag{5.50}$$

where $u_\nu^+(s_{2j-1})$ is the value of the first integral along a smooth curve on the first sheet which does not intersect the canonical cross-sections of the surface; $\tilde{u}_{\nu j}(t)$ gives the value of the second integral along the upper arc $s_{2j-1}t$ of the slit. If $t = t^-$ lies on the lower bank of the cut, then since $du_\nu(\tau^-) = -du_\nu(\tau^+),\ \tau \in s_{2j-1}s_{2j}$, we get

$$u_\nu^+(t^-) = \int_{s_8}^{s_{2j-1}} du_\nu(s) + \int_{s_{2j-1}}^{t} du_\nu(\tau^-) = u_\nu^+(s_{2j-1}) - \tilde{u}_{\nu j}(t). \tag{5.51}$$

By substituting (5.50) and (5.51) into (5.49) we transform the first integral in (5.49) as follows:

$$\oint_{\mathbf{a}_j} u_\nu^+(t) du_j(t) = \int_{s_{2j-1}}^{s_{2j}} \left[\left(u_\nu^+(s_{2j-1}) - \tilde{u}_{\nu j}(t)\right) du_j(t^-) - \left(u_\nu^+(s_{2j-1}) + \tilde{u}_{\nu j}(t)\right) du_j(t^+) \right]. \tag{5.52}$$

The differentials du_j have opposite signs on the upper and lower banks of the cuts: $du_j(t^+) = -du_j(t^-)$, $t \in s_{2j-1}s_{2j}$. Therefore

$$\oint_{\mathbf{a}_j} u_\nu^+(t) du_j(t) = 2u_\nu^+(s_{2j-1}) \int_{s_{2j-1}}^{s_{2j}} du_j(t^-) = u_\nu^+(s_{2j-1}) \oint_{\mathbf{a}_j} du_j(t) = u_\nu^+(s_{2j-1}), \tag{5.53}$$

and formula (5.49) becomes

$$k_\nu = -\frac{1}{2} + \frac{1}{2}\mathcal{B}_{\nu\nu} - \sum_{j=1,\, j\neq\nu}^{3} \left[u_\nu^+(s_{2j-1}) + \mathcal{B}_{\nu j} \right]. \tag{5.54}$$

It is possible to evaluate $u_\nu^+(s_{2j-1})$ explicitly. For $u_\nu^+(s_1)$ we get

$$u_\nu^+(s_1) = \int_{s_8}^{s_1} du_\nu(s) = -\int_{s_1}^{s_8} du_\nu(s). \tag{5.55}$$

The last integral is taken over the inner side of the cut $s_1 s_8$ on $\mathbb{C}_1$. Then the line of integration does not intersect the canonical cross-sections and

$$u_\nu^+(s_1) = -\left(\int_{s_1}^{s_2} + \int_{s_3}^{s_4} + \int_{s_5}^{s_6}\right) du_\nu(s^-) - \left(\int_{s_2}^{s_3} + \int_{s_4}^{s_5} + \int_{s_6}^{s_7}\right) du_\nu(s) - \int_{s_7}^{s_8} du_\nu(s^-), \tag{5.56}$$

where the first sum is equal to $\frac{1}{2}$, and the sum of the second group of integrals is equal to $\frac{1}{2}\mathcal{B}_{\nu 1}$. Since the differentials du_ν are linearly expressible through the differentials

$$d\omega_j(t) = \frac{t^{j-1}}{\xi(t)} dt, \quad j = 1, 2, 3, \tag{5.57}$$

and at infinity $(\infty, \infty_1) \in \mathbb{C}_1$ they behave as $t^{j-5} dt$, it follows that

$$\left(\oint_{\mathbf{a}_1} + \oint_{\mathbf{a}_2} + \oint_{\mathbf{a}_3}\right) u_\nu(s) + \oint_{s_7 s_8} du_\nu(s) = 0. \tag{5.58}$$

The sum of the first group of integrals is 1. Therefore

$$\int_{s_7}^{s_8} du_\nu(s^-) = -\frac{1}{2}, \quad \nu = 1, 2, 3. \tag{5.59}$$

The explicit expression follows from use of the formulae (5.56) and (5.59):

$$u_\nu^+(s_1) = -\tfrac{1}{2}\mathcal{B}_{\nu 1}, \quad \nu = 1, 2, 3. \tag{5.60}$$

Similarly,

$$u_\nu^+(s_3) = -\tfrac{1}{2}\mathcal{B}_{\nu 2} + \tfrac{1}{2}\delta_{\nu 1}, \quad u_\nu^+(s_5) = -\tfrac{1}{2}\mathcal{B}_{\nu 3} + \tfrac{1}{2}\left(1 - \delta_{\nu 3}\right), \quad \nu = 1, 2, 3, \tag{5.61}$$

where $\delta_{\nu r}$ is Kronecker's symbol. Now, formulae (5.54), (5.60), (5.61) and (5.30) enable us to obtain Riemann's constants of the surface $\mathcal{R}$:

$$k_1 = -\tfrac{3}{2}, \quad k_2 = -1, \quad k_3 = -\tfrac{1}{2}. \tag{5.62}$$

5.4.2 *Residues $R_{\nu r}$.* Let us derive the behaviour of the abelian integrals $\omega_\nu(q)$ at infinity and evaluate the residues (5.44).

Fix a point $q_* \in \mathbb{C}_r$ ($r = 1, 2$) on the ray $\arg s = -\theta_0$ such that $|q_*| \gg \rho_0$ and use the asymptotic expansion of the function $f^{-\frac{1}{2}}(s)$ at infinity

$$f^{-\frac{1}{2}}(s) = \frac{1}{s^4} + \frac{M_1}{s^8} + \frac{3M_1^2 - M_2}{2s^{12}} + O(s^{-16}), \quad s \to \infty. \tag{5.63}$$

Then we represent the abelian integral as follows:

$$\omega_\nu(q) = (-1)^{r-1}\left[\int_{q_0}^{q_*} \frac{s^{\nu-1} ds}{f^{\frac{1}{2}}(s)} + \int_{q_*}^{q} s^{\nu-1}\left(\frac{1}{s^4} + \frac{M_1}{s^8} + \frac{3M_1^2 - M_2}{2s^{12}} + \dots\right) ds\right], \quad q \in \mathbb{C}_r. \tag{5.64}$$

Here $q_0 = (s_8, 0)$. The contour of integration $q_0 q$ consists of points (s, w) of the segment

$$[q_0, q_*] = \{s : \rho_0 \leqslant |s| \leqslant \rho_*, \arg s = -\theta_0\} \quad (\rho_* = |q_*|),$$

and any curve $q_* q$ such that $|s| > \rho_*$. This choice guarantees that the path of integration does not cross the cross-sections $\mathbf{a}_\nu$ and $\mathbf{b}_\nu$ $(\nu = 1, 2, 3)$. The second integral in (5.64) can be found explicitly and we arrive at the asymptotic expansion of the abelian integrals $\omega_\nu(q)$ at infinity

$$\omega_\nu(q) = (-1)^{r-1}\left[I_\nu + \frac{s^{\nu-4}}{\nu-4} + \frac{M_1}{\nu-8}s^{\nu-8} + O(s^{\nu-12})\right], \quad s \to \infty \quad (q \in \mathbb{C}_r), \tag{5.65}$$

where

$$\begin{aligned} I_\nu = e^{i(4-\nu)\theta_0} &\int_{\rho_0}^{\rho_*} \frac{\rho^{\nu-1} d\rho}{\sqrt{(\rho^4 - \rho_0^4)(\rho^4 - \rho_0^4 e^{8i\theta_0})}} \\ &- \frac{q_*^{\nu-4}}{\nu-4} - \frac{M_1 q_*^{\nu-8}}{\nu-8} - \frac{(3M_1^2 - M_2) q_*^{\nu-12}}{2(\nu-12)} + O(\rho_*^{\nu-16}) \quad (\nu = 1, 2, 3). \end{aligned} \tag{5.66}$$

Here $q_* = \rho_* e^{-i\theta_0}$, $\rho_* \gg \rho_0$.

To compute the residues (5.44) we shall need expansions at infinity not only for the abelian integrals but also for their first derivative:

$$\omega_\nu'(q) = (-1)^{r-1}\left[s^{\nu-5} + M_1 s^{\nu-9} + \frac{3M_1^2 - M_2}{2}s^{\nu-13} + O(s^{\nu-17})\right], \quad s \to \infty, \quad q \in \mathbb{C}_r. \tag{5.67}$$

By substituting (5.65) into (5.47) and introducing the notation

$$\begin{aligned} \varkappa_r(l_1, l_2, l_3) = \pi[&\beta_1(l_1+l_2)^2 + \beta_2(l_1+l_3)^2 + \beta_1(l_2+l_3)^2] \\ &-2\pi i \sum_{\nu=1}^{3} l_\nu \left(\sum_{j=1}^{3}(-1)^{r-1}\mu_{\nu j} I_j - e_\nu\right) \end{aligned} \tag{5.68}$$

we get the asymptotic expansion for the function $\mathfrak{F}(q)$

$$\begin{aligned} \mathfrak{F}(q) = \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}\sum_{l_3=-\infty}^{\infty} &\exp\{-\varkappa_r(l_1, l_2, l_3) \\ &+2\pi i(-1)^{r-1}\sum_{\nu=1}^{3} l_\nu \sum_{j=1}^{3}\left[\mu_{\nu j}\frac{s^{j-4}}{j-4} + O(s^{j-7})\right]\bigg\}, \\ &q \to \infty \in \mathbb{C}_r, \quad r = 1, 2. \end{aligned} \tag{5.69}$$

The positive definiteness of the quadratic form $\mathrm{Im}(\boldsymbol{\mathcal{B}})$ is important for the series representation of the Θ-function $\mathfrak{F}(q)$ to be convergent. In fact, the convergence is extremely rapid and just the first few terms yield an accuracy of dozens of significant figures. For our purposes, we need an asymptotic expansion for the function $1/\mathfrak{F}(q)$ that follows from (5.69)

$$\frac{1}{\mathfrak{F}(q)} = \frac{1}{a_{0r}} + \frac{a_{1r}}{a_{0r}^2 s} + \frac{a_{0r}a_{2r} + a_{1r}^2}{a_{0r}^3 s^2} + O\left(\frac{1}{s^3}\right), \quad q \to \infty \in \mathbb{C}_r, \tag{5.70}$$

where

$$a_{0r} = \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \sum_{l_3=-\infty}^{\infty} e^{-\varkappa_r(l_1,l_2,l_3)},$$

$$a_{1r} = (-1)^{r-1} 2\pi i \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \sum_{l_3=-\infty}^{\infty} \tau_3 e^{-\varkappa_r(l_1,l_2,l_3)},$$

$$a_{2r} = \pi i \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \sum_{l_3=-\infty}^{\infty} \left[(-1)^{r-1}\tau_2 - 2\pi i \tau_3^2\right] e^{-\varkappa_r(l_1,l_2,l_3)}, \quad r = 1, 2. \tag{5.71}$$

Here we have introduced the following notation:

$$\tau_j = \sum_{\nu=1}^{3} \mu_{\nu j} l_\nu. \tag{5.72}$$

Next, we study the behaviour of the derivative of the Θ-function $\mathfrak{F}'(q)$

$$\mathfrak{F}'(q) = 2\pi i \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \sum_{l_3=-\infty}^{\infty} [l_1 u_1'(q) + l_2 u_2'(q) + l_3 u_3'(q)]$$
$$\times \exp\Biggl\{ -\pi[\beta_1(l_1+l_2)^2 + \beta_2(l_1+l_3)^2 + \beta_1(l_2+l_3)^2]$$
$$+ 2\pi i \sum_{\nu=1}^{3} l_\nu [u_\nu(q) - e_\nu] \Biggr\} \tag{5.73}$$

at infinity. By the same procedure that was applied for the function $1/\mathfrak{F}(q)$, we establish that the derivative $\mathfrak{F}'(q)$ enjoys the expansion

$$\mathfrak{F}'(q) = \frac{b_{2r}}{s^2} + \frac{b_{3r}}{s^3} + \frac{b_{4r}}{s^4} + O\left(\frac{1}{s^5}\right), \quad q \to \infty \in \mathbb{C}_r, \tag{5.74}$$

where

$$b_{2r} = a_{1r}, \quad b_{3r} = 2a_{2r},$$

$$b_{4r} = 2\pi i \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \sum_{l_3=-\infty}^{\infty} \{(-1)^{r-1}\tau_1 - 2\pi i \tau_2 \tau_3$$
$$-\pi i \tau_3 [\tau_2 + (-1)^r 2\pi i \tau_3^2]\} e^{-\varkappa_r(l_1,l_2,l_3)}, \quad r = 1, 2. \tag{5.75}$$

Thus, we find

$$\frac{s^\nu \mathfrak{F}'(q)}{\mathfrak{F}(q)} = s^\nu \left[\frac{b_{2r}}{s^2} + \frac{b_{3r}}{s^3} + \frac{b_{4r}}{s^4} + O\left(\frac{1}{s^5}\right)\right]$$
$$\times \left[\frac{1}{a_{0r}} + \frac{a_{1r}}{a_{0r}^2 s} + \frac{a_{0r}a_{2r} + a_{1r}^2}{a_{0r}^3 s^2} + O\left(\frac{1}{s^3}\right)\right], \quad q \to \infty \in \mathbb{C}_r. \tag{5.76}$$

Now it is straightforward to obtain the desired expressions for the residues of the function in the left-hand side in (5.76) at the infinite points on both sheets of the surface $\mathcal{R}$:

$$
\begin{aligned}
R_{1r} &= -\frac{b_{2r}}{a_{0r}},\\
R_{2r} &= -\frac{b_{3r}}{a_{0r}} - \frac{b_{2r}a_{1r}}{a_{0r}^2},\\
R_{3r} &= -\frac{b_{4r}}{a_{0r}} - \frac{a_{1r}b_{3r}}{a_{0r}^2} - \frac{(a_{0r}a_{2r} + a_{1r}^2)b_{2r}}{a_{0r}^3}.
\end{aligned}
\tag{5.77}
$$

Thus, the coefficients $\varepsilon_1, \varepsilon_2, \varepsilon_3$ in the cubic equation (5.45) are determined explicitly by formulae (5.43), (5.7), (5.8), (5.46) and (5.77).

5.5 *Explicit expressions for the integers n_ν and m_ν*

Now we proceed to the solution of the cubic equation (5.45). Clearly, the list of possible solutions possesses eight triples of the points q_1, q_2, q_3. For each set of the points, the numbers n_ν and m_ν ($\nu = 1, 2, 3$) are easily found from the linear system (4.17) and formulae (4.19):

$$
\begin{aligned}
n_1 &= \frac{(\beta_1 + 2\beta_2)\eta_1^\circ - \beta_1\eta_2^\circ + (\beta_1 - 2\beta_2)\eta_3^\circ}{4\beta_1\beta_2},\\
n_2 &= \frac{-\beta_1\eta_1^\circ + (\beta_1 + 2\beta_2)\eta_2^\circ - \beta_1\eta_3^\circ}{4\beta_1\beta_2},\\
n_3 &= \frac{(\beta_1 - 2\beta_2)\eta_1^\circ - \beta_1\eta_2^\circ + (\beta_1 + 2\beta_2)\eta_3^\circ}{4\beta_1\beta_2},\\
m_\nu &= \mathrm{Re}(\eta_\nu), \quad \nu = 1, 2, 3.
\end{aligned}
\tag{5.78}
$$

Here we have assumed the following notation:

$$
\eta_\nu^\circ = \mathrm{Im}(\eta_\nu), \quad \eta_\nu = d_\nu - \sum_{j=1}^{3} u_\nu(q_j), \tag{5.79}
$$

and used the property of the matrix $\boldsymbol{\mathcal{B}}$: $\mathrm{Re}(\mathcal{B}_{\nu j}) = 0$ ($\nu, j = 1, 2, 3$). The solution of the system (4.17) exists and is unique because the determinant of the system, $4\beta_1^2\beta_2$, is always positive.

Formulae (5.78) provide an efficient test to verify whether the set $\{q_\nu, n_\nu, m_\nu\ (\nu = 1, 2, 3)\}$ is a genuine solution of Jacobi's problem or a mock one.

Indeed, the parameters $n_\nu, m_\nu (\nu = 1, 2, 3)$ are integers, so must be the right-hand sides of formulae (5.78). This completes the solution of the Jacobi inversion problem (3.20) for the hyperelliptic surface of genus 3. The problem is solved by quadratures.

6. Solution of the vector Riemann–Hilbert problem

The purpose of this section is to derive an exact solution of the homogeneous Riemann–Hilbert problem (2.1). By substituting the factorization (2.10) of the matrix $\mathbf{G}(t)$ into the boundary condition (2.1) we obtain

$$
\mathbf{X}^+(t)\mathbf{\Phi}^+(t) = -\mathbf{X}^-(t)\mathbf{\Phi}^-(t) + \mathbf{X}^-(t)\mathbf{g}(t), \quad t \in L. \tag{6.1}
$$

The vector $\mathbf{X}^-(t)\mathbf{g}(t)$ satisfies the Hölder condition on the contour L and therefore admits the representation

$$\mathbf{X}^-(t)\mathbf{g}(t) = \mathbf{\Psi}^+(t) - \mathbf{\Psi}^-(t), \quad t \in L \tag{6.2}$$

through the limit values of the vector

$$\mathbf{\Psi}(s) = \frac{1}{2\pi i}\int_L \frac{\mathbf{X}^-(t)\mathbf{g}(t)}{t-s}dt. \tag{6.3}$$

Now the boundary condition becomes

$$\mathbf{X}^+(t)\mathbf{\Phi}^+(t) - \mathbf{\Psi}^+(t) = -\mathbf{X}^-(t)\mathbf{\Phi}^-(t) - \mathbf{\Psi}^-(t), \quad t \in L, \tag{6.4}$$

and it is ready for applying the generalized Liouville's theorem. It has been pointed out that the function $F(s, w)$ has three simple poles (δ_j, v_j) $(j = 1, 2, 3)$ on the first sheet $\mathbb{C}_1$ and three simple zeros q_j $(j = 1, 2, 3)$ on the surface $\mathcal{R}$. This means that in the vicinity of the points (δ_j, v_j) the solution of the Riemann–Hilbert problem (3.1) behaves as follows:

$$\begin{gathered} F(s, w) \sim D_j(s-\delta_j)^{-1}, \quad s \to \delta_j, \quad (s, w) \in \mathbb{C}_1, \quad D_j = \text{const}, \\ F(s, -w) = O(1), \quad s \to \delta_j, \quad (s, -w) \in \mathbb{C}_2, \quad j = 1, 2, 3. \end{gathered} \tag{6.5}$$

Assume that the zero q_j lies on the first sheet $\mathbb{C}_1$. Then

$$\begin{gathered} F(s, w) \sim E_j(s-\sigma_j), \quad \frac{1}{F(s, -w)} = O(1), \quad s \to \sigma_j, \\ E_j = \text{const}, \quad (s, w) \in \mathbb{C}_1, \quad j = 1, 2, 3. \end{gathered} \tag{6.6}$$

The situation changes symmetrically if $q_j \in \mathbb{C}_2$:

$$\frac{1}{F(s, w)} = O(1), \quad F(s, -w) \sim E_j(s-\sigma_j), \quad s \to \sigma_j, \quad (s, w) \in \mathbb{C}_1, \quad j = 1, 2, 3. \tag{6.7}$$

Let us derive the behaviour of the vectors $\mathbf{X}^\pm(s)\mathbf{\Phi}^\pm(s)$ at the points $s = \delta_j$. Since $w^2(s) = l^2(s) + m(s)n(s)$, the matrix $\mathbf{Y}(s, w)$ is singular and $\operatorname{rank}\mathbf{Y}(s, w) = 1$. Therefore by formulae (2.11), (2.12), (6.5)

$$\begin{aligned} \mathbf{X}^\pm(s)\mathbf{\Phi}^\pm(s) &\sim \frac{D_j}{s-\delta_j}\mathbf{Y}(\delta_j, v_j)\mathbf{\Phi}^\pm(\delta_j) = \frac{D_j}{2(s-\delta_j)} \\ &\times \left[\left(1 + \frac{l(\delta_j)}{v_j}\right)\Phi_1^\pm(\delta_j) + \frac{m(\delta_j)}{v_j}\Phi_2^\pm(\delta_j)\right]\begin{pmatrix}1\\ \zeta_j\end{pmatrix}, \quad s \to \delta_j \in \mathbb{C}^\pm, \end{aligned} \tag{6.8}$$

where $\Phi_1^\pm(s)$, $\Phi_2^\pm(s)$ are the components of the vector-functions $\mathbf{\Phi}^\pm(s)$, respectively, $v_j = w(\delta_j)$ and

$$\zeta_j = \frac{n(\delta_j)}{l(\delta_j) + v_j}. \tag{6.9}$$

At infinity, the matrices $\mathbf{X}^{\pm}(s)$ are bounded and the vectors $\mathbf{\Phi}^{\pm}(s)$, $\mathbf{\Psi}^{\pm}(s)$ vanish. By the continuity principle and the generalized Liouville's theorem the vectors

$$\mathbf{E}^{+}(s) = \mathbf{X}^{+}(s)\mathbf{\Phi}^{+}(s) - \mathbf{\Psi}^{+}(s), \quad s \in \mathbb{C}^{+},$$

$$\mathbf{E}^{-}(s) = -\mathbf{X}^{-}(s)\mathbf{\Phi}^{-}(s) - \mathbf{\Psi}^{-}(s), \quad s \in \mathbb{C}^{-} \tag{6.10}$$

constitute the analytical continuation of one another and they are a rational vector

$$\mathbf{E}^{+}(s) = \mathbf{E}^{-}(s) = \sum_{j=1}^{3} \frac{C_j}{s-\delta_j}\begin{pmatrix}1\\ \zeta_j\end{pmatrix}, \quad s \in \mathbb{C}, \tag{6.11}$$

where C_1, C_2, C_3 are arbitrary constants. Hence by inverting the matrices $\mathbf{X}^{+}(s)$, $\mathbf{X}^{-}(s)$ in (6.10), the solution of the problem (2.1) becomes

$$\mathbf{\Phi}^{+}(s) = [\mathbf{X}^{+}(s)]^{-1}\left[\mathbf{\Psi}^{+}(s) + \sum_{j=1}^{3} \frac{C_j}{s-\delta_j}\begin{pmatrix}1\\ \zeta_j\end{pmatrix}\right], \quad s \in \mathbb{C}^{+},$$

$$\mathbf{\Phi}^{-}(s) = -[\mathbf{X}^{-}(s)]^{-1}\left[\mathbf{\Psi}^{-}(s) + \sum_{j=1}^{3} \frac{C_j}{s-\delta_j}\begin{pmatrix}1\\ \zeta_j\end{pmatrix}\right], \quad s \in \mathbb{C}^{-}. \tag{6.12}$$

Let us analyse the solution at the points $s = \delta_r$ $(r = 1, 2, 3)$. First, we note that

$$\mathbf{X}^{-1}(s) \sim \frac{\mathbf{Y}(\delta_j, -v_j)}{F(\delta_j, -v_j)} + O(s-\delta_j), \quad s \to \delta_j \quad (j = 1, 2, 3). \tag{6.13}$$

It is straightforward to check that

$$\mathbf{Y}(\delta_j, -v_j)\begin{pmatrix}1\\ \zeta_j\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 - v_j^{-1}l(\delta_j) - v_j^{-1}m(\delta_j)\zeta_j\\ -v_j^{-1}n(\delta_j) + [1 + v_j^{-1}l(\delta_j)]\zeta_j\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \tag{6.14}$$

and, therefore, the points $s = \delta_1, \delta_2, \delta_3$ are removable singularities of the vector-functions $\mathbf{\Phi}^{\pm}(s)$. As far as the points $s = \sigma_1, \sigma_2, \sigma_3$ are concerned, they are simple poles of the function $[F(s, w)]^{-1}$ or of the function $[F(s, -w)]^{-1}$ on the sheet $\mathbb{C}_1$. Hence

$$[\mathbf{X}^{-1}(s)] \sim \frac{\mathbf{Y}(\sigma_r, w_r)}{E_r(s-\sigma_r)}, \quad s \to \sigma_r \quad (r = 1, 2, 3), \tag{6.15}$$

where $w_r = f^{\frac{1}{2}}(\sigma_r)$ for a point $q_j \in \mathbb{C}_1$, and $w_r = -f^{\frac{1}{2}}(\sigma_r)$ for a point $q_j \in \mathbb{C}_2$. In general, we would have six conditions eliminating the singularities of the vector

$$\mathbf{X}^{-1}(s)\left[\mathbf{\Psi}(s) + \sum_{j=1}^{3} \frac{C_j}{s-\delta_j}\begin{pmatrix}1\\ \zeta_j\end{pmatrix}\right] \tag{6.16}$$

at the points $s = \sigma_1, s = \sigma_2, s = \sigma_3$. However, because rank $\mathbf{Y}(s, w) = 1$, we have only three conditions:

$$\left(1 + \frac{l(\sigma_r)}{w_r}\right)\left(\Psi_1(\sigma_r) + \sum_{j=1}^{3} \frac{C_j}{\sigma_r - \delta_j}\right) + \frac{m(\sigma_r)}{w_r}\left(\Psi_2(\sigma_r) + \sum_{j=1}^{3} \frac{\zeta_j C_j}{\sigma_r - \delta_j}\right) = 0, \quad r = 1, 2, 3, \tag{6.17}$$

where $\Psi_1(s), \Psi_2(s)$ are the components of the vector $\mathbf{\Psi}(s)$. Thus, in order for the vector $\mathbf{\Phi}(s)$ to be analytic at the points $s = \sigma_r$ $(r = 1, 2, 3)$ it is necessary and sufficient that the constants C_1, C_2, C_3 provide the solution of the system of linear algebraic equations

$$\sum_{j=1}^{3} \chi_{rj} C_j = h_r \quad (r = 1, 2, 3), \tag{6.18}$$

where

$$\chi_{rj} = \frac{1}{\sigma_r - \delta_j}\left(1 + \frac{l(\sigma_r) + m(\sigma_r)\zeta_j}{w_r}\right),$$

$$h_r = -\left(1 + \frac{l(\sigma_r)}{w_r}\right)\Psi_1(\sigma_r) - \frac{m(\sigma_r)}{w_r}\Psi_2(\sigma_r). \tag{6.19}$$

The definition of the constants C_1, C_2, C_3 completes the solution of the vector Riemann–Hilbert boundary-value problem (2.1).

7. Canonical factorization and the partial indices

7.1 *Definition of the canonical matrix*

The factorization matrix $\mathbf{X}(s)$ we have constructed, possesses three poles at the points $\delta_1, \delta_2, \delta_3$. At the three points $\sigma_1, \sigma_2, \sigma_3$ the determinant of the matrix vanishes and at infinity the matrix is bounded. The matrix $\mathbf{X}(s)$ allows us to apply the generalized Liouville theorem and to find the exact solution of the non-homogeneous Riemann–Hilbert problem (2.1). However, it does not give an answer to the fundamental questions in matrix factorization theory.

(i) What are the partial indices of the problem (2.1)?
(ii) Are the partial indices stable or not?

The main purpose of this section is to construct the canonical matrix and determine the partial indices. We reproduce the basic definitions (**23** to **25**) that we need to proceed further.

Let $\Upsilon_j(s) = \Upsilon_j^\circ(s)(s - s_0)^{\alpha_j}$, $s \to s_0$ $(j = 1, \ldots, n)$, where $\Upsilon_j^\circ(s)$ is bounded and does not vanish at the point $s = s_0$. Then the real number α_j is called the order of the function $\Upsilon_j(s)$ at the point $s = s_0$. The order of the vector $\mathbf{\Upsilon}(s) = (\Upsilon_1(s), \ldots, \Upsilon_n(s))^\top$ is $\alpha = \min\{\alpha_1, \ldots, \alpha_n\}$.

Assume further that $\Upsilon_j(s) = \Upsilon_j^*(s)s^{-\alpha_j}$, $s \to \infty$ $(j = 1, \ldots, n)$, where $\Upsilon_j^*(s)$ is bounded at infinity and $\Upsilon_j^*(\infty) \neq 0$. Then, by the Gakhov definition (**24**), the order α of the vector $\mathbf{\Upsilon}(s)$ at infinity is defined in the same manner: $\alpha = \min\{\alpha_1, \ldots, \alpha_n\}$.

A matrix $\mathbf{Z}(s)$ is said to be in *normal form* at a point $s = s_0$ with respect to the columns if the order of the determinant at this point is equal to the sum of the orders of the columns.

A matrix

$$\mathbf{X}_0(s) = \begin{pmatrix} \chi_1^{(1)}(s) & \dots & \chi_1^{(n)}(s) \\ \dots & \dots & \dots \\ \chi_n^{(1)}(s) & \dots & \chi_n^{(n)}(s) \end{pmatrix} \tag{7.1}$$

consisting of n solutions $\boldsymbol{\chi}^{(j)}(s) = (\chi_1^{(j)}(s), \dots, \chi_n^{(j)}(s))^\top$ $(j = 1, \dots, n)$ of the homogeneous boundary-value problem

$$\mathbf{\Phi}^+(t) + \mathbf{D}(t)\mathbf{\Phi}^-(t) = 0, \quad t \in L, \tag{7.2}$$

which are analytic everywhere in a finite complex plane and admitting poles at infinity, is called the *canonical matrix* if it satisfies the following two properties:

(i) $\det \mathbf{X}_0(s) \neq 0$ everywhere in a finite complex plane;
(ii) the matrix $\mathbf{X}_0(s)$ is in normal form at infinity.

If the matrix $\mathbf{X}_0(s)$ possesses the property (i) only, then it is called a *normal matrix*.

The orders of the columns of the canonical matrix $\kappa_1 \leqslant \kappa_2 \leqslant \cdots \leqslant \kappa_n$ are called the *partial indices* of the boundary-value problem. The partial indices play an essential part in the theory of solvability of a vector boundary-value problem and in the theory of approximate methods for vector Riemann–Hilbert problems. Indeed, the partial indices can be unstable. By the stability criterion for the partial indices (**23**, **28**), the system of the partial indices $\kappa_1 \leqslant \cdots \leqslant \kappa_n$ is stable if and only if $\kappa_n - \kappa_1 \leqslant 1$. If the partial indices do not satisfy the above criterion, then (**27**) in any neighbourhood of the matrix $\mathbf{D}(t)$ there exists a matrix $\mathbf{D}_\varepsilon(t)$ with the partial indices $\{\kappa_1', \dots, \kappa_n'\}$ which are different from $\kappa_1, \dots, \kappa_n$. Therefore the factorization factors $\mathbf{X}_\varepsilon^+(t), \mathbf{X}_\varepsilon^-(t)$ for the matrix $\mathbf{D}_\varepsilon(t)$ cannot be too close to the matrices $\mathbf{X}_0^+(t), \mathbf{X}_0^-(t)$, respectively. This circumstance may not guarantee the convergence of an approximate solution to the exact one.

Let us outline the procedure of construction of the canonical matrix for the problem (2.1). First, we write down the homogeneous problem in the form (7.2), that is,

$$\mathbf{\Phi}^+(t) + [\mathbf{G}(t)]^{-1}\mathbf{\Phi}^-(t) = 0, \quad t \in L. \tag{7.3}$$

Thus, the original matrix we have to work with, is $[\mathbf{X}(s)]^{-1}$ given by (2.11). The matrix $\mathbf{X}(s)$ has poles at the points $s = \delta_1, s = \delta_2, s = \delta_3$ and $\det \mathbf{X}(s) = 0$ as $s = \sigma_1, s = \sigma_2, s = \sigma_3$. Therefore, its counterpart $[\mathbf{X}(s)]^{-1}$ possesses three poles σ_j and the determinant has three zeros: $\det[\mathbf{X}(\delta_j)]^{-1} = 0$ $(j = 1, 2, 3)$. To obtain the canonical matrix, we apply the Gakhov algorithm (**24**). The procedure consists of three steps. At the first stage, we reduce the matrix $[\mathbf{X}(s)]^{-1}$ to normal form at the points δ_j $(j = 1, 2, 3)$ and remove the zeros of its determinant. The second step of the algorithm eliminates the poles σ_j and transforms the matrix $[\mathbf{X}(s)]^{-1}$ into normal form at these points. Finally, we check whether the new matrix is in normal form at infinity and if not reduce it to normal form which, in fact, is the canonical matrix.

7.2 *A normal matrix*

We start with the point $s = \delta_1$, the zero of the function $\det[\mathbf{X}(s)]^{-1}$, and reduce the matrix to normal form at this point. In virtue of formula (6.5), in the vicinity of the point $s = \delta_1$, the matrix $[\mathbf{X}(s)]^{-1}$

admits the representation

$$[\mathbf{X}(s)]^{-1} = \begin{pmatrix} F_1\left(1-\dfrac{l_1}{v_1}\right)+F_{11}(s-\delta_1)+\ldots & -F_1\dfrac{m_1}{v_1}+F_{12}(s-\delta_1)+\ldots \\ -F_1\dfrac{n_1}{v_1}+F_{21}(s-\delta_1)+\ldots & F_1\left(1+\dfrac{l_1}{v_1}\right)+F_{22}(s-\delta_1)+\ldots \end{pmatrix}, \quad (7.4)$$

where $F_1 \neq 0$ and $F_{jr} \neq 0$ $(j, r = 1, 2)$ are constants. Here and later we use the notation

$$(v_j, l_j, m_j, n_j) = (w, l, m, n)(\delta_j) \quad (j = 1, 2, 3) \quad \text{and} \quad v_j^2 = l_j^2 + m_j n_j. \quad (7.5)$$

Clearly, the order of $\det[\mathbf{X}(s)]^{-1}$ at $s = \delta_1$ is 1 while the sum of the orders of the columns is 0. The matrix is not in normal form at this point. Introduce the matrix $\mathbf{T}_1^\circ$

$$\mathbf{T}_1^\circ = \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix}, \quad (7.6)$$

where $t_1 = (v_1 - l_1)m_1^{-1}$. The matrix $[\mathbf{X}(s)]^{-1}\mathbf{T}_1^\circ$ is in normal form at the point $s = \delta_1$ and the orders of the columns are 1 and 0. Therefore the matrix that eliminates the zero of $\det[\mathbf{X}(s)]^{-1}$ has the following form:

$$\mathbf{T}_1(s) = \begin{pmatrix} (s-\delta_1)^{-1} & 0 \\ t_1(s-\delta_1)^{-1} & 1 \end{pmatrix}. \quad (7.7)$$

The determinant of the new matrix $[\mathbf{X}(s)]^{-1}\mathbf{T}_1(s)$ does not vanish and the elements of this matrix are still analytic at the point $s = \delta_1$. We repeat the procedure and find the next matrix of transformation $\mathbf{T}_2(s)$

$$\mathbf{T}_2(s) = \begin{pmatrix} \dfrac{\delta_2-\delta_1}{s-\delta_2} & 0 \\ \dfrac{t_2-t_1}{s-\delta_2} & 1 \end{pmatrix} \quad (7.8)$$

with $t_2 = (v_2 - l_2)m_2^{-1}$. To proceed further we need the expression of the product of the two matrices $\mathbf{T}_1(s)$ and $\mathbf{T}_2(s)$, that is,

$$\mathbf{T}_1(s)\mathbf{T}_2(s) = \begin{pmatrix} \hat{z}_0(s) & 0 \\ z_0(s) & 1 \end{pmatrix}, \quad (7.9)$$

where

$$\hat{z}_0(s) = \frac{\delta_2-\delta_1}{(s-\delta_1)(s-\delta_2)}, \quad z_0(s) = \frac{t_1(\delta_2-\delta_1)}{(s-\delta_1)(s-\delta_2)} + \frac{t_2-t_1}{s-\delta_2}. \quad (7.10)$$

It is clear that $\det\{[\mathbf{X}(s)]^{-1}\mathbf{T}_1(s)\mathbf{T}_2(s)\} \neq 0$ at the points $s = \delta_1, \delta_2$, and the new matrix is analytic at these points. The last zero of the determinant of the matrix $[\mathbf{X}(s)]^{-1}$, the point $s = \delta_3$, can be eliminated by the matrix

$$\mathbf{T}_3(s) = \begin{pmatrix} \dfrac{(\delta_3-\delta_1)(\delta_3-\delta_2)}{(\delta_2-\delta_1)(s-\delta_3)} & 0 \\ \left(t_3 - \dfrac{\delta_3-\delta_1}{\delta_2-\delta_1}t_2 - \dfrac{\delta_2-\delta_3}{\delta_2-\delta_1}t_1\right)\dfrac{1}{s-\delta_3} & 1 \end{pmatrix}, \quad (7.11)$$

where $t_3 = (v_3 - l_3)m_3^{-1}$. Therefore, the matrix $[\mathbf{X}(s)]^{-1}\mathbf{T}(s)$ is not singular at the points δ_1, δ_2 and δ_3. Here we introduce the notation

$$\mathbf{T}(s) = \mathbf{T}_1(s)\mathbf{T}_2(s)\mathbf{T}_3(s) = \begin{pmatrix} \hat{z}_1(s) & 0 \\ z_1(s) & 1 \end{pmatrix}, \tag{7.12}$$

where

$$\hat{z}_1(s) = \frac{(\delta_3 - \delta_1)(\delta_3 - \delta_2)}{(s - \delta_1)(s - \delta_2)(s - \delta_3)},$$

$$z_1(s) = \frac{(\delta_3 - \delta_1)(\delta_3 - \delta_2)}{(s - \delta_2)(s - \delta_3)} \left(\frac{t_1}{s - \delta_1} + \frac{t_2 - t_1}{\delta_2 - \delta_1} \right) + \left(t_3 - \frac{\delta_3 - \delta_1}{\delta_2 - \delta_1} t_2 - \frac{\delta_2 - \delta_3}{\delta_2 - \delta_1} t_1 \right) \frac{1}{s - \delta_3}. \tag{7.13}$$

At the next stage, we remove the poles of the matrix $[\mathbf{X}(s)]^{-1}$, the points $\sigma_1, \sigma_2, \sigma_3$. The solution of the Riemann–Hilbert problem (3.1) has three zeros on the surface $\mathcal{R}$. Let the zero (σ_r, w_r) lie on the sheet $\mathbb{C}_j$ $(r = 1, 2, 3;\ j = 1, 2)$. Then

$$\frac{s - \sigma_r}{F(s, w)} = \hat{F}_j + O(s - \sigma_r), \quad (s, w) \to (\sigma_r, w_r) \in \mathbb{C}_j, \quad \hat{F}_j = \text{const} \neq 0,$$

$$\frac{s - \sigma_r}{F(s, -w)} = O(s - \sigma_r), \quad (s, -w) \to (\sigma_r, -w_r) \in \mathbb{C}_{3-j}. \tag{7.14}$$

The behaviour of the matrix $[\mathbf{X}(s)]^{-1}\mathbf{T}(s)(s - \sigma_1)$ at the point $s = \sigma_1$ is defined by

$$[\mathbf{X}(s)]^{-1}\mathbf{T}(s)(s - \sigma_1) = \frac{1}{2} \begin{pmatrix} \hat{F}_1 \left(1 + \frac{\hat{l}_1}{w_1} \right) + \hat{F}_{11}(s - \sigma_1) + \ldots & \hat{F}_1 \frac{\hat{m}_1}{w_1} + \hat{F}_{12}(s - \sigma_1) + \ldots \\ \hat{F}_1 \frac{\hat{n}_1}{w_1} + \hat{F}_{21}(s - \sigma_1) + \ldots & \hat{F}_1 \left(1 - \frac{\hat{l}_1}{w_1} \right) + \hat{F}_{22}(s - \sigma_1) + \ldots \end{pmatrix}, \tag{7.15}$$

where $\hat{F}_{jr} \neq 0$ $(j, r = 1, 2)$ are constants and

$$(w_j, \hat{l}_j, \hat{m}_j, \hat{n}_j) = (w, l, m, n)(\sigma_j) \quad (j = 1, 2, 3) \quad \text{and} \quad w_j^2 = \hat{l}_j^2 + \hat{m}_j \hat{n}_j. \tag{7.16}$$

Doubtless, the above algorithm leads to a new matrix, say, $[\mathbf{X}(s)]^{-1}\mathbf{R}(s)$ which is not singular at all the six points $s = \delta_r, s = \sigma_r$ $(r = 1, 2, 3)$. The following chain of formulae determines the matrix of transformation $\mathbf{R}(s)$:

$$\mathbf{R}(s) = (s - \sigma_1)(s - \sigma_2)(s - \sigma_3) \begin{pmatrix} \hat{z}_4(s) & 0 \\ z_4(s) & 1 \end{pmatrix}, \tag{7.17}$$

where

$$
\begin{aligned}
z_{r+1}(s) &= \left(\frac{z_r(s)-z_r(\sigma_r)}{\hat{z}_r(\sigma_r)}+\hat{t}_r\right)\frac{1}{s-\sigma_r}, \quad r=1,2,3,\\
\hat{z}_4(s) &= \frac{(\sigma_3-\delta_1)(\sigma_3-\delta_2)(\sigma_3-\delta_3)(\sigma_3-\sigma_1)(\sigma_3-\sigma_2)}{(s-\delta_1)(s-\delta_2)(s-\delta_3)(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)},\\
\hat{z}_3(s) &= \frac{(\sigma_2-\delta_1)(\sigma_2-\delta_2)(\sigma_2-\delta_3)(\sigma_2-\sigma_1)}{(s-\delta_1)(s-\delta_2)(s-\delta_3)(s-\sigma_1)(s-\sigma_2)},\\
\hat{z}_2(s) &= \frac{(\sigma_1-\delta_1)(\sigma_1-\delta_2)(\sigma_1-\delta_3)}{(s-\delta_1)(s-\delta_2)(s-\delta_3)(s-\sigma_1)},
\end{aligned}
\tag{7.18}
$$

and $\hat{t}_r = -(w_r+\hat{l}_r)\hat{m}_r^{-1}$, $r=1,2,3$. Without loss of generality, $\hat{t}_r \neq 0$. The matrix $[\mathbf{X}(s)]^{-1}\mathbf{R}(s)$ is analytic in any finite part of the complex plane, its columns are solutions of the homogeneous problem (7.3) and the determinant of the matrix does not vanish everywhere in a finite plane. By the definition, $[\mathbf{X}(s)]^{-1}\mathbf{R}(s)$ is a normal matrix of solutions.

7.3 *The canonical matrix*

To construct the canonical matrix of solutions, we have to reduce the normal matrix $\mathbf{X}_*(s) = [\mathbf{X}(s)]^{-1}\mathbf{R}(s)$ to normal form at infinity. For definiteness, we fix the behaviour of the polynomials $l(s)$, $m(s)$ and $n(s)$ at infinity:

$$
l(s) \sim s^4, \quad m(s) \sim m, \quad n(s) \sim n, \quad s \to \infty, \tag{7.19}
$$

where m, n are constants. The choice (7.19) is determined by the behaviour of the corresponding polynomials in the matrix arising in the problem of scattering by a perforated panel in section 8. Then the matrix $[\mathbf{X}(s)]^{-1}$ can be represented at infinity as follows:

$$
[\mathbf{X}(s)]^{-1} = \begin{pmatrix} F_0^{-1}+O(s^{-1}) & \frac{1}{2}m(F_0^{-1}-F_0)s^{-4}+O(s^{-5}) \\ \frac{1}{2}n(F_0^{-1}-F_0)s^{-4}+O(s^{-5}) & F_0+O(s^{-1}) \end{pmatrix}, \quad s \to \infty, \tag{7.20}
$$

where $F_0 \neq 0$ is the leading term in the expansion of the function $F(s,w)$ at infinity:

$$
F(s,w) \sim F_0^{(-1)^{j-1}} + O(s^{-1}), \quad (s,w) \to (\infty, \infty_j). \tag{7.21}
$$

Since the functions $\hat{z}_4(s)$ and $z_4(s)$ decay at infinity: $\hat{z}_4(s) = O(s^{-6})$ and $z_4(s) = O(s^{-1})$ as $s \to \infty$, it follows that the expansion of the normal matrix $\mathbf{X}_*(s)$ at infinity is given by

$$
\mathbf{X}_*(s) = \begin{pmatrix} a_{11}^{(1)}s^{-2}+a_{11}^{(2)}s^{-3}+\dots & a_{12}^{(1)}s^{-1}+a_{12}^{(2)}s^{-2}+\dots \\ a_{21}^{(1)}s^{2}+a_{21}^{(2)}s+\dots & a_{22}^{(1)}s^{3}+a_{22}^{(2)}s^{2}+\dots \end{pmatrix}, \quad s \to \infty, \tag{7.22}
$$

where

$$
a_{11}^{(1)} = \frac{m\hat{t}_3}{2}\left(\frac{1}{F_0}-F_0\right), \quad a_{12}^{(1)} = \frac{m}{2}\left(\frac{1}{F_0}-F_0\right),
$$

$$
a_{21}^{(1)} = \hat{t}_3 F_0, \quad a_{22}^{(1)} = F_0. \tag{7.23}
$$

Clearly, the other coefficients can be written down as well. It is seen that the orders of the columns at infinity are -2 and -3. At the same time, the order of the determinant at infinity equals 0. The matrix is not in normal form at infinity. Multiplying the matrix $\mathbf{X}_*(s)$ from the right by a polynomial matrix with a constant determinant enables us to increase the order of the second column up to -2. One of the matrices that does this is

$$\mathbf{U}_1(s) = \begin{pmatrix} 1 & \nu_1 s \\ 0 & 1 \end{pmatrix}, \quad \nu_1 = -\frac{a_{22}^{(1)}}{a_{21}^{(1)}}. \tag{7.24}$$

The above choice of the parameter ν_1 guarantees the relations

$$a_{j1}^{(1)}\nu_1 + a_{j2}^{(1)} = 0, \quad j = 1, 2. \tag{7.25}$$

Therefore, the new matrix $\mathbf{X}_*(s)\mathbf{U}_1(s)$ admits the following expansion at infinity:

$$\mathbf{X}_*(s)\mathbf{U}_1(s) = \begin{pmatrix} a_{11}^{(1)}s^{-2} + \dots & (\nu_1 a_{11}^{(2)} + a_{12}^{(2)})s^{-2} + \dots \\ a_{21}^{(1)}s^2 + \dots & (\nu_1 a_{21}^{(2)} + a_{22}^{(2)})s^2 + \dots \end{pmatrix}, \quad s \to \infty. \tag{7.26}$$

The sum of the orders of the columns at infinity equals -4. However, it still does not coincide with the order of the determinant, that is, 0. At the next stage, we multiply the matrix $\mathbf{X}_*(s)\mathbf{U}_1(s)$ from the right by the matrix

$$\mathbf{U}_2(s) = \begin{pmatrix} 1 & 0 \\ \nu_2 & 1 \end{pmatrix}, \quad \nu_2 = -\frac{a_{21}^{(1)}}{\nu_1 a_{21}^{(2)} + a_{22}^{(2)}} \tag{7.27}$$

and find that the orders at infinity of the matrix $\mathbf{X}_*(s)\mathbf{U}_1(s)\mathbf{U}_2(s)$ are different, namely, -1 and -2. The order of the determinant is still the same and is equal to zero. The leading terms of the expansion at infinity of the new matrix are given by

$$\begin{aligned} &\mathbf{X}_*(s)\mathbf{U}_1(s)\mathbf{U}_2(s) \\ &= \begin{pmatrix} \{a_{11}^{(1)} + \nu_2(\nu_1 a_{11}^{(2)} + a_{12}^{(2)})\}s^{-2} + \dots & (\nu_1 a_{11}^{(2)} + a_{12}^{(2)})s^{-2} + \dots \\ \{a_{21}^{(2)} + \nu_2(\nu_1 a_{21}^{(3)} + a_{22}^{(3)})\}s + \dots & (\nu_1 a_{21}^{(2)} + a_{22}^{(2)})s^2 + \dots \end{pmatrix}, \quad s \to \infty\cdot \end{aligned} \tag{7.28}$$

We carry on reducing the normal matrix to canonical form. Multiply successively from the right the matrix $\mathbf{X}_*(s)\mathbf{U}_1(s)\mathbf{U}_2(s)$ by the matrices $\mathbf{U}_3(s)$, $\mathbf{U}_4(s)$ and $\mathbf{U}_5(s)$, where

$$\mathbf{U}_j(s) = \begin{pmatrix} 1 & \nu_j s \\ 0 & 1 \end{pmatrix} \quad (j = 3, 5) \quad \text{and} \quad \mathbf{U}_4(s) = \begin{pmatrix} 1 & 0 \\ \nu_4 & 1 \end{pmatrix}. \tag{7.29}$$

The coefficients ν_3, ν_4, ν_5 are chosen as follows:

$$\begin{aligned} \nu_3 &= -\frac{\nu_1 a_{21}^{(2)} + a_{22}^{(2)}}{a_{21}^{(2)} + \nu_2(\nu_1 a_{21}^{(3)} + a_{22}^{(3)})}, \\ \nu_4 &= -\frac{a_{21}^{(2)} + \nu_2(\nu_1 a_{21}^{(3)} + a_{22}^{(3)})}{\nu_3[a_{21}^{(3)} + \nu_2(\nu_1 a_{21}^{(4)} + a_{22}^{(4)})] + \nu_1 a_{21}^{(3)} + a_{22}^{(3)}}, \\ \nu_5 &= -\frac{\nu_3[a_{21}^{(3)} + \nu_2(\nu_1 a_{21}^{(4)} + a_{22}^{(4)})] + \nu_1 a_{21}^{(3)} + a_{22}^{(3)}}{a_{21}^{(3)} + \nu_2(\nu_1 a_{21}^{(4)} + a_{22}^{(4)}) + \nu_4\{\nu_3[a_{21}^{(4)} + \nu_2(\nu_1 a_{21}^{(5)} + a_{22}^{(5)})] + \nu_1 a_{21}^{(4)} + a_{22}^{(4)}\}}. \end{aligned} \tag{7.30}$$

Now we have achieved the desired behaviour of the matrix at infinity:

$$\mathbf{X}_*(s)\mathbf{U}(s) = \begin{pmatrix} \hat{a}_{11}s^{-1} + \dots & \hat{a}_{12} + \dots \\ \hat{a}_{21} + \dots & \hat{a}_{22} + \dots \end{pmatrix}, \quad s \to \infty. \tag{7.31}$$

Here $\hat{a}_{rj} \neq 0$ $(r, j = 1, 2)$ are constants and

$$\mathbf{U}(s) = \prod_{j=1}^{5} \mathbf{U}_j(s)$$
$$= \begin{pmatrix} 1 + \nu_1\nu_2 s + \nu_4 s(\nu_1 + \nu_3 + \nu_1\nu_2\nu_3 s) & [\nu_5(1 + \nu_1\nu_2 s) + (1 + \nu_4\nu_5 s)(\nu_1 + \nu_3 + \nu_1\nu_2\nu_3 s)]s \\ \nu_2 + \nu_4(1 + \nu_2\nu_3 s) & \nu_2\nu_5 s + (1 + \nu_2\nu_3 s)(1 + \nu_4\nu_5 s) \end{pmatrix}. \tag{7.32}$$

The constructed matrix

$$\mathbf{X}_0(s) = [\mathbf{X}(s)]^{-1}\mathbf{R}(s)\mathbf{U}(s) \tag{7.33}$$

possesses the following properties:

(i) it is a normal matrix of solutions;
(ii) $\lim_{s\to\infty} \det \mathbf{X}_0(s) = -\hat{a}_{12}\hat{a}_{21} \neq 0$;
(iii) the orders of both columns at infinity are equal to zero.

Hence the matrix $\mathbf{X}_0(s)$ is the canonical matrix of solutions and the partial indices are $\kappa_1 = 0$, $\kappa_2 = 0$. By the stability criterion, the partial indices are stable. The total index of the vector Riemann–Hilbert problem (2.1) is $\kappa = \operatorname{ind} \det \mathbf{G}(t) = \operatorname{ind} \lambda_1(t) + \operatorname{ind} \lambda_2(t) = 0$. At the same time, it is the sum of the partial indices, and therefore, again, $\kappa = \kappa_1 + \kappa_2 = 0$.

8. Scattering by a perforated sandwich panel

8.1 *Formulation*

As an illustration of the proposed technique, we find an exact solution of the problem of scattering of sound waves by the edges of a sandwich panel. The panel consists of two thin semi-infinite elastic plates. The first plate $\{-\infty < x < 0, y = \pm 0\}$ is a rigid screen. The second one $\{0 < x < \infty, y = \pm 0\}$ is a perforated panel with acoustically rigid walls (Fig. 5). The two plates are clamped in such a way that the displacement and gradient are zero at the point $x = 0$, $y = 0$.

Let the primary source be an incident plane wave of potential $\phi_{\text{inc}} = \exp\{ik(x\sin\theta + y\cos\theta)\}$, $y < 0$, where $k = \omega_0/c_0$ is the acoustic wave number, ω_0 is an angular frequency, c_0 is the sound speed in the fluid. The total velocity potential can be represented in the form

$$(\phi_{\text{inc}} + \phi_{\text{ref}} + \phi_0)e^{-i\omega_0 t} \quad (y < 0), \qquad \phi_1 e^{-i\omega_0 t} \quad (y > 0), \tag{8.1}$$

where ϕ_{ref} is the potential of a reflected wave $\phi_{\text{ref}} = \exp\{ik(x\sin\theta - y\cos\theta)\}$, $y < 0$, and the potentials ϕ_0, ϕ_1 are solutions of the Helmholtz equation

$$(\Delta + k^2)\phi_0 = 0, \quad |x| < \infty, \quad y < 0,$$

$$(\Delta + k^2)\phi_1 = 0, \quad |x| < \infty, \quad y > 0, \tag{8.2}$$

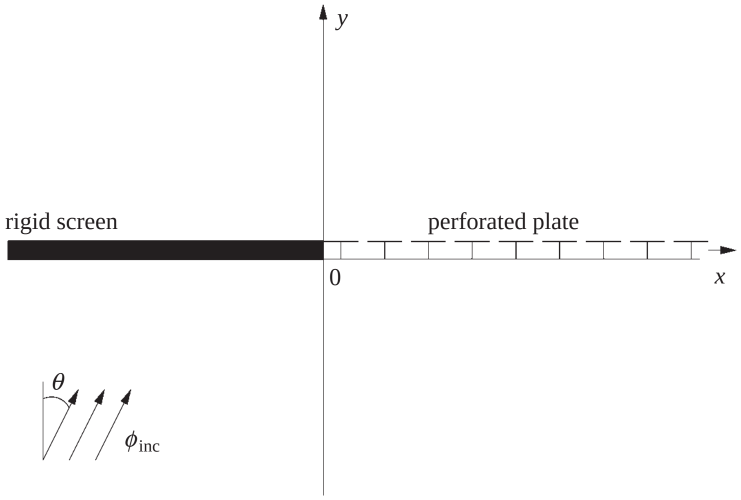

Fig. 5 Geometry of the physical problem

with $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The suffices 0 and 1 refer respectively to the unperforated and perforated sides of the plate. On the rigid screen, the potentials satisfy the boundary conditions

$$\frac{\partial \phi_0}{\partial y} = \frac{\partial \phi_1}{\partial y} = 0, \quad -\infty < x < 0, \quad y = 0. \tag{8.3}$$

On the perforated panel, the Leppington boundary conditions are imposed (**33**)

$$\left(\frac{1}{\mu^4}\frac{\partial^4}{\partial x^4} - 1\right)\frac{\partial \phi_0}{\partial y} + \alpha(\phi_1 - \phi_0) = 2\alpha e^{ikx \sin\theta}, \quad 0 < x < \infty, \quad y = 0,$$

$$\frac{\partial \phi_0}{\partial y} - \frac{\partial \phi_1}{\partial y} - \tau k \phi_1 = 0, \quad 0 < x < \infty, \quad y = 0, \tag{8.4}$$

where $\mu^4 = M_p \omega_0^2 / B_p$, $\alpha = \rho / M_p$, ρ is the mean fluid density, M_p is the mass per unit area of the plate and B_p is the bending stiffness of the plate, τ is the Leppington parameter:

$$\tau = \frac{kd}{1 - k^2 V/(2r_a)}. \tag{8.5}$$

Here r_a is the aperture radius, V is the cell volume, d is the plate separation of the panel. In addition, the edge conditions are stipulated as

$$\frac{\partial \phi_0}{\partial y} = \frac{\partial^2 \phi_0}{\partial x \partial y} = 0, \quad x = y = 0. \tag{8.6}$$

This problem was stated and reduced to a vector Riemann–Hilbert problem by Jones (**34**). We write down those intermediate relations that are essential for the further analysis. Assume that $k = k' + ik''$ $(0 < k'' \ll k'$ and $k'' < \operatorname{Im}(s_1)$, $s_1 = \rho_0 \exp\{i\theta_1\})$. Let $\mu = \mu' + i\mu''$ $(0 < \mu'' \ll \mu')$. Applying the Laplace transform

$$\Phi_{j+}(y;s) = \int_0^\infty \phi_j(x,y)e^{isx}dx, \quad \Phi_{j-}(y;s) = \int_{-\infty}^0 \phi_j(x,y)e^{isx}dx \quad (j=0,1) \tag{8.7}$$

to the boundary-value problem (8.2), (8.3), (8.4) reduces it to a new problem for the ordinary differential equation

$$\left(\frac{d^2}{dy^2} - \gamma^2\right)[\Phi_{j-}(y;s) + \Phi_{j+}(y;s)] = 0, \quad j=0 \quad \text{if} \quad y<0 \quad \text{and} \quad j=1 \quad \text{if} \quad y>0,$$

$$\frac{d}{dy}\Phi_{0-}(0;s) = \frac{d}{dy}\Phi_{1-}(0;s) = 0,$$

$$(s^4-\mu^4)\frac{d}{dy}\Phi_{0+}(0;s) + \alpha\mu^4[\Phi_{1+}(0;s) - \Phi_{0+}(0;s)] = \frac{2i\alpha\mu^4}{s+k\sin\theta} - N(s)$$

$$\frac{d}{dy}\Phi_{0+}(0;s) - \frac{d}{dy}\Phi_{1+}(0;s) - \tau k\Phi_{1+}(0;s) = 0, \tag{8.8}$$

where

$$\gamma^2 = s^2 - k^2, \quad N(s) = is\phi'_{0xx}(0,0) - \phi'_{0xxx}(0,0) \tag{8.9}$$

and

$$\phi'_{0xx}(0,0) = \frac{\partial^3\phi_0}{\partial x^2\partial y}(0,0), \quad \phi'_{0xxx}(0,0) = \frac{\partial^4\phi_0}{\partial x^3\partial y}(0,0) \tag{8.10}$$

are constants to be determined. The problem (8.8) is reducible to the following Riemann–Hilbert problem:

$$\mathbf{G}(t)\mathbf{\Phi}^+(t) + \mathbf{\Phi}^-(t) = \mathbf{g}(t), \quad t \in L, \tag{8.11}$$

where

$$\mathbf{\Phi}^+(s) = \begin{pmatrix} \Phi_{0+}(0;s) + 2i(s+k\sin\theta)^{-1} + N_1 s + N_2 \\ \Phi_{1+}(0;s) \end{pmatrix}, \quad \mathbf{\Phi}^-(s) = \begin{pmatrix} \Phi_{0-}(0;s) \\ \Phi_{1-}(0;s) \end{pmatrix},$$

$$\mathbf{G}(t) = \frac{1}{\gamma(t)(t^4-\mu^4)}\begin{pmatrix} \gamma(t)(t^4-\mu^4) - \alpha\mu^4 & \alpha\mu^4 \\ \alpha\mu^4 & (\gamma(t)-\tau k)(t^4-\mu^4) - \alpha\mu^4 \end{pmatrix},$$

$$\mathbf{g}(t) = \left(\frac{2i}{t+k\sin\theta} + N_1 t + N_2\right)\mathbf{J}, \quad \mathbf{J} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$N_1 = -\frac{i\phi'_{0xx}(0,0)}{\alpha\mu^4}, \quad N_2 = \frac{\phi'_{0xxx}(0,0)}{\alpha\mu^4}. \tag{8.12}$$

Here and subsequently the s-plane is cut along the straight lines from $k = k' + ik''$ to $ik'' + \infty$ and from $-k = -k' - ik''$ to $-ik'' - \infty$. When $k'' \to 0$, the contour L becomes the real axis, passing above the branch point at $s = -k$, below the branch point at $s = k$ and above the point $s = -k\sin\theta$. The branch of the function $\gamma(s)$ is chosen such that $\gamma(0) = -ik$.

8.2 *Solution by quadratures*

It is seen that the matrix coefficient $\mathbf{G}(s)$ possesses the structure (2.2). Indeed, if we put

$$b(t)=\frac{(\gamma(t)-\tau k/2)(t^4-\mu^4)-\alpha\mu^4}{\gamma(t)(t^4-\mu^4)},\quad c(t)=\frac{\tau k}{2\gamma(t)(t^4-\mu^4)},$$

$$l(t)=t^4-\mu^4,\quad m(t)=n(t)=\frac{2\alpha\mu^4}{\tau k},\tag{8.13}$$

then the matrix (8.12) coincides with (2.2). The characteristic polynomial becomes

$$f(t)=l^2(t)+m(t)n(t)=t^8-2\mu^4t^4+\mu^8\left(1+\frac{4\alpha^2}{\tau^2k^2}\right).\tag{8.14}$$

Letting $k''=+0$, $\mu''=+0$, we achieve

$$M_1=\mu^4,\quad M_2=\mu^8\left(1+\frac{4\alpha^2}{\tau^2k^2}\right)>M_1^2,\tag{8.15}$$

that is, the condition (2.4) is satisfied. Since the functions

$$\lambda_1(t)=b(t)+c(t)f^{\frac{1}{2}}(t),\quad \lambda_2(t)=b(t)-c(t)f^{\frac{1}{2}}(t)\tag{8.16}$$

are even with respect to t, it follows that the indices of the functions $\lambda_1(t)$, $\lambda_2(t)$ are zero and come to agree with the conditions (2.5). The integrals (3.17) admit the following simplification:

$$d_\nu^\circ=-\frac{1}{\pi i}\int_{L^+}\log\frac{\lambda_1(t)}{\lambda_2(t)}\frac{t^{\nu-1}}{f^{\frac{1}{2}}(t)}dt,\quad \nu=1,3,\quad d_2^\circ=0,\tag{8.17}$$

with $L^+=\{\mathrm{Re}(s)\in(0,k'),\mathrm{Im}(s)=0\}\cup\{\mathrm{Re}(s)\in(k',\infty),\mathrm{Im}(s)=-0\}$. The factorization of the matrix $\mathbf{G}(t)$ is given by formulae (2.10), (2.11), (3.5), (3.13).

We note that if $k''=+0$ and $\mu''=+0$ then the first integral in (3.13)

$$\Lambda_0(s)=\frac{1}{4\pi i}\int_L\log\{\lambda_1(t)\lambda_2(t)\}\frac{dt}{t-s}\tag{8.18}$$

has singularities at the points $s=-k,-\mu,k,\mu$. Indeed, from (8.13), (8.16) we get

$$\lambda_1(t)\lambda_2(t)=\frac{1}{(t^2-k^2)(t^4-\mu^4)}[\gamma(t)(\gamma(t)-\tau k)(t^4-\mu^4)-2\alpha\mu^4\gamma(t)+\alpha\mu^4\tau k].\tag{8.19}$$

Let us select a branch of the function $\log(s+k)$ on the upper side of the cut $(-\infty,-k)$ and pick up a branch of $\log(s-k)$ on the lower side of the cut (k,∞). Then

$$\begin{aligned}&\log(t+k)=[\log(t+k)]^+,\quad [\log(t+k)]^-=\log(t+k)-2\pi i,\\&\log(t-k)=[\log(t-k)]^-,\quad [\log(t-k)]^+=\log(t-k)+2\pi i.\end{aligned}\tag{8.20}$$

For the chosen branches we obtain (**35**)

$$\Lambda_0(s) = \begin{cases} \frac{1}{2}\log(s+a) + \Omega_1(s), & s \to -a,\ s \in \mathbb{C}^+, \\ \Omega_1(s), & s \to -a,\ s \in \mathbb{C}^-, \end{cases}$$

$$\Lambda_0(s) = \begin{cases} \Omega_2(s), & s \to a,\ s \in \mathbb{C}^+, \\ -\frac{1}{2}\log(s-a) + \Omega_2(s), & s \to a,\ s \in \mathbb{C}^-, \end{cases} \tag{8.21}$$

where $a = k, \mu$ and $\Omega_1(s), \Omega_2(s)$ are analytic in the vicinity of the points $s = -a, s = a$, respectively. Next, by substituting formulae (8.21) into (3.13), in virtue of relations (2.11), (2.12) we get the behaviour of the matrices $\mathbf{X}^{\pm}(s)$ at the singular points

$$\begin{aligned} &\mathbf{X}^+(s) \sim (s+a)^{1/2}\mathbf{I}, \quad s \to -a \quad (a = k, \mu), \\ &\mathbf{X}^-(s) \sim (s-a)^{-1/2}\mathbf{I}, \quad s \to a \quad (a = k, \mu) \end{aligned} \tag{8.22}$$

and the matrices $\mathbf{X}^+(s)$ and $\mathbf{X}^-(s)$ are analytic at the points $s = a$ and $s = -a$, respectively.

In the particular case (8.11) of the vector $\mathbf{g}(t)$ there is no need to introduce the Cauchy integral (6.3). As before, in section 6, to construct the solution of the non-homogeneous boundary-value problem (8.11), instead of the canonical matrix $\mathbf{X}_0(s)$ that possesses rather cumbersome form, we use the factorization matrix $\mathbf{X}(s)$. Eliminating the pole at the point $s = -k\sin\theta$ and substituting the factorization (2.10) into the boundary condition (8.11) we obtain

$$\begin{aligned} &\mathbf{X}^+(t)\mathbf{\Phi}^+(t) - \frac{2i}{t + k\sin\theta}\mathbf{X}^-(-k\sin\theta)\mathbf{J} \\ &\quad = -\mathbf{X}^-(t)\mathbf{\Phi}^-(t) + \mathbf{X}^-(t)(N_1 t + N_2)\mathbf{J} + \frac{2i}{t + k\sin\theta}[\mathbf{X}^-(t) - \mathbf{X}^-(-k\sin\theta)]\mathbf{J}\cdot \end{aligned} \tag{8.23}$$

Observe now that the function $\varphi(s, w)$ defined by (3.13) satisfies the conditions (3.16) and therefore the function $F(s, w)$ is bounded at infinity:

$$\begin{aligned} &F(s, w) = F_0 + O(s^{-1}), \quad s \to \infty, \quad (s, w) \in \mathbb{C}_1, \\ &F(s, -w) = F_0^{-1} + O(s^{-1}), \quad s \to \infty, \quad (s, -w) \in \mathbb{C}_2, \\ &F_0 = \exp\left\{-\frac{1}{2}\sum_{j=1}^{3}\left(\int_{(\delta_j, v_j)}^{(\sigma_j, w_j)} \frac{t^3 dt}{\xi(t)} + m_j A_{4j} + n_j B_{4j}\right)\right\}, \end{aligned} \tag{8.24}$$

with A_{4j} defined by (5.46) and B_{4j} given by

$$B_{41} = -\frac{i}{8}\int_{-\theta_0}^{\theta_0} \frac{e^{2i\theta}}{|\xi_0(\theta)|}d\theta, \quad B_{42} = -2B_{41}, \quad B_{43} = -B_{41}. \tag{8.25}$$

Therefore, the behaviour of the factorization matrix $\mathbf{X}(s)$ at infinity becomes

$$\mathbf{X}^{\pm}(s) \sim \mathrm{diag}\{F_0, F_0^{-1}\}, \quad s \to \infty, \quad s \in \mathbb{C}^{\pm}. \tag{8.26}$$

Then we follow the scheme of section 6 and use that $\Phi_1^+(s) = N_1 s + N_2 + O(s^{-1}),\ s \to \infty$. The desired solution is given by

$$\mathbf{\Phi}^{+}(s) = [\mathbf{X}^{+}(s)]^{-1}\left[\mathbf{\Psi}(s) + \sum_{j=1}^{3}\frac{C_j}{s-\delta_j}\begin{pmatrix}1\\ \zeta_j\end{pmatrix}\right], \quad s \in \mathbb{C}^{+},$$

$$\begin{aligned}\mathbf{\Phi}^{-}(s) = -[\mathbf{X}^{-}(s)]^{-1}&\left[\mathbf{\Psi}(s) + \sum_{j=1}^{3}\frac{C_j}{s-\delta_j}\begin{pmatrix}1\\ \zeta_j\end{pmatrix}\right]\\ &+\left(\frac{2i}{s+k\sin\theta} + N_1 s + N_2\right)\mathbf{J}, \quad s \in \mathbb{C}^{-},\end{aligned} \tag{8.27}$$

where

$$\mathbf{\Psi}(s) = \frac{2i}{s+k\sin\theta}\mathbf{X}^{-}(-k\sin\theta)\mathbf{J} + F_0(N_1 s + N_2)\mathbf{J}, \tag{8.28}$$

and the constants C_1, C_2, C_3 are defined by (6.18) with the elements $\Psi_1(s), \Psi_2(s)$ of the vector (8.28). Finally, we find the two unknown constants N_1 and N_2. These constants should be fixed from the two conditions

$$\Phi_{0+}(0; s) - \Phi_{1+}(0; s) + \frac{2i}{s+k\sin\theta} - \frac{N(s)}{\alpha\mu^4} = 0 \qquad \text{as} \quad s = \mu, i\mu \in \mathbb{C}^{+} \tag{8.29}$$

which follow from the definition of the function $d\Phi_{0+}(0; s)/dy$:

$$\frac{d}{dy}\Phi_{0+}(0; s) = \frac{\mu^4\alpha}{s^4-\mu^4}\left[\Phi_{0+}(0; s) - \Phi_{1+}(0; s) + \frac{2i}{s+k\sin\theta} - \frac{N(s)}{\alpha\mu^4}\right], \tag{8.30}$$

and ensure the analyticity of the right-hand side in (8.30) at the points $s = \mu, i\mu$. The definition of the constants $\phi'_{0xx}(0, 0)$ and $\phi'_{0xxx}(0, 0)$ from (8.12) completes the solution of the problem.

9. Concluding remarks

This paper has derived a closed-form solution of a vector Riemann–Hilbert boundary-value problem when the matrix coefficient admits the Chebotarev–Khrapkov form and the corresponding polynomial $f(s)$ is of degree eight. The procedure involves reducing the vector problem to a scalar Riemann–Hilbert boundary-value problem on a hyperelliptic surface of genus 3. A meromorphic solution has been constructed in terms of Weierstrass and abelian integrals. The upper limits of some abelian integrals are arbitrary points of the Riemann surface. In addition, the solution possesses six arbitrary integers. An algebraic behaviour of the solution at infinity has been achieved by stipulating three conditions for the three points of the surface and the six integers which gives rise to the classical Jacobi inversion problem. A solution of this problem was found in terms of the zeros of Riemann's Θ-function. The zeros have been determined by quadratures for the surface of genus $h = 3$. It is clear that the zeros are evaluated easier if the genus h is less than 3. In the general case, when $h > 3$ the zeros of Riemann's Θ-function can be found either from an algebraic equation of degree h, or by implementing the direct numerical procedure proposed in Appendix B. Both approaches find the zeros with any accuracy.

Three factorizations of the matrix coefficient have been constructed. The first is commutative:

$\mathbf{G}(t) = \mathbf{X}^+(t)[\mathbf{X}^-(t)]^{-1} = [\mathbf{X}^-(t)]^{-1}\mathbf{X}^+(t)$, $t \in L$. It was explicitly shown that the matrix $\mathbf{X}(s)$ is bounded at infinity. In a finite complex plane, the matrix $\mathbf{X}(s)$ has three simple poles and three points where it is singular: $\det \mathbf{X}(s) = 0$. By the generalized Liouville's theorem, this factorization leads to exact formulae for the solution of the original vector Riemann–Hilbert problem and three arbitrary constants. The constants were easily found from a linear algebraic system of three equations eliminating the poles of the solution. The second factorization relates to the problem $\mathbf{\Phi}^+(t) + [\mathbf{G}(t)]^{-1}\mathbf{\Phi}^-(t) = \mathbf{g}_0(t)$, $t \in L$, where $\mathbf{g}_0(t) = [\mathbf{G}(t)]^{-1}\mathbf{g}(t)$. The factorization is not commutative and is given by the normal matrix $[\mathbf{X}(s)]^{-1}\mathbf{R}(s)$, where $\mathbf{R}(s)$ is a special rational matrix. The normal matrix does not have poles and is not singular in any finite part of the complex plane. However, it is not in normal form at infinity. We needed the normal matrix to construct the third factorization provided by the canonical matrix of the vector Riemann–Hilbert problem: $\mathbf{X}_0(s) = [\mathbf{X}(s)]^{-1}\mathbf{R}(s)\mathbf{U}(s)$, where $\mathbf{U}(s)$ is a specified polynomial matrix. The matrix $\mathbf{X}_0(s)$ is in normal form at infinity, does not have poles and is not singular in a finite part of the complex plane. The orders of its columns at infinity are equal to 0. The canonical matrix gives rise to the partial indices of the vector boundary-value problem, which are $\kappa_1 = 0$ and $\kappa_2 = 0$. This circumstance indicates the stability of the vector Riemann–Hilbert problem (8.11).

The problem of scattering by a semi-infinite perforated sandwich panel has been solved by quadratures for any range of the parameters involved in the matrix coefficient.

To summarize, we list the main steps of the algorithm building the closed-form solution to the vector Riemann–Hilbert problem corresponding to the analysed scattering problem.

1. Factorizing the matrix $\mathbf{G}(t)$ by (2.10), (2.11) in terms of the solution (3.5), (3.13) to the scalar Riemann–Hilbert problem on the surface $\mathcal{R}$.
2. Eliminating the essential singularity at infinity of the factorizing matrix by setting Jacobi's inversion problem (3.20) for three unknown points $(\sigma_j, w_j) \in \mathcal{R}$ and six integers n_j, m_j $(j = 1, 2, 3)$.
3. Evaluating the A- and B-periods of the abelian integrals by (5.17). Normalizing the basis of these integrals by (5.36) and computing the B-periods of the canonical basis by (5.30).
4. Deriving the cubic equation (5.45) for the unknowns σ_j $(j = 1, 2, 3)$. Finding Riemann's constants by (5.62) and the residues (5.44) by (5.77) that are crucial for constructing the coefficients of the cubic equation.
5. Determining the six integers n_j, m_j $(j = 1, 2, 3)$ by (5.78).
6. Constructing a solution of the vector Riemann–Hilbert problem by (6.12). This solution possesses three simple poles at $s = \sigma_j$ and three arbitrary constants C_j $(j = 1, 2, 3)$. Removing these poles fixes the constants by (6.18) and completes the procedure.

At the next stage, the authors aim

(i) to analyse how the order of Jacobi's inversion problem reflects the symmetry of the Riemann surface and properties of the characteristic functions;

(ii) to find a closed-form solution of the problem on scattering by a perforated sandwich panel with acoustically transparent walls. In this case, the second boundary condition is replaced by (**33**)

$$\left[1 - \frac{d}{N\sigma}(\Delta + k^2)\right]\left(\frac{\partial \phi_1}{\partial y} - \frac{\partial \phi_0}{\partial y}\right) + d(\Delta + k^2)\phi_1 = 0,$$

where Δ is the Laplace operator, d is the plate separation, N is the number of apertures per unit area and σ is the complex conductivity of an aperture. The corresponding vector Riemann–Hilbert problem is equivalent to a scalar Riemann–Hilbert problem on a hyperelliptic surface of genus 5.

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APPENDIX A

Derivation of the quadric equation for genus 4

We describe the procedure[†] of reduction of the system (4.25) for $h = 4$

$$\sigma_1^\nu + \sigma_2^\nu + \sigma_3^\nu + \sigma_4^\nu = \varepsilon_\nu, \quad \nu = 1, 2, 3, 4, \tag{A.1}$$

[†] Essentially, this device is employed by A. Y. Zemlyanova.

to the quadric equation

$$\sigma^4 - c_1\sigma^3 + c_2\sigma^2 - c_3\sigma + c_4 = 0 \tag{A.2}$$

with the coefficients c_j ($j = 1, \ldots, 4$) to be determined. The derivation will use the Viéte theorem:

$$\begin{aligned}
&\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = c_1,\\
&\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_1\sigma_4 + \sigma_2\sigma_3 + \sigma_2\sigma_4 + \sigma_3\sigma_4 = c_2,\\
&\sigma_1\sigma_2\sigma_3 + \sigma_1\sigma_2\sigma_4 + \sigma_1\sigma_3\sigma_4 + \sigma_2\sigma_3\sigma_4 = c_3,\\
&\sigma_1\sigma_2\sigma_3\sigma_4 = c_4.
\end{aligned} \tag{A.3}$$

By comparing (A.1) for $\nu = 1$ and $\nu = 2$ with (A.3) we get

$$c_1 = \varepsilon_1, \quad (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)^2 = \varepsilon_2 + 2c_2 \tag{A.4}$$

and therefore $c_2 = \frac{1}{2}(\varepsilon_1^2 - \varepsilon_2)$. To find c_3 we notice that

$$\begin{aligned}
(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)^3 - (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_1\sigma_4 + \sigma_2\sigma_3 + \sigma_2\sigma_4 + \sigma_3\sigma_4)&\\
= \varepsilon_3 + 2c_2\varepsilon_1 - 3c_3.&
\end{aligned} \tag{A.5}$$

On the other hand, the left-hand side of (A.5) equals $\varepsilon_1^3 - c_2\varepsilon_1$. Comparing these results gives

$$c_3 = \frac{\varepsilon_1^3}{6} - \frac{\varepsilon_1\varepsilon_2}{2} + \frac{\varepsilon_3}{3}. \tag{A.6}$$

Finally, the find c_4. Consider the expression

$$\begin{aligned}
S = {}&(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)^4\\
&-(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)^2(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_1\sigma_4 + \sigma_2\sigma_3 + \sigma_2\sigma_4 + \sigma_3\sigma_4)\\
&+(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)(\sigma_1\sigma_2\sigma_3 + \sigma_1\sigma_2\sigma_4 + \sigma_1\sigma_3\sigma_4 + \sigma_2\sigma_3\sigma_4).
\end{aligned} \tag{A.7}$$

Opening the brackets enables us to express S in terms of ε_ν and c_ν ($\nu = 1, 2, 3, 4$):

$$S = \varepsilon_4 + 3\varepsilon_1(c_2\varepsilon_1 - c_3) - 2c_2^2 + 4c_4. \tag{A.8}$$

In addition to this result, use of (A.3) yields $S = \varepsilon_1^4 - \varepsilon_1^2 c_2 + \varepsilon_1 c_3$. Therefore,

$$c_4 = \frac{\varepsilon_1^4}{24} - \frac{\varepsilon_1^2\varepsilon_2}{4} + \frac{\varepsilon_1\varepsilon_3}{3} + \frac{\varepsilon_2^2}{8} - \frac{\varepsilon_4}{4}. \tag{A.9}$$

Clearly, this technique can be extended for any $h > 4$.

APPENDIX B

Alternative numerical procedure for Jacobi's problem

We have shown that the nonlinear algebraic system (4.25) is equivalent to an algebraic equation of degree h. Obviously, for $h \leqslant 4$ this equation is solvable by radicals. If the genus of the surface $\mathcal{R}$ is higher than 4, we are unlikely to find the roots of the system by radicals. We propose an alternative way to define the zeros of the Riemann θ-function which provide the solution to the Jacobi inverse problem. This is a direct numerical procedure based on the argument principle for an analytic function and on analysing the real and imaginary parts of the function $\mathfrak{F}(q)$. This function is analytic and single-valued on $\hat{\mathcal{R}}$. It has precisely h zeros on $\hat{\mathcal{R}}$ (see section 4.1). Let the two sheets of the surface $\mathcal{R}$ be glued crosswise along arcs of the circle Σ_{ρ_0} of radius ρ_0, centred at the origin as, for instance, in section 3 (Fig. 1). Then the zeros of the function $\mathfrak{F}(q)$ can be found by executing the following steps.

1. Find the zeros of the function $\mathfrak{F}(q)$ which lie inside of the circle Σ_{ρ_0} on the first sheet $\mathbb{C}_1$. Start with the origin and verify whether the function $\mathfrak{F}(q)$ vanishes at the point $q_0 = (0, f^{\frac{1}{2}}(0)) \in \mathbb{C}_1$. If $\mathfrak{F}(q_0) = 0$, define the order of the zero. Let the order be n_0. Obviously, $n_0 = 0$ if $\mathfrak{F}(q_0) \neq 0$. Choosing a sufficiently small positive $r \ll \rho_0$ evaluate the increment of the argument of the function $\mathfrak{F}(q)$ $(q = (s, w),\ s = r\exp(\psi))$ when q traverses the circle Σ_r of radius r, centred at the origin. It is clear that there exists such a number $r = \varepsilon$ that $[\arg \mathfrak{F}(q)]_{\Sigma_\varepsilon} = 2\pi n_0$. This means that the only zero of the function $\mathfrak{F}(q)$ inside of the circle Σ_ε is the point $q = q_0$. Next, increase r and define such a number $r = r_* \in (\varepsilon, \rho_0)$ providing $[\arg \mathfrak{F}(q)]_{\Sigma_r} > 2\pi n_0$ as $r = r_*$. This means that the circle Σ_{r*} has 'jumped' over the next zero of the function, and in the interior of the circle there is a zero different from q_0. By varying r we find such $r = r_1$ when the graph of the parametrically defined function $\mathfrak{F}(q)$ on the plane $\{\mathrm{Re}(\mathfrak{F}), \mathrm{Im}(\mathfrak{F})\}$ (q depends on ψ, r is fixed) passes through the point $\mathrm{Re}(\mathfrak{F}) = 0, \mathrm{Im}(\mathfrak{F}) = 0$. The corresponding value $\psi = \psi_1$ defines the next zero $\sigma_1 = r_1 \exp(i\psi_1)$. Repeating the process, evaluate all the zeros of the function $\mathfrak{F}(q)$ inside of the circle $\Sigma_{\rho_0} \subset \mathbb{C}_1$. Note that r may not coincide with ρ_0. In other words, the contour Σ_r must avoid collision with the cross-sections $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_h$ through which the function $\mathfrak{F}(q)$ is discontinuous.
2. Verify whether the infinite point $q_\infty^{(1)} = (\infty, \infty_1) \in \mathbb{C}_1$ is a zero of the function $\mathfrak{F}(q)$ and if it is, find its order n_∞ ($n_\infty = 0$ if $\mathfrak{F}(q_\infty^{(1)}) \neq 0$). After that fix such a sufficiently large number $R \gg \rho_0$ providing $[\arg \mathfrak{F}(q)]_{\Sigma_R} = -2\pi n_\infty$. This test indicates that the only zero of the function $\mathfrak{F}(q)$ outside the circle Σ_R is q_∞. Now, by successively decreasing the radius R, the same machinery as before gives all the zeros of the function $\mathfrak{F}(q)$ in the exterior of the circle Σ_{ρ_0} on the first sheet $\mathbb{C}_1$.
3. Find those zeros lying on the arcs of the circle Σ_{ρ_0} on the sheet $\mathbb{C}_1$ outside the junction slits.

4 to 6. Repeat the procedure for the second sheet $\mathbb{C}_2$.

The process is over if the number of the zeros is equal to h, where in determining the number h, the orders of the zeros are counted. Note that in the case of the problem (3.20) by varying the location of the initial points (δ_j, v_j) $(j = 1, 2, 3)$ it is always possible to avoid implementing steps 3 and 6 and achieve a situation when the zeros lie neither on $\Sigma_{\rho_0} \in \mathbb{C}_1$ nor on $\Sigma_{\rho_0} \in \mathbb{C}_2$.

We emphasize the rapid convergence of the series representation (4.7) of the Θ-function. This circumstance guarantees efficiency of the numerical procedure. It is clear that if the junction lines of the surface $\mathcal{R}$ are not arcs of the same circle, the above procedure remains valid if the circles Σ_r are appropriately replaced by closed simple curves crossing neither each other nor the junction lines of the surface.