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# Vector functional-difference equation in electromagnetic scattering

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A vector functional-difference equation of the first order with a special matrix coefficient is analysed. It is shown how it can be converted into a Riemann–Hilbert boundary-value problem on a union of two segments on a hyper-elliptic surface. The genus of the surface is defined by the number of zeros and poles of odd order of a characteristic function in a strip. An even solution of a symmetric Riemann–Hilbert problem is also constructed. This is a key step in the procedure for diffraction problems. The proposed technique is applied for solving in closed form a new model problem of electromagnetic scattering of a plane wave obliquely incident on an anisotropic impedance half-plane (all the four impedances are assumed to be arbitrary).

*Keywords*: electromagnetic scattering; Riemann–Hillbert problem; Riemann surfaces; vector difference equation.

## 1. Introduction

The most powerful and general methods for exact solution of model problems in acoustic and electromagnetic scattering are those of Wiener & Hopf (1931) and Maliuzhinets (1958). The former method leads to the Riemann–Hilbert boundary-value problem on an infinite straight line $L$ (it splits the complex plane into two half-planes $\mathcal{D}^+$ and $\mathcal{D}^-$):

$$\boldsymbol{\Phi}^+(t) = \mathbf{G}(t)\,\boldsymbol{\Phi}^-(t) + \mathbf{g}(t), \quad t \in L, \tag{1.1}$$

where the unknown vectors (functions) $\boldsymbol{\Phi}^{\pm}(t)$ are analytic in $\mathcal{D}^{\pm}$. The matrix (function) $\mathbf{G}(t)$ and the vector (function) $\mathbf{g}(t)$ are given. The Maliuzhinets method gives rise to a functional-difference equation (a particular case of the Carleman boundary-value problem of the theory of analytic functions):

$$\boldsymbol{\Phi}(\sigma) = \mathbf{G}(\sigma)\,\boldsymbol{\Phi}(\sigma - h) + \mathbf{g}(\sigma), \quad \sigma \in \Omega = \{\mathrm{Re}(s) = \omega\}, \tag{1.2}$$

where $\boldsymbol{\Phi}(\sigma)$ is an unknown vector (function) analytic in the strip $\Pi = \{\omega - h < \mathrm{Re}(s) < \omega\}$. The matrix (function) $\mathbf{G}(\sigma)$ and the vector (function) $\mathbf{g}(\sigma)$ are supposed to be known.

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The method of exact solution of (1.1), (1.2) rests on our ability to factorize the coefficient **G** of the problems, i.e. to split the matrix (function) **G** into two factors:

$$\mathbf{G}(t) = \mathbf{X}^{+}(t)[\mathbf{X}^{-}(t)]^{-1}, \quad t \in L, \tag{1.3}$$

in the case of (1.1), and

$$\mathbf{G}(\sigma) = \mathbf{X}(\sigma)[\mathbf{X}(\sigma - h)]^{-1}, \quad \sigma \in \Omega, \tag{1.4}$$

for (1.2). Here $\mathbf{X}^{\pm}(z)$ are analytic and non-singular in the domains $\mathcal{D}^{\pm}$, and $\mathbf{X}(s)$ is analytic and non-singular in the strip $\Pi$.

If the aforementioned equations are scalar, then in either case there is an exact device for factorisation which is, essentially, based on the Sokhotski–Plemelj formulae. Thus, practically all conceivable scalar equations (1.1), (1.2) corresponding to applied problems can be solved exactly (for a survey see Noble, 1988; Osipov & Norris, 1999).

It is known that for a system of functional equations (1.1) or (1.2) such a general procedure is not available. In comparison with the difference matrix factorisation (1.4), there are significantly more studies on the Wiener–Hopf matrix factorisation (1.3). We mention the papers by Khrapkov (1971), Jones (1984) and Moiseyev (1989). The paper by Jones also provides some references to other results on the Wiener–Hopf matrix factorisation and their applications to physical models.

As for the vector functional-difference equation (1.2), to the best of the authors' knowledge, classes of matrices which admit the constructive difference factorisation (1.4) have not been studied. We, of course, discard those cases when the matrix coefficient **G** can be diagonalized by multiplying the left- and right-hand sides of (1.2) by a constant matrix.

In this paper, we study the vector functional-difference equation (1.2) with the matrix coefficient of the form

$$\mathbf{G}(\sigma) = \begin{pmatrix} a_1(\sigma) + a_2(\sigma) f_1(\sigma) & a_2(\sigma) \\ a_2(\sigma) f_2(\sigma) & a_1(\sigma) - a_2(\sigma) f_1(\sigma) \end{pmatrix}, \quad \sigma \in \Omega, \tag{1.5}$$

where $a_1(\sigma), a_2(\sigma)$ are arbitrary Hölder functions on every finite segment of the contour $\Omega$, $f_1(\sigma)$, $f_2(\sigma)$ are arbitrary single-valued meromorphic functions in the strip $\Pi$ such that $f_j(\sigma) = f_j(\sigma - h)$, $\sigma \in \Omega$, $j = 1, 2$. It is assumed that the function $f_1(s)$ and the characteristic function $f(s) = f_1^2(s) + f_2(s)$ have finite numbers of poles in the strip $\Pi$. The number of zeros of the function $f(s)$ in the strip $\Pi$ is also finite.

We propose a procedure for exact solution of the vector functional-difference equation (1.2) with the matrix coefficient (1.5). The method consists of the following steps:

(i) reducing the initial equation (1.2) to two separate functional-difference equations of the first order and a system of boundary conditions for the unknown functions on a system of cuts. The cuts join the branch points in the strip of the function $f^{1/2}(s)$;

(ii) converting the problem to a vector Riemann–Hilbert problem on a system of open curves;

(iii) setting up a Riemann–Hilbert problem on the contour $\mathcal{L} = L_1 \cup L_2$, $L_j = (-1, 1) \subset \mathbb{C}_j$, on a hyper-elliptic surface $\mathcal{R}$ formed from the two copies $\mathbb{C}_1$ and $\mathbb{C}_2$ of the cut complex plane;

(iv) constructing a solution of the Riemann–Hilbert problem on the surface growing at infinity;

(v) solving the Jacobi inversion problem (Springer, 1956; Zverovich, 1971; Farkas & Kra, 1991; Antipov & Silvestrov, 2002) and removing the growth at infinity of the solution;

(vi) writing down the general solution of the Riemann–Hilbert problem on the surface and, afterwards, the general solution of the vector functional-difference equation (1.2).

If the function $f^{1/2}(s)$ has no branch points in the strip $\Pi$, then one can find a closed-form solution of the vector equation (1.2) by analysing a standard Riemann–Hilbert problem on the segment $(-1, 1)$ of the complex plane. In general, however, it is necessary to formulate and solve a Riemann–Hilbert problem on a two-sheeted surface of genus $\rho$, with $2\rho+2$ being the number of the branch points of the function $f^{1/2}(s)$ in the strip $\Pi$ (the number of these points is always even). If the function $f^{1/2}(s)$ has only two branch points in the strip $\Pi$, then the genus of the surface is zero and the solution of the Jacobi inversion problem can be bypassed. For $\rho \geqslant 1$, the analysis of the Riemann–Hilbert problem requires solving the Jacobi inversion problem in terms of either the Riemann $\theta$-function (see, for instance, Farkas & Kra, 1991) if $\rho \geqslant 2$, or elliptic functions (see, for example, Hancock, 1968) if $\rho = 1$.

It turns out that applying the Maliuzhinets technique to problems of diffraction needs a special solution which meets the symmetry condition:

$$\boldsymbol{\Phi}(\omega + \mathrm{i}\tau) = \boldsymbol{\Phi}(\omega - h - \mathrm{i}\tau), \quad -\infty < \tau < \infty. \tag{1.6}$$

The above relation not only narrows the class of solutions but also imposes some necessary conditions on the matrix $\mathbf{G}(\sigma)$ and the vector $\mathbf{g}(\sigma)$. If those conditions are satisfied, then one needs to seek an even solution of the Riemann–Hilbert problem on a surface of genus $\rho' = [\rho/2]$ ($[a]$ is the entire part of a number $a$). Therefore, in this case, there is no need to solve Jacobi's problem if the number of the branch points in the strip $\Pi$ is not greater than 4. Otherwise, for the number of the branch points not greater than 8, the problem is solvable in terms of elliptic functions. We note that the number of the branch points of the function $f^{1/2}(s)$ is a topological characteristics of the problem. To decrease the genus of the corresponding surface we need an additional symmetry of the problem. Recently, Senior & Legault (2000) analysed a second-order scalar functional-difference equation in the case when it is solvable by elliptic functions. Although their method is different, it also uses some elements of the theory of Riemann surfaces (a torus in their case).

To show how the proposed technique works, we choose a new canonical problem of electromagnetic scattering by an anisotropic impedance half-plane. Senior (1978) formulated the problem for four different impedance parameters using both Wiener–Hopf and Maliuzhinets methods. The Wiener–Hopf formulation leads to a $1\times4$ vector Riemann–Hilbert boundary-value problem for an infinite contour on a plane. The particular case, when the impedances meet the restriction $\eta_j^+ = \eta_j^-$ ($j = 1, 2$), was analysed by Hurd & Lüneberg (1985). They chose the Wiener–Hopf formulation and found a closed-form solution of the corresponding $1 \times 2$ vector Riemann–Hilbert problem on the real axis in terms of elliptic functions. On the other hand, the Maliuzhinets formulation of the general problem gives a second-order functional-difference equation. As it was pointed out by Senior (1978), it was beyond known techniques.

In this paper, we present a closed-form solution of this most general case of the scattering problem. Mathematically, it converts into a Riemann–Hilbert problem on a hyper-elliptic surface of genus three that is solvable in terms of the Riemann $\theta$-function of genus three (Antipov & Silvestrov, 2002).

The paper is organized as follows. In Section 2, we define sufficient conditions for the matrix coefficient $\mathbf{G}(\sigma)$ to be imposed in order that the proposed method works. We reduce the initial functional-difference equation (1.2) to a scalar Riemann–Hilbert problem on an open contour of a Riemann surface in Section 3. A canonical solution of this problem is constructed in Section 4. The general solution of the Riemann–Hilbert problem on the surface is written down in Section 5. In Section 6, we construct and analyse a closed-form solution of the vector functional-difference equation (1.2). We also specify it for the case when all the poles are simple.

For problems of scattering, it is crucial to know how to construct a solution that meets the symmetry condition (1.6). This is the main aim of Section 7.

Section 8 is devoted to the problem of diffraction by an anisotropic impedance half-plane (all the four impedances are assumed to be arbitrary). In Section 8.1, we reduce the problem to a vector functional-difference equation of the first order. The general case (the corresponding surface is of genus three) is analysed in Section 8.2. Finally, in Section 8.3, a special case, when there are no branch points, is considered. We emphasize that in this case the impedances are not necessarily the same, and the solution of the Jacobi inversion problem is bypassed.

## 2. Vector functional-difference equation of the first order

Let $\Pi$ be a strip in the plane of a complex variable $s$: $\Pi = \{s \in \mathbb{C} : \omega - h < \mathrm{Re}(s) < \omega\}$, where $\omega$ is real and $h > 0$. Let $\Omega$, $\Omega_{-1}$ be the boundaries of the strip: $\Omega = \{\mathrm{Re}(s) = \omega\}$, $\Omega_{-1} = \{\mathrm{Re}(s) = \omega - h\}$. Consider the following boundary-value problem of the theory of analytic functions.

*Given a* $2 \times 2$ *matrix* $\mathbf{G}(\sigma)$ *and a vector* $\mathbf{g}(\sigma)$ *find a vector* $\boldsymbol{\Phi}(s)$ *analytic in the strip* $\Pi$, *continuous up to the boundary* $\Omega \cup \Omega_{-1}$ *apart from a finite number of poles* $\beta_1, \beta_2, \ldots, \beta_t \in \Pi$ *of orders* $\tau_1, \tau_2, \ldots, \tau_t$ *and satisfying the boundary condition*

$$\boldsymbol{\Phi}(\sigma) = \mathbf{G}(\sigma)\,\boldsymbol{\Phi}(\sigma - h) + \mathbf{g}(\sigma), \quad \sigma \in \Omega. \tag{2.1}$$

*At the ends of the strip, i.e. as* $\mathrm{Im}(s) \to \pm\infty$, $\boldsymbol{\Phi}(s) = O(\mathrm{e}^{b^{\pm}\,\mathrm{Im}(s)})$ *with* $b^{\pm}$ *being real, finite and prescribed. The matrix* $\mathbf{G}(\sigma)$ *and the vector* $\mathbf{g}(\sigma)$ *satisfy the Hölder condition on every finite segment of* $\Omega$. *At infinity, i.e. as* $\sigma \to \omega \pm i\infty$, *the components of the* $\mathbf{G}(\sigma)$ *and* $\mathbf{g}(\sigma)$ *may have a finite exponential growth not necessarily the same. The matrix* $\mathbf{G}(\sigma)$ *is also nonsingular on* $\Omega$.

This problem is a vector generalization of Carleman's boundary-value problem (Carleman, 1932, p.148) $\Phi(\sigma) = \mathcal{G}(\sigma)\Phi(\alpha(\sigma)) + g(\sigma)$, $\sigma \in \Omega \cup \Omega_{-1}$ with the shift function $\alpha(\sigma) = \sigma - h$ on $\Omega$ and $\alpha(\sigma) = \sigma + h$ on $\Omega_{-1}$. Obviously, the function $\alpha$ meets the Carleman condition $\alpha(\alpha(\sigma)) = \sigma$, $\sigma \in \Omega \cup \Omega_{-1}$. The other Carleman conditions $\mathcal{G}(\alpha(\sigma))\mathcal{G}(\sigma) = 1$ and $\mathcal{G}(\sigma)g(\alpha(\sigma)) + g(\sigma) = 0$, $\sigma \in \Omega \cup \Omega_{-1}$ are satisfied identically if we put $\mathcal{G}(\sigma) = \mathbf{G}(\sigma)$, $g(\sigma) = \mathbf{g}(\sigma)$, $\sigma \in \Omega$, and $\mathcal{G}(\sigma) = [\mathbf{G}(\sigma + h)]^{-1}$, $g(\sigma) = -[\mathbf{G}(\sigma + h)]^{-1}\mathbf{g}(\sigma + h)\ \sigma \in \Omega_{-1}$.

Note that, at the same time, the boundary condition (2.1) can be regarded as a vector functional-difference equation.

Let $\lambda_1(\sigma)$, $\lambda_2(\sigma)$ be the eigenvalues of the matrix $\mathbf{G}(\sigma)$ and let $\lambda_1(\sigma) \neq \lambda_2(\sigma)$. In this section we define a class of matrices representable in the form

$$\mathbf{G}(\sigma) = \mathbf{T}(\sigma)\boldsymbol{\Lambda}(\sigma)[\mathbf{T}(\sigma - h)]^{-1}, \quad \sigma \in \Omega, \tag{2.2}$$

where $\boldsymbol{\Lambda}(\sigma) = \operatorname{diag}\{\lambda_1(\sigma), \lambda_2(\sigma)\}$, and the matrix $\mathbf{T}(\sigma)$ admits a two-valued analytical continuation from the contour $\Omega$ into the strip apart from a finite number of poles, branch points and points where $\det \mathbf{T}(s) = 0$. It is also required that $\mathbf{T}(\sigma) = \mathbf{T}(\sigma - h)$, $\sigma \in \Omega$. The eigenvalues of the matrix

$$\mathbf{G}(\sigma) = \begin{pmatrix} G_{11}(\sigma) & G_{12}(\sigma) \\ G_{21}(\sigma) & G_{22}(\sigma) \end{pmatrix} \tag{2.3}$$

are given by

$$\lambda_1(\sigma) = \tfrac{1}{2}[G_{11}(\sigma) + G_{22}(\sigma) + \Delta^{1/2}(\sigma)], \quad \lambda_2(\sigma) = \tfrac{1}{2}[G_{11}(\sigma) + G_{22}(\sigma) - \Delta^{1/2}(\sigma)], \tag{2.4}$$

where

$$\Delta(\sigma) = [G_{11}(\sigma) - G_{22}(\sigma)]^2 + 4G_{12}(\sigma)G_{21}(\sigma). \tag{2.5}$$

Take the diagonalising matrix $\mathbf{T}(\sigma)$ in the form

$$\mathbf{T}(\sigma) = \begin{pmatrix} 1 & 1 \\ \frac{G_{22}(\sigma)-G_{11}(\sigma)+\Delta^{1/2}(\sigma)}{2G_{12}(\sigma)} & \frac{G_{22}(\sigma)-G_{11}(\sigma)-\Delta^{1/2}(\sigma)}{2G_{12}(\sigma)} \end{pmatrix}, \quad \sigma \in \Omega, \tag{2.6}$$

with $\det \mathbf{T}(\sigma) = -\Delta^{1/2}(\sigma)[G_{12}(\sigma)]^{-1}$. In order for the matrix $\mathbf{T}(\sigma)$ to be meromorphic and two-valued, it is sufficient that the functions

$$\frac{G_{22}(s) - G_{11}(s)}{G_{12}(s)} \quad \text{and} \quad \frac{\Delta(s)}{G_{12}^2(s)}, \quad s \in \Pi, \tag{2.7}$$

are single-valued meromorphic functions. Clearly, if the functions (2.7) are meromorphic, then the function $G_{21}(s)/G_{12}(s)$ is also meromorphic. To clarify the structure of the matrix $\mathbf{G}(s)$ that meets the above conditions, introduce the functions

$$f_1(s) = \frac{G_{11}(s) - G_{22}(s)}{2G_{12}(s)}, \quad f_2(s) = \frac{G_{21}(s)}{G_{12}(s)}, \quad s \in \Pi, \tag{2.8}$$

which are single-valued meromorphic functions in $\Pi$. Then the original matrix has the form

$$\mathbf{G}(\sigma) = \begin{pmatrix} G_{11}(\sigma) & G_{12}(\sigma) \\ f_2(\sigma)G_{12}(\sigma) & G_{11}(\sigma) - 2f_1(\sigma)G_{12}(\sigma) \end{pmatrix}, \quad \sigma \in \Omega. \tag{2.9}$$

Note that, the elements $G_{ij}(\sigma)$ are not required to be meromorphic in the strip $\Pi$. Finally, we transform the matrix $\mathbf{G}(\sigma)$ into the form

$$\mathbf{G}(\sigma) = a_1(\sigma)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_2(\sigma)\begin{pmatrix} f_1(\sigma) & 1 \\ f_2(\sigma) & -f_1(\sigma) \end{pmatrix}, \quad \sigma \in \Omega, \tag{2.10}$$

where

$$a_1(\sigma) = \tfrac{1}{2}[G_{11}(\sigma) + G_{22}(\sigma)], \quad a_2(\sigma) = G_{12}(\sigma). \tag{2.11}$$

In the new notation, the eigenvalues $\lambda_1, \lambda_2$ and the matrix of transformation $\mathbf{T}$ become

$$\lambda_1(\sigma) = a_1(\sigma) + a_2(\sigma) f^{1/2}(\sigma), \quad \lambda_2(\sigma) = a_1(\sigma) - a_2(\sigma) f^{1/2}(\sigma), \tag{2.12}$$

$$\mathbf{T}(s) = \begin{pmatrix} 1 & 1 \\ -f_1(s) + f^{1/2}(s) & -f_1(s) - f^{1/2}(s) \end{pmatrix}, \tag{2.13}$$

where $f(s) = f_1^2(s) + f_2(s)$. Here $a_1(\sigma)$, $a_2(\sigma)$ are *arbitrary Hölder functions on* $\Omega$ (although they may be *discontinuous at infinity*), and $f_1(s)$, $f_2(s)$ are *arbitrary single-valued meromorphic functions in the strip* $\Pi$. They do not have poles on $\Omega$. In the strip $\Pi$, the functions $f_1(s)$, $f(s)$ have finite numbers of poles. It is assumed that the number of zeros of the function $f(s)$ in the strip $\Pi$ is also finite. We emphasize that the elements of the matrix $\mathbf{T}(s)$ are $h$-periodic or, equivalently, the functions $f_1(s)$, $f^{1/2}(s)$ are $h$-periodic.

Formula (2.10) can be treated as an analogue of the Chebotarev–Khrapkov matrix (Chebotarev, 1956; Khrapkov, 1971) for the functional-difference equation (2.1).

## 3. Scalar Riemann–Hilbert problem on a hyper-elliptic surface

In this section we reduce the vector functional-difference equation (2.1) with the matrix coefficient (2.10) to a scalar Riemann–Hilbert problem on a Riemann surface. First, substitute the relation (2.2) into (2.1):

$$[\mathbf{T}(\sigma)]^{-1}\boldsymbol{\Phi}(\sigma) = \boldsymbol{\Lambda}(\sigma)[\mathbf{T}(\sigma-h)]^{-1}\boldsymbol{\Phi}(\sigma-h) + [\mathbf{T}(\sigma)]^{-1}\mathbf{g}(\sigma), \quad \sigma \in \Omega, \tag{3.1}$$

and introduce a new vector function

$$\boldsymbol{\phi}(s) = [\mathbf{T}(s)]^{-1}\boldsymbol{\Phi}(s), \quad s \in \Pi, \tag{3.2}$$

with the components

$$\begin{aligned} \phi_1(s) &= \left(\frac{f_1(s)}{2f^{1/2}(s)} + \frac{1}{2}\right)\Phi_1(s) + \frac{\Phi_2(s)}{2f^{1/2}(s)}, \\ \phi_2(s) &= \left(-\frac{f_1(s)}{2f^{1/2}(s)} + \frac{1}{2}\right)\Phi_1(s) - \frac{\Phi_2(s)}{2f^{1/2}(s)}, \quad s \in \Pi. \end{aligned} \tag{3.3}$$

These formulae indicate that the functions $\phi_1(s)$ and $\phi_2(s)$ are multi-valued. They have branch points at the zeros and poles of odd order of the function $f(s)$.

Among these points there can also be the two infinite points at the upper and lower ends of the strip. From the theory of periodic meromorphic functions, by definition, the upper end $x + \mathrm{i}\infty$ ($\omega - h \leqslant x \leqslant \omega$) of the strip is called a zero of order $\nu$ of a function $f(s)$ if $f(s) \sim A\mathrm{e}^{2\pi \mathrm{i} s\nu/h}$ as $\mathrm{Im}(s) \to +\infty$ ($A$ =const$\neq 0$). The point $x + \mathrm{i}\infty$ is a pole of order $\nu$ if $f(s) \sim A\mathrm{e}^{-2\pi \mathrm{i} s\nu/h}$ as $\mathrm{Im}(s) \to +\infty$. The lower end $x - \mathrm{i}\infty$ is treated similarly. It is known (Hancock, 1968) that any $h$-periodic meromorphic function has the same number of poles and zeros in the strip of the periods (the poles and zeros including the upper and lower infinite points are counted according to the multiplicity). Indeed, by the conformal mapping $z = e^{-2\pi i s/h}$, the strip $\Pi$ is transformed into $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and an $h$-periodic function in the $s$-plane becomes a rational function in the extended $z$-plane with the same number of poles and zeros in $\bar{\mathbb{C}}$.

Therefore, the function $f^{1/2}(s)$ has an even number of the branch points (the infinite points $x \pm i\infty$ can be branch points as well). Let the branch points be $s_0, s_1, \dots, s_{2\rho+1}$. In the case $\rho = -1$, the function $f(s)$ is either a constant, or all its poles and zeros are of even order. Henceforth, it is assumed that $\rho \geqslant 0$. Apart from the branch points $s_0, s_1, \dots, s_{2\rho+1}$, the functions $\phi_1(s)$ and $\phi_2(s)$ admit a finite number of poles in the strip $\Pi$. In addition to the prescribed poles $\beta_1, \beta_2, \dots, \beta_t$ of the vector function $\boldsymbol{\Phi}(s)$, the functions $\phi_1$ and $\phi_2$ have new poles. Their multiplicity and location are entirely defined by the poles of the function $f_1(s)$ and the zeros of even order of the function $f(s)$. Let all the poles of the functions $\phi_1(s)$ and $\phi_2(s)$ be $a_1, a_2, \dots, a_m$ of orders $\nu_1, \nu_2, \dots, \nu_m$.

By using (3.2) the coupled difference equation (3.1) reduces to two separate equations:

$$\begin{aligned} \phi_1(\sigma) &= \lambda_1(\sigma)\phi_1(\sigma - h) + g_1^\circ(\sigma), \quad \sigma \in \Omega, \\ \phi_2(\sigma) &= \lambda_2(\sigma)\phi_2(\sigma - h) + g_2^\circ(\sigma), \quad \sigma \in \Omega, \end{aligned} \tag{3.4}$$

$$\begin{aligned} g_1^\circ(\sigma) &= \left(\frac{f_1(\sigma)}{2f^{1/2}(\sigma)} + \frac{1}{2}\right) g_1(\sigma) + \frac{g_2(\sigma)}{2f^{1/2}(\sigma)}, \\ g_2^\circ(\sigma) &= \left(-\frac{f_1(\sigma)}{2f^{1/2}(\sigma)} + \frac{1}{2}\right) g_1(\sigma) - \frac{g_2(\sigma)}{2f^{1/2}(\sigma)}, \quad \sigma \in \Omega. \end{aligned} \tag{3.5}$$

and $\lambda_1$, $\lambda_2$ being the functions (2.12). To fix a branch of the function $f^{1/2}(s)$ we cut the strip $\Pi$ by smooth curves $\Gamma_j \subset \Pi$ ($j = 0, 1, \dots, \rho$) which do not intersect each other and join the branch points so that $\Gamma_j = s_{2j}s_{2j+1}$ ($j = 0, 1, \dots, \rho$). The positive direction of $\Gamma_j$ is chosen from $s_{2j}$ to $s_{2j+1}$. Denote the limit value of the fixed branch on the left and the right sides of the cut as $[f^{1/2}(\sigma)]^+$ and $[f^{1/2}(\sigma)]^-$, respectively. Clearly, $[f^{1/2}(\sigma)]^+ = -[f^{1/2}(\sigma)]^-$, $\sigma \in \Gamma_j$.

Since the vector function $\boldsymbol{\Phi}(s)$ must be single-valued in the strip $\Pi$, from (3.2), in addition, we get the following boundary condition on the system of curves $\Gamma_j$ ($j = 0, 1, \dots, \rho$):

$$\mathbf{T}^+(\sigma)\phi^+(\sigma) = \mathbf{T}^-(\sigma)\phi^-(\sigma), \quad \sigma \in \Gamma_j. \tag{3.6}$$

This requirement recovers the linear relations between the limit values of the functions $\phi_1$ and $\phi_2$ on the curves $\Gamma_j$:

$$\phi_1^+(\sigma) = \phi_2^-(\sigma), \quad \phi_1^-(\sigma) = \phi_2^+(\sigma),$$

$$\sigma \in \Gamma_j \quad (j = 0, 1, \dots, \rho). \tag{3.7}$$

Therefore, the original vector functional-difference equation (2.1) with the matrix coefficient (2.10) is equivalent to the system of two separate difference equations (3.4) and the two relations of Riemann–Hilbert type (3.7).

To reduce this new problem to a vector Riemann–Hilbert problem on a system of open contours, we map the $s$-strip $\Pi$ onto a $z$-plane cut along the segment $[-1, 1]$. The mapping function and the inverse map are defined by

$$z = -\mathrm{i}\tan\frac{\pi}{h}(s-\omega), \quad s = \omega + \frac{\mathrm{i}h}{2\pi}\log\frac{1+z}{1-z}. \tag{3.8}$$

The contour $\Omega$ is mapped onto the upper side of the cut $[-1, 1]$ (the left bank with respect to the positive direction), the second side of the strip, $\Omega_{-1}$, is mapped onto the lower side of the cut. The images of the upper and the lower infinite points of the strip $\Pi$, $x - \mathrm{i}\infty$ and $x + \mathrm{i}\infty$ ($\omega - h \leqslant x \leqslant \omega$), are the points $z = -1$ and $z = 1$, respectively. The function $\log[(1+z)(1-z)^{-1}]$ is real on the upper side of the cut. Introduce the following functions:

$$F_j(z) = \phi_j\left(\omega + \frac{\mathrm{i}h}{2\pi}\log\frac{1+z}{1-z}\right), \quad z \in \mathbb{C},$$

$$l_j(t) = \lambda_j\left(\omega + \frac{\mathrm{i}h}{2\pi}\log\frac{1+t}{1-t}\right), \quad t \in [-1, 1], \tag{3.9}$$

$$g_j^*(t) = g_j^\circ\left(\omega + \frac{\mathrm{i}h}{2\pi}\log\frac{1+t}{1-t}\right), \quad t \in [-1, 1], \quad j = 1, 2,$$

and also the notation for the images of the branch points $s_j$ and the poles $a_k$:

$$\begin{aligned} z_j &= -\mathrm{i}\tan\frac{\pi}{h}(s_j-\omega), \quad j = 0, 1, \ldots, 2\rho+1, \\ \alpha_k &= -\mathrm{i}\tan\frac{\pi}{h}(a_k-\omega), \quad k = 1, 2, \ldots, m. \end{aligned} \tag{3.10}$$

Let the cuts $\Gamma_j$ be mapped onto curves $\Gamma_j^*$ ($j = 0, 1, \ldots, \rho$). The curves $\Gamma_j^* \subset \mathbb{C}$ and do not intersect each other and the segment $[-1, 1]$.

Thus, the system of equations (3.4), (3.7) is equivalent to the following vector Riemann–Hilbert problem:

$$\begin{aligned} F_1^+(t) &= l_1(t)F_1^-(t) + g_1^*(t), \quad t \in (-1, 1), \\ F_2^+(t) &= l_2(t)F_2^-(t) + g_2^*(t), \quad t \in (-1, 1), \\ F_1^+(t) &= F_2^-(t), \quad t \in \Gamma_j^*, \\ F_2^+(t) &= F_1^-(t), \quad t \in \Gamma_j^*, \quad j = 0, 1, \ldots, \rho. \end{aligned} \tag{3.11}$$

Finally, we reduce this vector problem on the complex plane to a scalar problem on a Riemann surface. Let $\mathcal{R}$ be the two-sheeted surface of the algebraic equation

$$w^2 = q(z), \quad q(z) = (z-z_0)(z-z_1)\cdots(z-z_{2\rho+1}), \tag{3.12}$$

formed by gluing two copies $\mathbb{C}_1$ and $\mathbb{C}_2$ of the extended complex plane $\mathbb{C} \cup \infty$ cut along the system of the curves $\Gamma_j^*$ ($j = 0, 1, \ldots, \rho$). The positive (left) sides of the cuts $\Gamma_j^*$ on

$\mathbb{C}_1$ are glued with the negative (right) sides of the curves $\Gamma_j^*$ on $\mathbb{C}_2$, and vice versa. This gives rise to a two-sheeted Riemann surface $\mathcal{R}$ of genus $\rho$. Then the function $w$, defined by (3.12), becomes single-valued on the surface $\mathcal{R}$:

$$w = \begin{cases} q^{1/2}(z), & z \in \mathbb{C}_1 \\ -q^{1/2}(z), & z \in \mathbb{C}_2, \end{cases} \tag{3.13}$$

where $q^{1/2}(z)$ is the branch chosen such that $q^{1/2}(z) \sim z^{\rho+1}$, $z \to \infty$.

Denote a point of the surface $\mathcal{R}$ with affix $z$ on $\mathbb{C}_1$ by the pair $(z, q^{1/2}(z))$, and its counterpart on $\mathbb{C}_2$ by the pair $(z, -q^{1/2}(z))$. Introduce a function on the surface $\mathcal{R}$,

$$F(z, w) = \begin{cases} F_1(z), & (z, w) \in \mathbb{C}_1 \\ F_2(z), & (z, w) \in \mathbb{C}_2. \end{cases} \tag{3.14}$$

Because of the third and fourth conditions in (3.11), the function $F(z, w)$ is meromorphic everywhere on the surface except for the contour $\mathcal{L} = L_1 \cup L_2$, where $L_1 = (-1, 1) \subset \mathbb{C}_1$ and $L_2 = (-1, 1) \subset \mathbb{C}_2$. Therefore, the system (3.11) is equivalent to a scalar Riemann–Hilbert problem on the surface $\mathcal{R}$,

$$F^+(t, \xi) = l(t, \xi)F^-(t, \xi) + g^*(t, \xi), \quad (t, \xi) \in \mathcal{L}, \tag{3.15}$$

where

$$l(t, \xi) = \begin{cases} l_1(t), & (t, \xi) \in L_1 \\ l_2(t), & (t, \xi) \in L_2, \end{cases} \qquad g^*(t, \xi) = \begin{cases} g_1^*(t), & (t, \xi) \in L_1 \\ g_2^*(t), & (t, \xi) \in L_2, \end{cases} \tag{3.16}$$

and $\xi = w(t)$.

Without loss of generality, the Hölder function $l(t, \xi)$ does not vanish on the contour $\mathcal{L}$ and has definite limits at the end-points $t = \pm 1$. The function $g^*(t, \xi)$ is also a Hölder function on $\mathcal{L}$ except possibly the ends:

$$|g^*(t, \xi)| \leqslant A_0^{(\mu)} |t \mp 1|^{-\tilde{\nu}_\mu^\pm}, \quad (t, \xi) \in L_\mu, \quad \mu = 1, 2, \quad t \to \pm 1, \tag{3.17}$$

where $A_0^{(\mu)} = \text{const}$. The parameters $\tilde{\nu}_\mu^\pm$ are defined from (3.5) by the behaviour at the points $\omega \pm \mathrm{i}\infty$ of the functions $f_1(\sigma)$, $f^{1/2}(\sigma)$, $g_1(\sigma)$ and $g_2(\sigma)$.

## 4. Canonical solution to the Riemann–Hilbert problem on a hyper-elliptic surface

### 4.1 *Class of solutions*

First, describe a class of solutions for the problem (3.15). Clearly, the function $F(z, w)$ admits poles at the points $(\alpha_k, q^{1/2}(\alpha_k))$ and $(\alpha_k, -q^{1/2}(\alpha_k))$ of orders $\nu_k$ ($k = 1, 2, \dots, m$). In addition, this function may have poles at the branch points $z_j$ of order, say, $\mu_j \geqslant 0$ ($j = 0, 1, \dots, 2\rho + 1$). If one of these points $z_j$ is a removable singularity, then $\mu_j = 0$. Obviously, if $\mu_j > 0$, then $\mu_j$ is odd. Recall (Springer, 1956) that a branch point $z_j$ of a Riemann surface is called a pole of order $\mu_j$ for a function $F(z, w)$ if $F(z, w) \sim A\zeta^{-\mu_j}$, $\zeta \to 0$, $A = \text{const}$, and $\zeta = (z - z_j)^{1/2}$ is a local uniformizing parameter of the point $z_j$.

Formulae (3.3), (3.9) and (3.14) indicate that at the end-points of the contour $\mathcal{L}$, the function $F(z, w)$ may have singularities:

$$|F(z,w)| \leqslant A_1^{(\mu)}|z \mp 1|^{-\nu_\mu^\pm}, \quad (z,w) \in \mathbb{C}_\mu, \quad \mu = 1, 2, \quad z \to \pm 1, \tag{4.1}$$

where $A_1^{(\mu)} = \text{const}$, and $\nu_\mu^\pm \geqslant \tilde{\nu}_\mu^\pm$. The numbers $\nu_\mu^\pm$ are defined by the parameters $\tilde{\nu}_\mu^\pm$, by the prescribed growth at the ends of the strip of the functions $\Phi_1(s)$, $\Phi_2(s)$, i.e. by the numbers $b^\pm$, and also by the behaviour of the functions $f_1(s) f^{-1/2}(s)+1$, $f_1(s) f^{-1/2}(s)-1$ and $f^{-1/2}(s)$ as $s \to x \pm \mathrm{i}\infty$ $(\omega - h \leqslant x \leqslant \omega)$.

The key step of the solution technique is to factorize the function $l(t, \xi)$ or to construct a special, canonical function. We say that the function $X(z, w)$ is a *canonical solution* of the problem (3.15) if it provides a solution to the following homogeneous problem on an open contour of the surface $\mathcal{R}$.

*Find a function $X(z, w)$ which is meromorphic on $\mathcal{R}\backslash\mathcal{L}$, admits a finite number of poles and zeros and has non-zero boundary values $X^\pm(t, \xi)$ satisfying the boundary condition*

$$X^+(t,\xi) = l(t,\xi)X^-(t,\xi), \quad (t,\xi) \in \mathcal{L} \subset \mathcal{R}, \tag{4.2}$$

*where the contour $\mathcal{L}$ consists of the contours $L_1 = (-1, 1) \subset \mathbb{C}_1$ and $L_2 = (-1, 1) \subset \mathbb{C}_2$. At the ends of the contours $L_\mu$,*

$$|X(z,w)| \leqslant A_2^{(\mu)}|z \mp 1|^{-\nu_\mu^\pm}, \quad (z,w) \in \mathbb{C}_\mu, \quad z \to \pm 1, \quad A_2^{(\mu)} = \text{const}, \quad \mu = 1, 2. \tag{4.3}$$

### 4.2 *Solution to the problem growing at infinity*

We start with constructing a system of canonical cross-sections of the surface $\mathcal{R}$: $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_\rho$ and $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_\rho$. If $\rho = 0$, then the surface $\mathcal{R}$ is topologically equivalent to a sphere, and there are no cross-sections. Let $\rho > 0$. The cross-section $\mathbf{a}_j$ is a closed smooth curve built up from the banks of the cut $\Gamma_j^* = z_{2j}z_{2j+1}$. As $\mathbf{a}_j$ is traced in the positive direction, the first sheet $\mathbb{C}_1$ is to the left (Fig. 1).

The cross-section $\mathbf{b}_j$ is a smooth closed curve that consists of two parts. The first one (the solid line in Fig. 1) lies on the first sheet $\mathbb{C}_1$, its starting point is $z_{2j}$, and the ending point is $z_1$. The second part lies on the second sheet (the dashed line in Fig. 1), starts at the point $z_1$ (it belongs to both sheets $\mathbb{C}_1$ and $\mathbb{C}_2$) and goes to the point $z_{2j}$ at which it returns to the first sheet. The contour $\mathbf{b}_j$ crosses the cross-section $\mathbf{a}_j$ from right to the left and does not cross the other sections $\mathbf{a}_k$ and $\mathbf{b}_k$ $(k \neq j)$ and the contour $\mathcal{L}$. We mention that the choice of the system of the cross-sections is not unique. Another possibility, that under some circumstances can be more convenient, is to take the cross-section $\mathbf{b}_\rho$ as a loop joining the points $z_{2\rho+1}$ and $z_0$ and passing through the infinite points of both sheets of the surface (Fig. 2).

Choose Weierstrass' kernel (Zverovich, 1971)

$$\mathrm{d}W = \frac{w+\xi}{2\xi}\frac{\mathrm{d}t}{t-z}, \quad w = w(z), \quad \xi = w(t), \tag{4.4}$$

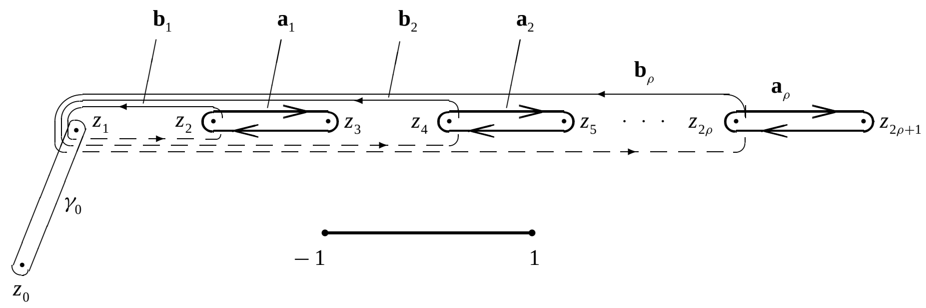


FIG. 1. Canonical cross-sections $\mathbf{a}_j, \mathbf{b}_j$ $(j = 1, 2, \ldots, \rho)$. The loop $\mathbf{b}_\rho$ joins the points $z_1$ and $z_{2\rho}$.

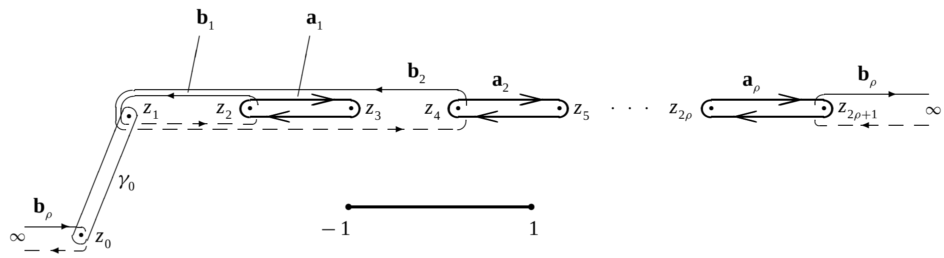


FIG. 2. Canonical cross-sections $\mathbf{a}_j, \mathbf{b}_j$ $(j = 1, 2, \ldots, \rho)$. The loop $\mathbf{b}_\rho$ joins the points $z_{2\rho+1}$ and $z_0$.

as an analogue of the Cauchy kernel on the surface $\mathcal{R}$. We next show that the function

$$X(z, w) = \exp\{\chi(z, w)\}, \quad (z, w) \in \mathcal{R} \tag{4.5}$$

provides a partial solution of the problem (4.2). Here

$$\chi(z, w) = \frac{1}{2\pi i} \int_{\mathcal{L}} \log l(t, \xi)\, dW + \sum_{\mu=1}^{2} \left( \operatorname{sgn} \kappa_\mu^+ \sum_{j=1}^{|\kappa_\mu^+|} \int_{p'_{\mu 0}}^{p'_{\mu j}} dW + \operatorname{sgn} \kappa_\mu^- \sum_{j=1}^{|\kappa_\mu^-|} \int_{p''_{\mu 0}}^{p''_{\mu j}} dW \right)$$

$$+ \sum_{j=1}^{\rho} \left( \int_{p_j}^{r_j} dW + m_j \oint_{\mathbf{a}_j} dW + n_j \oint_{\mathbf{b}_j} dW \right), \tag{4.6}$$

where

$$\begin{aligned} &p'_{10} = (1, q^{1/2}(1)), \quad p'_{20} = (1, -q^{1/2}(1)),\\ &p''_{10} = (-1, q^{1/2}(-1)), \quad p''_{20} = (-1, -q^{1/2}(-1)),\\ &p_j = (\delta_j, v_j) \in \mathbb{C}_1, \quad v_j = q^{1/2}(\delta_j), \quad j = 1, 2, \ldots, \rho,\\ &p'_{\mu j} = (\delta'_{\mu j}, (-1)^{\mu-1} v'_{\mu j}) \in \mathbb{C}_\mu, \quad v'_{\mu j} = q^{1/2}(\delta'_{\mu j}), \quad j = 1, 2, \ldots, |\kappa^+_\mu|, \ \ \mu = 1, 2,\\ &p''_{\mu j} = (\delta''_{\mu j}, (-1)^{\mu-1} v''_{\mu j}) \in \mathbb{C}_\mu, \quad v''_{\mu j} = q^{1/2}(\delta''_{\mu j}), \quad j = 1, 2, \ldots, |\kappa^-_\mu|, \ \ \mu = 1, 2, \end{aligned} \tag{4.7}$$

are arbitrary fixed distinct points of the surface $\mathcal{R}$ which do not lie on the contour $\mathcal{L}$ and the canonical cross-sections. Also, they coincide with none of the branch points of the surface $\mathcal{R}$ and the poles of the function $F(z, w)$. The final formulae for the solution do not depend upon the choice of the points $p'_{\mu j}$, $p''_{\mu j}$ and $p_j$.

As far as the points $r_j = (\sigma_j, w_j)$ ($w_j = w(\sigma_j)$, $j = 1, 2, \ldots, \rho$) are concerned, they are unknown and may lie on either sheet of the surface. The points $r_j$ are also assumed to be different from the branch points $z_0, z_1, \ldots z_{2\rho+1}$ and the poles with affixes $\alpha_1, \alpha_2, \ldots, \alpha_m$. The numbers $\kappa^\pm_1, \kappa^\pm_2, m_j$ and $n_j$ ($j = 1, 2, \ldots, \rho$) are unknown integers. Branches of the function $\log l(t, \xi)$ on the contours $L_1$ and $L_2$ are chosen in an arbitrary way and will be fixed afterwards. The points $r_j$ and the integers $\kappa^\pm_1, \kappa^\pm_2, m_j, n_j$ will be chosen later to make the function $X(z, w)$ bounded at infinity and to satisfy the condition (4.3) at the ending points of the contour $\mathcal{L}$. The integrals in (4.6), apart from the integrals over $\mathcal{L}$ and around $\mathbf{a}_j$, $\mathbf{b}_j$, are taken over smooth curves joining the end-points and which do not cross the cross-sections $\mathbf{a}_j$, $\mathbf{b}_j$ and the contour $\mathcal{L}$. The values of these integrals are independent of the shape of the path. The first integral in (4.6),

$$\chi_0(z, w) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \log l(t, \xi)\, \mathrm{d}W, \tag{4.8}$$

is discontinuous through the contour $\mathcal{L}$ with the jump $\log l(t, \xi)$. The other integrals are also discontinuous through the curves of integration. However, the corresponding jumps are $2\pi \mathrm{i} k$ ($k$ is an integer), and therefore, the function $X(z, w)$ satisfies the homogeneous boundary condition (4.2).

The second and the third terms in (4.6) are taken to achieve the prescribed behaviour (4.3) of the canonical solution at the ends $z = \pm 1$ of the contours $L_1$ and $L_2$ (see Section 4.3). Analysis of the term $\exp\{\chi(z, w)\}$ in the vicinity of the points $p'_{\mu j}$ shows that the function $X(z, w)$ has simple poles at these points if $\kappa^+_\mu < 0$ and simple zeros if $\kappa^+_\mu > 0$. Clearly, for $\kappa^+_\mu = 0$ there is no singularity at the point $p'_{\mu j}$. The same rule is applicable to the integrals over the curves with the ending point $p''_{\mu j}$.

We emphasize that, in general, the function (4.5) has an essential singularity at infinity for the Weierstrass kernel having the algebraic growth at infinity. To eliminate the essential singularity, the last sum in (4.6) is added (see Section 4.4). At the starting points $p_j$, the function $X(z, w)$ possesses simple poles, and at the ending points $r_j$, it has simple zeros.

### 4.3 *Choice of a branch of* $\log l(t,\xi)$ *and integers* $\kappa_1^\pm, \kappa_2^\pm$

Let us fix a branch of the function $\log l(t,\xi)$ such that

$$-\pi < \arg l_\mu(0) \leqslant \pi, \quad \mu = 1, 2. \tag{4.9}$$

Then

$$\log l_\mu(1) = \log l_\mu(0) + \mathrm{i}\Delta'_\mu, \quad \log l_\mu(-1) = \log l_\mu(0) - \mathrm{i}\Delta''_\mu, \quad \mu = 1, 2, \tag{4.10}$$

where $\Delta'_\mu$ and $\Delta''_\mu$ are the increments of the arguments of the functions $l_\mu(t)$ as $t$ traces the contours $[0, 1]$ and $[0, -1]$, respectively, with $t = 0$ as a starting point.

With a view towards recovering the property (4.3) of the function $X(z, w)$ in neighbourhoods of the end-points, we choose the integers $\kappa_1^\pm$ and $\kappa_2^\pm$. To do this, first, rewrite the integral (4.8) in the form

$$\chi_0(z, w) = \frac{1}{4\pi\mathrm{i}}\int_{-1}^{1}[\log l_1(t) + \log l_2(t)]\frac{\mathrm{d}t}{t - z}$$

$$+\frac{w(z)}{4\pi\mathrm{i}}\int_{-1}^{1}[\log l_1(t) - \log l_2(t)]\frac{\mathrm{d}t}{q^{1/2}(t)(t - z)} \tag{4.11}$$

and analyse its behaviour at $z = \pm 1$. The first term in (4.11) has the logarithmic singularity $B^\pm \log(z \mp 1)$ at the points $z = \pm 1$ of both sheets of the surface, where

$$B^\pm = \pm\frac{1}{4\pi\mathrm{i}}[\log l_1(\pm 1) + \log l_2(\pm 1)]. \tag{4.12}$$

As for the second integral, its behaviour depends on whether or not the points $z = \pm 1$ coincide with the branch points of the surface. If $z = \pm 1$ are not the branch points, then the second integral has the logarithmic singularity $B_\mu^\pm \log(z \mp 1)$ on the sheet $\mathbb{C}_\mu$, where

$$B_\mu^\pm = \mp\frac{(-1)^\mu}{4\pi\mathrm{i}}[\log l_1(\pm 1) - \log l_2(\pm 1)], \quad \mu = 1, 2. \tag{4.13}$$

If $z = 1$ or $z = -1$ is a branch point, then the second integral is bounded as $z \to 1$ or $z \to -1$ on both sheets $\mathbb{C}_1$ and $\mathbb{C}_2$.

We thus obtain that if $z = 1$ is a branch point, regardless of which sheet the point $z = 1$ belongs to, the function $\chi_0(z, w)$ behaves as

$$\chi_0(z, w) \sim \frac{\log l_1(1) + \log l_2(1)}{4\pi\mathrm{i}}\log(z - 1), \quad (z, w) \in \mathcal{R}, \quad z \to 1. \tag{4.14}$$

In the vicinity of the second end-point, if it is a branch point, then

$$\chi_0(z, w) \sim -\frac{\log l_1(-1) + \log l_2(-1)}{4\pi\mathrm{i}}\log(z + 1), \quad (z, w) \in \mathcal{R}, \quad z \to -1. \tag{4.15}$$

If $z=\pm 1$ are regular points of the surface,

$$\chi_0(z,w)\sim\pm\frac{\log l_\mu(\pm 1)}{2\pi\mathrm{i}}\log(z\mp 1),\quad (z,w)\in\mathbb{C}_\mu,\quad z\to\pm 1,\quad \mu=1,2. \tag{4.16}$$

Substituting formulae (4.14), (4.15), (4.16) into (4.5), (4.6) yields

$$X(z,w)=O(|z\mp 1|^{\beta_\mu^\pm}),\quad (z,w)\in\mathbb{C}_\mu,\quad z\to\pm 1,\quad \mu=1,2, \tag{4.17}$$

where

$$\beta_\mu^\pm=\pm\frac{1}{2\pi}\arg l_\mu(\pm 1)-\kappa_\mu^\pm,\quad \mu=1,2. \tag{4.18}$$

This is true if $z=\pm 1$ are regular points of the surface. If, however, $z=1$ or $z=-1$ coincides with a branch point, then

$$X(z,w)=O(|z-1|^{\beta^+}),\quad (z,w)\in\mathcal{R},\quad z\to 1 \tag{4.19}$$

or

$$X(z,w)=O(|z+1|^{\beta^-}),\quad (z,w)\in\mathcal{R},\quad z\to -1, \tag{4.20}$$

with

$$\beta^\pm=\pm\frac{1}{4\pi}\left[\arg l_1(\pm 1)+\arg l_2(\pm 1)\right]-\kappa_1^\pm. \tag{4.21}$$

In this case we put $\kappa_2^+=0$ or $\kappa_2^-=0$. Obviously, the function $X(z,w)$ meets the condition (4.3) if the numbers $\beta_1^\pm$, $\beta^\pm$ satisfy the inequalities

$$-\nu_\mu^\pm\leqslant\beta_\mu^\pm<1-\nu_\mu^\pm\quad(\mu=1,2),\quad -\nu_1^\pm\leqslant\beta^\pm<1-\nu_1^\pm. \tag{4.22}$$

Hence, if $z=\pm 1$ are regular points of the surface $\mathcal{R}$, then

$$\kappa_\mu^\pm=\nu_\mu^\pm+\left[\pm\frac{1}{2\pi}\arg l_\mu(\pm 1)\right],\quad \mu=1,2. \tag{4.23}$$

Here $[a]$ is the entire part of a number $a$. If $z=1$ or $z=-1$ is a branch point of the surface, then

$$\kappa_1^+=\nu_1^++\left[\frac{1}{4\pi}\left(\arg l_1(1)+\arg l_2(1)\right)\right],\quad \kappa_2^+=0 \tag{4.24}$$

or

$$\kappa_1^-=\nu_1^-+\left[-\frac{1}{4\pi}\left(\arg l_1(-1)+\arg l_2(-1)\right)\right],\quad \kappa_2^-=0. \tag{4.25}$$

4.4 *Jacobi's inversion problem*

If the genus $\rho$ of the surface $\mathcal{R}$ is zero, then the last sum in (4.6) vanishes, and the function $X(z, w)$ given by (4.5) is a solution (bounded as $z \to \infty$) to the homogeneous problem (4.2). The choice (4.23) or (4.24), (4.25) provides the prescribed behaviour of the solution to the original vector functional equation at the ends $x \pm \mathrm{i}\infty$ ($h - \omega \leqslant x \leqslant \omega$) of the strip.

Let us concentrate on the elliptic ($\rho = 1$) and hyper-elliptic ($\rho \geqslant 2$) cases. In general, for arbitrary $r_j, m_j, n_j$, because of the pole of order $\rho$ of the Weierstrass kernel at infinity, the function $X(z, w)$ has an essential singularity at infinity. The presence of the points $r_j$ and the integers $m_j, n_j$ makes it possible to eliminate this singularity. To do this, we rewrite the representation (4.6) for the function $\chi(z, w)$ as follows:

$$\chi(z, w) = \chi_1(z) + w(z)\chi_2(z), \tag{4.26}$$

where

$$\chi_1(z) = \frac{1}{4\pi\mathrm{i}}\int_{-1}^{1}[\log l_1(t) + \log l_2(t)]\frac{\mathrm{d}t}{t-z}$$

$$+\frac{1}{2}\sum_{\mu=1}^{2}\left(\operatorname{sgn}\kappa_\mu^+\sum_{j=1}^{|\kappa_\mu^+|}\int_{1}^{\delta'_{\mu j}}\frac{\mathrm{d}t}{t-z} + \operatorname{sgn}\kappa_\mu^-\sum_{j=1}^{|\kappa_\mu^-|}\int_{-1}^{\delta''_{\mu j}}\frac{\mathrm{d}t}{t-z}\right) + \frac{1}{2}\sum_{j=1}^{\rho}\int_{\delta_j}^{\sigma_j}\frac{\mathrm{d}t}{t-z},$$

$$\chi_2(z) = \frac{1}{4\pi\mathrm{i}}\int_{-1}^{1}[\log l_1(t) - \log l_2(t)]\frac{\mathrm{d}t}{q^{1/2}(t)(t-z)}$$

$$-\frac{1}{2}\sum_{\mu=1}^{2}(-1)^\mu\left(\operatorname{sgn}\kappa_\mu^+\sum_{j=1}^{|\kappa_\mu^+|}\int_{1}^{\delta'_{\mu j}}\frac{\mathrm{d}t}{q^{1/2}(t)(t-z)} + \operatorname{sgn}\kappa_\mu^-\sum_{j=1}^{|\kappa_\mu^-|}\int_{-1}^{\delta''_{\mu j}}\frac{\mathrm{d}t}{q^{1/2}(t)(t-z)}\right)$$

$$+\frac{1}{2}\sum_{j=1}^{\rho}\left(\int_{(\delta_j, v_j)}^{(\sigma_j, w_j)} + m_j\oint_{\mathbf{a}_j} + n_j\oint_{\mathbf{b}_j}\right)\frac{\mathrm{d}t}{\xi(t)(t-z)}. \tag{4.27}$$

By use of the identity

$$\frac{1}{t-z} = -\frac{1}{z} - \frac{t}{z^2} - \ldots - \frac{t^{\rho-1}}{z^\rho} + \frac{t^\rho}{z^\rho(t-z)} \tag{4.28}$$

we obtain the following asymptotic expansion of the function $\chi(z, w)$ at infinity:

$$\chi(z, w) = -\frac{1}{2}\sum_{\nu=1}^{\rho}\left\{\frac{1}{2\pi\mathrm{i}}\int_{-1}^{1}[\log l_1(t) - \log l_2(t)]\frac{t^{\nu-1}\mathrm{d}t}{q^{1/2}(t)}\right.$$

$$- \sum_{\mu=1}^{2} (-1)^{\mu} \left( \operatorname{sgn} \kappa_\mu^+ \sum_{j=1}^{|\kappa_\mu^+|} \int_{1}^{\delta'_{\mu j}} \frac{t^{\nu-1} \mathrm{d}t}{q^{1/2}(t)} + \operatorname{sgn} \kappa_\mu^- \sum_{j=1}^{|\kappa_\mu^-|} \int_{-1}^{\delta''_{\mu j}} \frac{t^{\nu-1} \mathrm{d}t}{q^{1/2}(t)} \right)$$

$$+ \sum_{j=1}^{\rho} \left( \int_{(\delta_j, v_j)}^{(\sigma_j, w_j)} + m_j \oint_{\mathbf{a}_j} + n_j \oint_{\mathbf{b}_j} \right) \frac{t^{\nu-1} \mathrm{d}t}{\xi(t)} \Bigg\} \frac{w(z)}{z^\nu} + O(1), \quad z \to \infty. \tag{4.29}$$

The function $\chi(z, w)$ is bounded at infinity if and only if the following $\rho$ conditions hold:

$$\sum_{j=1}^{\rho} \left( \int_{(\delta_j, v_j)}^{(\sigma_j, w_j)} \mathrm{d}\omega_\nu + m_j \oint_{\mathbf{a}_j} \mathrm{d}\omega_\nu + n_j \oint_{\mathbf{b}_j} \mathrm{d}\omega_\nu \right) = d_\nu^\circ, \quad \nu = 1, 2, \ldots, \rho, \tag{4.30}$$

where

$$d_\nu^\circ = -\frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} [\log l_1(t) - \log l_2(t)] \frac{t^{\nu-1} \mathrm{d}t}{q^{1/2}(t)}$$

$$+ \sum_{\mu=1}^{2} (-1)^{\mu} \left( \operatorname{sgn} \kappa_\mu^+ \sum_{j=1}^{|\kappa_\mu^+|} \int_{1}^{\delta'_{\mu j}} \frac{t^{\nu-1} \mathrm{d}t}{q^{1/2}(t)} + \operatorname{sgn} \kappa_\mu^- \sum_{j=1}^{|\kappa_\mu^-|} \int_{-1}^{\delta''_{\mu j}} \frac{t^{\nu-1} \mathrm{d}t}{q^{1/2}(t)} \right),$$

$$\mathrm{d}\omega_\nu = \frac{t^{\nu-1} \mathrm{d}t}{\xi(t)}. \tag{4.31}$$

The differentials $\mathrm{d}\omega_1, \mathrm{d}\omega_2, \ldots, \mathrm{d}\omega_\rho$ form a basis of Abelian differentials of the first kind on the surface $\mathcal{R}$. The integrals

$$A_{\nu j} = \oint_{\mathbf{a}_j} \frac{t^{\nu-1} \mathrm{d}t}{\xi(t)}, \quad B_{\nu j} = \oint_{\mathbf{b}_j} \frac{t^{\nu-1} \mathrm{d}t}{\xi(t)} \tag{4.32}$$

are the $A$- and $B$-periods of the Abelian integrals (Springer, 1956):

$$\omega_\nu = \omega_\nu(z, w) = \int_{(z_0, 0)}^{(z, w)} \frac{t^{\nu-1} \mathrm{d}t}{\xi(t)}, \quad \nu = 1, 2, \ldots, \rho. \tag{4.33}$$

By use of the notation (4.32) and (4.33), equation (4.30) becomes

$$\sum_{j=1}^{\rho} [\omega_\nu(\sigma_j, w_j) + m_j A_{\nu j} + n_j B_{\nu j}] = d_\nu^*, \quad \nu = 1, 2, \ldots, \rho, \tag{4.34}$$

where

$$d_\nu^* = d_\nu^\circ + \sum_{j=1}^{\rho} \omega_\nu(\delta_j, v_j). \tag{4.35}$$

The nonlinear system (4.34) with respect to the points $(\sigma_j, w_j) \in \mathcal{R}$ and the integers $m_j, n_j$ ($j = 1, 2, \ldots, \rho$) is the classical Jacobi inversion problem (Springer, 1956; Zverovich, 1971; Farkas, 1992). It is known that its solution always exists.

In the elliptic case, $\rho = 1$, the problem is equivalent to the inversion of the elliptic integral

$$\int_{(z_0,0)}^{(\sigma_1,w_1)} \frac{dt}{\sqrt{(t-z_0)(t-z_1)(t-z_2)(t-z_3)}} + m_1 A_{11} + n_1 B_{11} = d_1^*. \tag{4.36}$$

This is solvable in terms of elliptic functions (Hancock, 1968). In the hyper-elliptic case, $\rho \geqslant 2$, the inversion problem gives rise to a system of $\rho$ algebraic equations (Zverovich, 1971) that is equivalent to one algebraic equation of order $\rho$ (Antipov & Silvestrov, 2002). To provide a guideline to the reader, we describe the main steps of the procedure for the inversion problem (Antipov & Silvestrov, 2002):

(i) normalizing the basis of the Abelian integrals of the first kind (4.33);

(ii) setting up Jacobi's inversion problem for the normalised basis;

(iii) reducing the problem to an algebraic equation of order $\rho$;

(iv) evaluating the coefficients of the algebraic equation in terms of Riemann's $\theta$-function.

The function $X(z, w)$ defined by (4.5), (4.26) is a canonical solution of the problem (3.15) provided the continuous branches of the functions $\log l_1(t)$, $\log l_2(t)$ are chosen as in (4.9); the integers $\kappa_1^\pm, \kappa_2^\pm$ are fixed by (4.23) to (4.25). The points $(\delta_j, v_j) \in \mathcal{R}$ ($j = 1, 2, \ldots, \rho$) are fixed in an arbitrary manner. The points $(\sigma_j, w_j) \in \mathcal{R}$ and the integers $m_j, n_j$ ($j = 1, 2, \ldots, \rho$) should be found from the Jacobi inversion problem (4.34). We note that it is always possible to avoid (by changing the location of the points $(\delta_j, v_j)$) the case when either some of the points $(\sigma_j, w_j)$ coincide, or some of them fall on the poles $(\alpha_k, \pm q^{1/2}(\alpha_k))$, or on the branch points of the surface $\mathcal{R}$.

## 5. Non-homogeneous Riemann–Hilbert problem

Use of the canonical solution enables us to find the general solution of the non-homogeneous problem (3.15). First, by splitting the function

$$l(t, \xi) = \frac{X^+(t, \xi)}{X^-(t, \xi)}, \quad (t, \xi) \in \mathcal{L}, \tag{5.1}$$

we obtain

$$\frac{F^+(t, \xi)}{X^+(t, \xi)} - \frac{F^-(t, \xi)}{X^-(t, \xi)} = \frac{g^*(t, \xi)}{X^+(t, \xi)}, \quad (t, \xi) \in \mathcal{L}. \tag{5.2}$$

It follows from (3.17), (4.17), (4.19), (4.20), (4.22) that the function $[X^+(t,\xi)]^{-1}g^*(t,\xi)$ may have integrable singularities at the ends $t=\pm 1$ of the contour $\mathcal{R}$:

$$\left|\frac{g^*(t,\xi)}{X^+(t,\xi)}\right| \leqslant A'|t\mp 1|^{-\delta}, \quad A'=\text{const}, \quad 0\leqslant \delta<1, \quad t\to\pm 1. \tag{5.3}$$

Hence a partial solution of the problem (3.15) is the function $X(z,w)\Psi(z,w)$, where

$$\Psi(z,w)=\psi_1(z)+w(z)\psi_2(z),$$

$$\psi_1(z)=\frac{1}{4\pi\mathrm{i}}\int_{\mathcal{L}}\frac{g^*(t,\xi)}{X^+(t,\xi)}\frac{\mathrm{d}t}{t-z}, \quad \psi_2(z)=\frac{1}{4\pi\mathrm{i}}\int_{\mathcal{L}}\frac{g^*(t,\xi)}{\xi(t)X^+(t,\xi)}\frac{\mathrm{d}t}{t-z}. \tag{5.4}$$

Then the general solution of the problem (3.15) becomes

$$F(z,w)=X(z,w)[\Psi(z,w)+R(z,w)], \tag{5.5}$$

where $R(z,w)$ is the meromorphic function on $\mathcal{R}$ whose poles are defined by the class of solutions described in Section 4.1 and, also, by the properties of the canonical function $X(z,w)$. The function $R(z,w)$ has poles of orders $\nu_1,\nu_2,\dots,\nu_m$ at the points with the affixes $\alpha_1,\alpha_2,\dots\alpha_m$ on both sheets of the surface. It also has simple poles at the points $r_j=(\sigma_j,w_j)$ $(j=1,2,\dots,\rho)$ and poles of orders $\mu_0,\mu_1,\dots,\mu_{2\rho+1}$ ($\mu_j$ are either zero, or odd positive numbers) at the branch points $z_0,z_1,\dots,z_{2\rho+1}$, respectively.

If $\kappa_\mu^+>0$ $(\mu=1$ or $\mu=2)$, then at the points $p'_{\mu j}=(\delta'_{\mu j},(-1)^{\mu-1}v'_j)\in\mathbb{C}_\mu$ $(j=1,2,\dots,\kappa_\mu^+)$ the canonical solution has simple zeros, and, therefore, the function $R(z,w)$ may have simple poles at these points. In the case $\kappa_\mu^+<0$, the canonical function $X(z,w)$ has simple poles at the points $p'_{\mu j}$ $(j=1,2,\dots,-\kappa_\mu^+)$. Eventually, this causes the presence of inadmissible poles of the function $F(z,w)$. In order for the solution to be bounded at the points $p'_{\mu j}$ it is necessary and sufficient that the function $\Psi(z,w)+R(z,w)$ vanish at these points. Analysis of the structure of the function $R(z,w)$ at the points $p''_{\mu j}=(\delta''_{\mu j},(-1)^{\mu-1}v''_{\mu j})\in\mathbb{C}_\mu$ $(j=1,2,\dots,|\kappa_\mu^-|)$ is employed similarly.

In addition, the function $\Psi(z,w)+R(z,w)$ has simple zeros at the points $p_j=(\delta_j,v_j)$ $(j=1,2,\dots,\rho)$ and has to be bounded at infinity on both sheets (if of course none of the above poles coincides with one of the two infinite points of the surface). The meromorphic function $R(z,w)$ with the described poles has the form

$$R(z,w)=R_1(z)+w(z)R_2(z), \tag{5.6}$$

where

$$R_1(z) = C_0 + \sum_{j=1}^{\rho} \frac{C_j w_j}{z-\sigma_j} + \sum_{k=1}^{m}\sum_{j=1}^{\nu_k} \frac{D'_{kj}}{(z-\alpha_k)^j}$$
$$+ \sum_{k=0}^{2\rho+1}\sum_{j=1}^{(\mu_k-1)/2} \frac{E'_{kj}}{(z-z_k)^j} - \sum_{\mu=1}^{2}(-1)^{\mu}\left(\sum_{j=1}^{\kappa'_\mu} \frac{H'_{\mu j} v'_{\mu j}}{z-\delta'_{\mu j}} + \sum_{j=1}^{\kappa''_\mu} \frac{H''_{\mu j} v''_{\mu j}}{z-\delta''_{\mu j}}\right),$$
$$R_2(z) = \sum_{j=1}^{\rho} \frac{C_j}{z-\sigma_j} + \sum_{k=1}^{m}\sum_{j=1}^{\nu_k} \frac{D''_{kj}}{(z-\alpha_k)^j}$$
$$+ \sum_{k=0}^{2\rho+1}\sum_{j=1}^{(\mu_k+1)/2} \frac{E''_{kj}}{(z-z_k)^j} + \sum_{\mu=1}^{2}\left(\sum_{j=1}^{\kappa'_\mu} \frac{H'_{\mu j}}{z-\delta'_{\mu j}} + \sum_{j=1}^{\kappa''_\mu} \frac{H''_{\mu j}}{z-\delta''_{\mu j}}\right). \tag{5.7}$$

Here $\kappa'_\mu = \max\{\kappa^+_\mu, 0\}$, $\kappa''_\mu = \max\{\kappa^-_\mu, 0\}$ $(\mu = 1, 2)$, $v'_{\mu j} = q^{1/2}(\delta'_{\mu j})$, $v''_{\mu j} = q^{1/2}(\delta''_{\mu j})$, $w_j = w(\sigma_j)$. If the upper index is less than the lower one, then the corresponding sum is assumed to be zero. The constants $C_j$ $(j = 0, 1, \dots, \rho)$, $D'_{kj}$, $D''_{kj}$ $(k = 1, 2, \dots, m;\ j = 1, 2, \dots, \nu_k)$, $E'_{kj}$ $(k = 0, 1, \dots, 2\rho+1;\ j = 1, 2, \dots, (\mu_k - 1)/2)$, $E''_{kj}$ $(k = 0, 1, \dots, 2\rho+1;\ j = 1, 2, \dots, (\mu_k+1)/2)$, $H'_{\mu j}$ $(j = 1, 2, \dots, \kappa'_\mu;\ \mu = 1, 2)$ and $H''_{\mu j}$ $(j = 1, 2, \dots, \kappa''_\mu;\ \mu = 1, 2)$ are arbitrary. The same choice of the constants $C_j$ in the representations for the rational functions $R_1(z)$ and $R_2(z)$ is explained by the fact that the canonical function $X(z, w)$ has simple poles at the points $r_j = (\sigma_j, w_j)$ which lie either on the first sheet $\mathbb{C}_1$ or on the second one. The constants $D'_{kj}$ and $D''_{kj}$ are not the same because the general solution has to have poles at the points $\alpha_1, \alpha_2, \dots, \alpha_m$, and the functions 1, $w(z)$ are linearly independent. For the same reason the constants $E'_{kj}$, $E''_{kj}$ and $H'_{mj}$, $H''_{mj}$ are different for the functions $R_1(z)$ and $R_2(z)$.

The procedure of solution of the Riemann–Hilbert problem (3.15) will be accomplished if the conditions

$$\lim_{z\to\infty} z^k[\psi_2(z) + R_2(z)] = 0, \quad k = 1, 2, \dots, \rho, \tag{5.8}$$

$$\Psi(\delta_k, v_k) + R(\delta_k, v_k) = 0, \quad k = 1, 2, \dots, \rho, \tag{5.9}$$

are satisfied. In addition,

$$\Psi(\delta'_{\mu j}, (-1)^{\mu-1}v'_{\mu j}) + R(\delta'_{\mu j}, (-1)^{\mu-1}v'_{\mu j}) = 0, \quad j = 1, 2, \dots, -\kappa^+_\mu, \quad \mu = 1, 2, \tag{5.10}$$

and

$$\Psi(\delta''_{\mu j}, (-1)^{\mu-1}v''_{\mu j}) + R(\delta''_{\mu j}, (-1)^{\mu-1}v''_{\mu j}) = 0, \quad j = 1, 2, \dots, -\kappa^-_\mu, \quad \mu = 1, 2, \tag{5.11}$$

are effective if the upper bounds are positive. Condition (5.8) provides the boundness of the function $\Psi(z, w) + R(z, w)$ at infinity. The next group of the conditions lends itself to eliminating the poles at the points $(\delta_k, v_k)$. The relations (5.10), (5.11) guarantee the boundness of the function $F(z, w)$ at the points $(\delta'_{\mu j}, (-1)^{\mu-1}v'_{\mu j})$ and $(\delta''_{\mu j}, (-1)^{\mu-1}v''_{\mu j})$ when $\kappa^+_\mu < 0$ and $\kappa^-_\mu < 0$, respectively.

*Remark.* Formulae (5.7) are written down under the assumption that the poles $\alpha_k$ and the branch points $z_k$ lie in a finite part of the complex plane. Otherwise these formulae and the conditions (5.8) should be corrected in the appropriate manner. Alternatively, the conformal mapping (3.8) can be changed by another mapping of the strip $\Pi$ into the complex plane with a cut different from $[-1, 1]$ to make all the points $\alpha_k$ and $z_k$ finite.

## 6. Exact solution to the vector functional-difference equation

### 6.1 *General case*

Now we define the solution to the initial equation (2.1) with the matrix $G(\sigma)$ given by (2.10). Use of the relations (3.2), (2.13), (3.9) and (3.14) gives

$$\Phi_1(s) = F(z, w) + F(z, -w),$$

$$\Phi_2(s) = -f_1(s)[F(z, w) + F(z, -w)] + f^{1/2}(s)[F(z, w) - F(z, -w)], \quad s \in \Pi, \tag{6.1}$$

where

$$\begin{aligned} z &= -\mathrm{i}\tan\frac{\pi}{h}(s-\omega), \quad f(s) = f_1^2(s) + f_2(s), \\ w &= q^{1/2}(z), \quad q(z) = (z-z_0)(z-z_1)\ldots(z-z_{2\rho+1}). \end{aligned} \tag{6.2}$$

The functions $f_1(s)$, $f_2(s)$ are defined by (2.8). To analyse the behaviour of the solution at the singular points, let us transform formulae (6.1). First, by making use of relations (5.4)–(5.6), (4.5) and (4.26), the solution to the Riemann–Hilbert problem (3.15) becomes

$$F(z, w) = \mathrm{e}^{\chi_1(z)+w(z)\chi_2(z)}[Y_1(z) + w(z)Y_2(z)], \tag{6.3}$$

where

$$Y_1(z) = \psi_1(z) + R_1(z), \quad Y_2(z) = \psi_2(z) + R_2(z), \tag{6.4}$$

and the functions $\chi_1$, $\chi_2$, $\psi_1$, $\psi_2$ and $R_1(z)$, $R_2(z)$ are defined by (4.27), (5.4) and (5.7). Substituting the expression (6.3) into (6.1) gives the resulting formulae for the solution:

$$\Phi_1(s) = 2e^{\chi_1(z)}[\cosh\{w(z)\chi_2(z)\}Y_1(z) + w(z)\sinh\{w(z)\chi_2(z)\}Y_2(z)],$$

$$\begin{aligned} \Phi_2(s) = &-f_1(s)\Phi_1(s) \\ &+2f^{1/2}(s)\mathrm{e}^{\chi_1(z)}[\sinh\{w(z)\chi_2(z)\}Y_1(z) + w(z)\cosh\{w(z)\chi_2(z)\}Y_2(z)]. \end{aligned} \tag{6.5}$$

The functions (6.5) satisfy (2.1). However, for arbitrary chosen constants in (5.7), they have poles in the strip $\Pi$. Indeed, the function $F(z, w)$ has poles at the points of both sheets of the surface with affixes $\alpha_1, \alpha_2, \dots \alpha_m$ and $z_0, z_1, \dots, z_{2\rho+1}$. Their images in the strip $\Pi$, the points $a_1, a_2, \dots a_m$, and $s_0, s_1, \dots s_{2\rho+1}$, respectively, are the poles of the functions $\Phi_1(s)$, $\Phi_2(s)$. The factors $f_1(s)$ and $f^{1/2}(s)$ may change the order of poles or add new ones to the set of poles of the function $\Phi_2(s)$. The conditions of analyticity of the functions $\Phi_1(s)$, $\Phi_2(s)$ at their superfluous singular points provide additional conditions which together with (5.8)–(5.11) are used to fix some of the arbitrary constants in (5.7).

## 6.2 *The case of simple poles*

Let all the poles $\alpha_k$ ($k = 1, 2, \dots, m$) and the branch points $z_k$ ($k = 0, 1, \dots, 2\rho + 1$) be simple, i.e. $\nu_k = 1$ ($k = 1, 2, \dots, m$), and $\mu_k = 1$ ($k = 0, 1, \dots, 2\rho + 1$). Then, obviously,

$$R_1(z) = C_0 + \sum_{j=1}^{\rho} \frac{C_j w_j}{z - \sigma_j} + \sum_{j=1}^{m} \frac{D'_j}{z - \alpha_j} - \sum_{\mu=1}^{2} (-1)^{\mu} \left( \sum_{j=1}^{\kappa'_\mu} \frac{H'_{\mu j} v'_{\mu j}}{z - \delta'_{\mu j}} + \sum_{j=1}^{\kappa''_\mu} \frac{H''_{\mu j} v''_{\mu j}}{z - \delta''_{\mu j}} \right),$$

$$R_2(z) = \sum_{j=1}^{\rho} \frac{C_j}{z - \sigma_j} + \sum_{j=1}^{m} \frac{D''_j}{z - \alpha_j} + \sum_{j=0}^{2\rho+1} \frac{E_j}{z - z_j} + \sum_{\mu=1}^{2} \left( \sum_{j=1}^{\kappa'_\mu} \frac{H'_{\mu j}}{z - \delta'_{\mu j}} + \sum_{j=1}^{\kappa''_\mu} \frac{H''_{\mu j}}{z - \delta''_{\mu j}} \right). \tag{6.6}$$

Therefore, the solution (6.5) possesses $3\rho + 2m + \kappa'_1 + \kappa'_2 + \kappa''_1 + \kappa''_2 + 3$ arbitrary constants. Now we write down all the conditions for the functions $\Phi_1(s)$, $\Phi_2(s)$ to be within the prescribed class. Assume that the point $(\sigma_k, w_k) \in \mathbb{C}_1$. Then from (6.6) the function $F(z, -w)$ is analytic at this point. Because of the simple zero for $X(z, w)$ at $(\sigma_k, w_k)$, the function $F(z, w)$ has a removable singularity at this point. A similar result follows for $(\sigma_k, w_k) \in \mathbb{C}_2$. The $2\rho$ conditions (5.8), (5.9) provide the required behaviour of the solution at infinity and remove the simple poles of the canonical function $X(z, w)$ at the points $(\delta_k, v_k) \in \mathbb{C}_1$ ($k = 1, 2, \dots, \rho$). Let $\alpha_1, \alpha_2, \dots, \alpha_t$ ($t \leqslant m$) be the prescribed poles of the solution. Then to eliminate the other poles $\alpha_{t+1}, \dots, \alpha_m$, we require

$$\operatorname*{res}_{z=\alpha_k} \Phi_1(s) = 0, \qquad \operatorname*{res}_{z=\alpha_k} \Phi_2(s) = 0 \quad (k = t+1, t+2, \dots, m) \tag{6.7}$$

where

$$s = \omega + \frac{\mathrm{i}h}{2\pi} \log \frac{1+z}{1-z}. \tag{6.8}$$

The above relations provide the additional $2(m - t)$ conditions. As for the poles $z = z_k$ ($k = 0, 1, \dots, 2\rho+1$) of the function $R_2(z)$, they become removable points of the solution. This is because the functions

$$w(z) \sinh\{w(z)\chi_2(z)\}, \qquad f^{1/2}(z) \sinh\{w(z)\chi_2(z)\}, \qquad f^{1/2}(z) w(z) \cosh\{w(z)\chi_2(z)\} \tag{6.9}$$

have simple zeros at the points $z = z_k$. The relations (5.10), (5.11) give $\hat{\kappa}_1 + \hat{\kappa}_2 + \tilde{\kappa}_1 + \tilde{\kappa}_2$ conditions, where $\hat{\kappa}_\mu = \max\{0, -\kappa_\mu^+\}$, $\tilde{\kappa}_\mu = \max\{0, -\kappa_\mu^-\}$ $(\mu = 1, 2)$. Finally, the function $\Phi_2(s)$ defined from (3.3) by

$$\Phi_2(s) = [-f_1(s) + f^{1/2}(s)]\phi_1(s) - [f_1(s) + f^{1/2}(s)]\phi_2(s) \tag{6.10}$$

may have inadmissible poles at the poles of the functions $f_1(s)$ and $f^{1/2}(s)$. Let these poles be $s_1^\circ, s_2^\circ, \ldots, s_{n^\circ}^\circ$. Assuming that all the poles are simple, write down the regularity conditions for the function $\Phi_2(s)$ at these points

$$\operatorname*{res}_{s=s_j^\circ} \Phi_2(s) = 0, \quad j = 1, 2, \ldots, n^\circ. \tag{6.11}$$

Therefore, the total number of additional conditions providing the functions $\Phi_1(s)$, $\Phi_2(s)$ to belong to the prescribed class, is $2\rho + 2m - 2t + n^\circ + \hat{\kappa}_1 + \hat{\kappa}_2 + \tilde{\kappa}_1 + \tilde{\kappa}_2$. Thus, the difference between the number of the arbitrary constants in (6.6) and the number of conditions for them is $\rho + 2t - n^\circ + \kappa_1^+ + \kappa_2^+ + \kappa_1^- + \kappa_2^- + 3$. Note that $\rho$ is the genus of the surface $\mathcal{R}$ (the number of the branch points of the function $f^{1/2}(s)$ in the strip $\Pi$ is $2\rho + 2$); $t$ is the number of the prescribed poles of the solution in the strip $\Pi$; $n^\circ$ is the number of the inadmissible poles $s_j^\circ$; the integers $\kappa_1^\pm$ and $\kappa_2^\pm$ are defined by (4.23)–(4.25) and depend on the elements of the matrix $\mathbf{G}(\sigma)$.

## 7. Even solution of the Riemann–Hilbert problem

In this section we aim to analyse a particular case of the Riemann–Hilbert problem (3.15) when its solution is even, i.e. satisfies the condition

$$F_\mu(z) = F_\mu(-z), \quad z \in \mathbb{C}_\mu \backslash [-1, 1], \quad \mu = 1, 2. \tag{7.1}$$

Since the points $s$ and $2\omega - h - s$ of the $s$-plane correspond to the points $z$ and $-z$ of the plane, respectively, the relation (7.1) holds, if simultaneously

$$\Phi_\mu(s) = \Phi_\mu(2\omega - h - s), \ \mu = 1, 2,$$

$$f_1(s) = f_1(2\omega - h - s), \ f^{1/2}(s) = f^{1/2}(2\omega - h - s), \quad s \in \Pi. \tag{7.2}$$

We also describe an algorithm for this case. To construct such an even solution is a crucial step in solving problems of electromagnetic scattering (see Section 8).

### 7.1 *Formulation*

Assume that the poles $\alpha_k$ $(k = 1, 2, \ldots, 2m';\ m = 2m')$ of the functions $F_1(s)$, $F_2(s)$ and the branch points $z_k$ $(k = 0, 1, \ldots, 2\rho + 1)$ of the surface $\mathcal{R}$ are simple and located symmetrically with respect to the origin:

$$\alpha_{m'+k} = -\alpha_k, \quad k = 1, 2, \ldots, m',$$

$$z_{\rho+1+k} = -z_k, \quad k = 0, 1, \dots, \rho. \tag{7.3}$$

Let also the points $z = \pm 1$ not coincide with the branch points.

Define a class of the Riemann–Hilbert problems (3.15) with additional condition of symmetry (7.1) which have a solution. The relation (7.1) implies

$$F_\mu^+(-t) = F_\mu^-(t), \quad F_\mu^-(-t) = F_\mu^+(t), \quad t \in (-1, 1), \quad \mu = 1, 2. \tag{7.4}$$

Replacing $t$ for $-t$ in the equation

$$F_\mu^+(t) = l_\mu(t) F_\mu^-(t) + g_\mu^*(t), \quad t \in (-1, 1), \quad \mu = 1, 2, \tag{7.5}$$

that follows from (3.15), and using formulae (7.4) gives

$$F_\mu^+(t) = \frac{1}{l_\mu(-t)} F_\mu^-(t) - \frac{g_\mu^*(-t)}{l_\mu(-t)}, \quad t \in (-1, 1), \quad \mu = 1, 2. \tag{7.6}$$

By comparison of relations (7.5) and (7.6) we get the following necessary conditions for a solution of the problem (3.15), (7.1) to exist:

$$l_\mu(t) l_\mu(-t) = 1, \quad g_\mu^*(t) + l_\mu(t) g_\mu^*(-t) = 0, \quad t \in (-1, 1), \quad \mu = 1, 2. \tag{7.7}$$

Note that the above conditions are equivalent to the relations

$$\lambda_\mu(\sigma)\lambda_\mu(\bar\sigma) = 1, \quad g_\mu^\circ(\sigma) + \lambda_\mu(\sigma) g_\mu^\circ(\bar\sigma) = 0, \quad \sigma \in \Omega, \quad \mu = 1, 2. \tag{7.8}$$

This is because the points $\sigma$ and $\bar\sigma$ of the contour $\Omega$ correspond to the points $t$ and $-t$ on the segment $[-1, 1]$, respectively.

Thus, we have two possibilities: $l_\mu(0) = 1$ and $l_\mu(0) = -1$. We will henceforth assume that the functions $l_\mu(t)$ and $g_\mu^*(t)$ meet the conditions (7.7). By the relation (7.1), the functions $F_\mu(z)$ have the same singularities at the points $z = \pm 1$, and the inequality (4.1) becomes

$$|F_\mu(z)| \leqslant A_1^{(\mu)} |z \mp 1|^{-\nu_\mu}, \quad z \to \pm 1, \quad A_1^{(\mu)} = \text{const} \quad (\mu = 1, 2). \tag{7.9}$$

### 7.2 *Even canonical function*

Choose a branch of the functions $\log l_\mu(t)$, $t \in [-1, 1]$ $(\mu = 1, 2)$ such that $-\pi < \arg l_\mu(0) \leqslant \pi$ $(\mu = 1, 2)$. Then, because of the conditions (7.7)

$$\log l_\mu(-t) = -\log l_\mu(t) + 2\pi \mathrm{i} \epsilon_\mu, \quad t \in [-1, 1], \quad \mu = 1, 2, \tag{7.10}$$

where

$$\epsilon_\mu = \begin{cases} 0 & \text{if } l_\mu(0) = 1 \\ 1 & \text{if } l_\mu(0) = -1, \end{cases} \tag{7.11}$$

and also since $q^{1/2}(-t) = q^{1/2}(t)$, $t \in [0, 1]$, the integral (4.11) has the form

$$\chi_0(z, w) = \frac{1}{2\pi \mathrm{i}} \int_0^1 [\log l_1(t) + \log l_2(t)] \frac{t\,\mathrm{d}t}{t^2 - z^2}$$

$$+\frac{w(z)}{2\pi \mathrm{i}}\int_0^1 [\log l_1(t)-\log l_2(t)]\frac{t\,\mathrm{d}t}{q^{1/2}(t)(t^2-z^2)}+\tilde{\chi}_0(z,w), \tag{7.12}$$

where

$$\tilde{\chi}_0(z,w)=-\frac{\epsilon_1+\epsilon_2}{2}\int_0^1\frac{\mathrm{d}t}{t+z}-w(z)\,\frac{\epsilon_1-\epsilon_2}{2}\int_0^1\frac{\mathrm{d}t}{q^{1/2}(t)(t+z)}. \tag{7.13}$$

The function $\tilde{\chi}_0(z,w)$ is continuous everywhere on the surface $\mathcal{R}$ apart from the segments $[-1,0]\subset\mathbb{C}_\mu,\ \ \mu=1,2$. On these segments, for the function $\tilde{\chi}_0(z,w)$, the following boundary condition holds:

$$\tilde{\chi}_0^+(t,\xi)-\tilde{\chi}_0^-(t,\xi)=2\pi\mathrm{i}\epsilon_\mu. \tag{7.14}$$

Hence the function $\exp\{\tilde{\chi}_0(z,w)\}$ is continuous everywhere on the surface $\mathcal{R}$. So, without loss of generality, we can take the function $\chi_0(z,w)$ without the last term $\tilde{\chi}_0(z,w)$, i.e. as

$$\chi_0(z,w)=\frac{1}{2\pi\mathrm{i}}\int_0^1[\log l_1(t)+\log l_2(t)]\frac{t\,\mathrm{d}t}{t^2-z^2}$$

$$+\frac{w(z)}{2\pi\mathrm{i}}\int_0^1[\log l_1(t)-\log l_2(t)]\frac{t\,\mathrm{d}t}{q^{1/2}(t)(t^2-z^2)}. \tag{7.15}$$

Introduce, next, a new algebraic function

$$p(\zeta)=(\zeta-\zeta_0)(\zeta-\zeta_1)\dots(\zeta-\zeta_\rho),\quad \zeta_j=z_j^2,\quad j=0,1,\dots,\rho. \tag{7.16}$$

Then, in view of the symmetry (7.3) of the branch points $z_j$,

$$q(z)=(z^2-z_0^2)(z^2-z_1^2)\dots(z^2-z_\rho^2)=p(z^2). \tag{7.17}$$

Rewrite now formula (7.15) as follows:

$$\begin{aligned}\chi_0(z,w)&=\frac{1}{4\pi\mathrm{i}}\int_0^1[\log l_1(\sqrt{\tau})+\log l_2(\sqrt{\tau})]\frac{\mathrm{d}\tau}{\tau-\zeta}\\&\quad+\frac{u(\zeta)}{4\pi\mathrm{i}}\int_0^1[\log l_1(\sqrt{\tau})-\log l_2(\sqrt{\tau})]\frac{\mathrm{d}\tau}{p^{1/2}(\tau)(\tau-\zeta)}\\&=\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}^*}\log l_*(\tau,\eta)\,\mathrm{d}U=\chi_{0*}(\zeta,u),\end{aligned} \tag{7.18}$$

where $\zeta = z^2$, $\mathcal{L}^* = L_1^* \cup L_2^*$, $L_1^* = [0,1] \subset \mathbb{C}_1$, $L_2^* = [0,1] \subset \mathbb{C}_2$, $\log l_*(\tau,\eta) = \log l_\mu(\sqrt{\tau})$ on the contour $L_\mu^*$ $(\mu = 1, 2)$, and

$$\mathrm{d}U = \frac{u+\eta}{2\eta}\frac{\mathrm{d}\tau}{\tau - \zeta} \tag{7.19}$$

is the Weierstrass kernel on a Riemann surface $\mathcal{R}'$ of the algebraic function $u^2 = p(z)$. Here $u(\zeta) = w(\sqrt{\zeta})$, $\eta = u(\tau)$.

On the other hand, the function $\exp \chi_{0*}(\zeta, u)$ is a particular solution of the homogeneous Riemann–Hilbert problem

$$\exp \chi_{0*}^+(\tau,\eta) = l_*(\tau,\eta)\exp\chi_{0*}^-(\tau,\eta), \quad (\tau,\eta) \in \mathcal{L}^* \tag{7.20}$$

on the surface $\mathcal{R}'$ of genus $\rho' = [\rho/2]$ with the branch points $\zeta_0, \zeta_1, \dots, \zeta_\rho$. This solution is bounded at the points $(0, \pm p^{1/2}(0))$. It may have a power singularity at the points $(1, \pm p^{1/2}(1))$ and an essential singularity at infinity. By the device proposed in Section 4, remove these singularities by adding a new function that does not affect the boundary condition (7.20):

$$\chi_*(\zeta,u) = \chi_{0*}(\zeta,u) + \sum_{\mu=1}^{2} \operatorname{sgn}\kappa_\mu \sum_{j=1}^{|\kappa_\mu|} \int_{p_{\mu 0}^*}^{p_{\mu j}^*} \mathrm{d}U + \sum_{j=1}^{\rho'} \left( \int_{p_j^*}^{r_j^*} \mathrm{d}U + m_j \oint_{\mathbf{a}_j^*} \mathrm{d}U + n_j \oint_{\mathbf{b}_j^*} \mathrm{d}U \right). \tag{7.21}$$

Here $\mathbf{a}_j^*$, $\mathbf{b}_j^*$ $(j = 1, 2, \dots, \rho')$ are the canonical cross-sections of the surface $\mathcal{R}'$ which are the images of the $\rho'$ cross-sections $\mathbf{a}_j$, $\mathbf{b}_j$ $(j = 1, 2, \dots, \rho')$ of the surface $\mathcal{R}$ by mapping $\zeta = z^2$. The integers $\kappa_\mu$ and the points $p_{\mu 0}^*$ are given by

$$\kappa_\mu = \nu_\mu + \left[\frac{1}{2\pi}\arg l_\mu(1)\right] = \nu_\mu + \left[\frac{1}{2\pi}\Delta_\mu^*\right], \quad \mu = 1, 2,$$

$$p_{\mu 0}^* = (1, (-1)^{\mu-1}p^{1/2}(1)) = (1, (-1)^{\mu-1}q^{1/2}(1)) \in \mathbb{C}_\mu, \quad \mu = 1, 2, \tag{7.22}$$

where $\Delta_\mu^*$ is the increment of the argument of the function $l_\mu(t)$ as $t$ traces the contour $[0, 1]$ with $t = 0$ as a starting point. The other points $p_{\mu j}^*$ and $p_j^*$ are arbitrary, distinct and fixed:

$$\begin{gathered} p_{\mu j}^* = (\gamma_{\mu j}^2, (-1)^{\mu-1}v_{\mu j}) \in \mathbb{C}_\mu, \quad v_{\mu j} = p^{1/2}(\gamma_{\mu j}^2) = q^{1/2}(\gamma_{\mu j}), \\ j = 1, 2, \dots, |\kappa_\mu|, \; \mu = 1, 2, \\ p_j^* = (\delta_j^2, v_j) \in \mathbb{C}_1, \quad v_j = p^{1/2}(\delta_j^2) = q^{1/2}(\delta_j), \quad j = 1, 2, \dots, \rho'. \end{gathered} \tag{7.23}$$

The integers $m_j, n_j$ $(j = 1, 2, \dots, \rho')$ and the points

$$r_j^* = (\sigma_j^2, u_j) \in \mathcal{R}', \quad u_j = u(\sigma_j^2) = w(\sigma_j) = w_j, \quad j = 1, 2, \dots, \rho', \tag{7.24}$$

are defined from the following Jacobi inversion problem on the Riemann surface $\mathcal{R}'$:

$$\sum_{j=1}^{\rho'}[\omega_\nu^*(\sigma_j^2, u(\sigma_j^2)) + m_j A_{\nu j}^* + n_j B_{\nu j}^*] = d_\nu^*, \quad \nu = 1, 2, \dots, \rho', \tag{7.25}$$

where

$$\begin{aligned}
\omega_\nu^* = \omega_\nu^*(\zeta, u) &= \int_{(\zeta_0,0)}^{(\zeta,u)} \frac{\tau^{\nu-1}\mathrm{d}\tau}{\eta(\tau)} = 2\int_{(z_0,0)}^{(z,w)} \frac{t^{2\nu-1}\mathrm{d}t}{\xi(t)} = 2\omega_{2\nu}(z, w),\\
A_{\nu j}^* = \oint_{\mathbf{a}_j^*} \mathrm{d}\omega_\nu^* &= 2A_{2\nu\, j}, \quad B_{\nu j}^* = \oint_{\mathbf{b}_j^*} \mathrm{d}\omega_\nu^* = 2B_{2\nu\, j}, \quad j = 1, 2, \dots, \rho',\\
d_\nu^* = 2\sum_{j=1}^{\rho'} \omega_{2\nu}(\delta_j, v_j) &- \frac{1}{\pi \mathrm{i}}\int_0^1 [\log l_1(t) - \log l_2(t)]\frac{t^{2\nu-1}\mathrm{d}t}{q^{1/2}(t)}\\
&+2\sum_{\mu=1}^{2}(-1)^\mu \operatorname{sgn}\kappa_\mu \sum_{j=1}^{|\kappa_\mu|}\int_1^{\gamma_{\mu j}} \frac{t^{2\nu-1}\mathrm{d}t}{q^{1/2}(t)}, \quad \nu = 1, 2, \dots, \rho'.
\end{aligned} \tag{7.26}$$

We emphasize that the new Jacobi problem (7.25) related to the symmetric problem (3.15), (7.1) consists of $\rho' = [\rho/2]$ non-linear algebraic equations, and can be reduced to an algebraic equation of degree $\rho'$ (Antipov & Silvestrov, 2002). Recall that in the general non-symmetric case, there are $\rho$ equations.

Next, by replacing in (7.18), (7.19), (7.21) $\zeta$ and $\tau$ for $z^2$ and $t^2$ respectively, we obtain the even canonical function in the form (4.26), where

$$\begin{aligned}
\chi_1(z) &= \frac{1}{2\pi \mathrm{i}}\int_0^1 [\log l_1(t)+\log l_2(t)]\frac{t\,\mathrm{d}t}{t^2 - z^2} + \sum_{\mu=1}^{2}\operatorname{sgn}\kappa_\mu \sum_{j=1}^{|\kappa_\mu|}\int_1^{\gamma_{\mu j}} \frac{t\,\mathrm{d}t}{t^2 - z^2} + \sum_{j=1}^{\rho'}\int_{\delta_j}^{\sigma_j} \frac{t\,\mathrm{d}t}{t^2 - z^2},\\
\chi_2(z) &= \frac{1}{2\pi \mathrm{i}}\int_0^1 [\log l_1(t) - \log l_2(t)]\frac{t\,\mathrm{d}t}{q^{1/2}(t)(t^2 - z^2)}\\
&\quad - \sum_{\mu=1}^{2}(-1)^\mu \operatorname{sgn}\kappa_\mu \sum_{j=1}^{|\kappa_\mu|}\int_1^{\gamma_{\mu j}} \frac{t\,\mathrm{d}t}{q^{1/2}(t)(t^2 - z^2)}\\
&\quad + \sum_{j=1}^{\rho'}\left(\int_{(\delta_j,v_j)}^{(\sigma_j,w_j)} + m_j\oint_{\mathbf{a}_j} + n_j\oint_{\mathbf{b}_j}\right)\frac{t\,\mathrm{d}t}{\xi(t)(t^2 - z^2)}.
\end{aligned} \tag{7.27}$$

### 7.3 *General even solution*

By use of the function $\chi(z, w)$ we can find the general solution of the even problem (3.15), (7.1). Let us write it down in the case of simple poles (analysed in Section 6.2):

$$F(z, w) = \chi(z, w)\{\psi_1(z) + R_1(z) + w(z)[\psi_2(z) + R_2(z)]\}, \tag{7.28}$$

where

$$\psi_1(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}^*} \frac{g^*(t,\xi)}{X^+(t,\xi)} \frac{t\,\mathrm{d}t}{t^2 - z^2},$$

$$\psi_2(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}^*} \frac{g^*(t,\xi)}{\xi(t) X^+(t,\xi)} \frac{t\,\mathrm{d}t}{t^2 - z^2},$$

$$R_1(z) = C_0 + \sum_{j=1}^{\rho'} \frac{C_j w_j}{z^2 - \sigma_j^2} + \sum_{j=1}^{m'} \frac{D'_j}{z^2 - \alpha_j^2} + \sum_{j=1}^{\kappa'} \frac{H'_j v_{1j}}{z^2 - \gamma_{1j}^2} - \sum_{j=1}^{\kappa''} \frac{H''_j v_{2j}}{z^2 - \gamma_{2j}^2},$$

$$R_2(z) = \sum_{j=1}^{\rho'} \frac{C_j}{z^2 - \sigma_j^2} + \sum_{j=1}^{m'} \frac{D''_j}{z^2 - \alpha_j^2} + \sum_{j=0}^{\rho} \frac{E_j}{z^2 - z_j^2} + \sum_{j=1}^{\kappa'} \frac{H'_j}{z^2 - \gamma_{1j}^2} + \sum_{j=1}^{\kappa''} \frac{H''_j}{z^2 - \gamma_{2j}^2},$$

$$\kappa' = \max\{\kappa_1, 0\}, \quad \kappa'' = \max\{\kappa_2, 0\}, \quad w_j = w(\sigma_j), \quad v_{\mu j} = q^{1/2}(\gamma_{\mu j}). \tag{7.29}$$

Here we used formulae (5.4), (7.7) and also

$$X^+(-t,\xi) = X^-(t,\xi) = X^+(t,\xi)/l(t,\xi), \quad (t,\xi) \in \mathcal{L}. \tag{7.30}$$

The solution (7.28) possesses $\rho + \rho' + 2m' + \kappa' + \kappa'' + 2$ arbitrary constants, and it has to meet the $2\rho'$ conditions

$$\lim_{z\to\infty} z^{2j} [\psi_2(z) + R_2(z)] = 0, \quad j = 1, 2, \ldots, \rho', \tag{7.31}$$

$$\psi_1(\delta_j) + R_1(\delta_j) + v_j [\psi_2(\delta_j) + R_2(\delta_j)] = 0, \quad j = 1, 2, \ldots, \rho', \tag{7.32}$$

which follow from (5.8), (5.9). As in the general case, it should also satisfy the relations

$$\psi_1(\gamma_{1j}) + R_1(\gamma_{1j}) + v_{1j}\ [\psi_2(\gamma_{1j}) + R_2(\gamma_{1j})] = 0, \quad j = 1, 2, \ldots, -\kappa_1 \quad (\text{if } \kappa_1 < 0) \tag{7.33}$$

$$\psi_1(\gamma_{2j}) + R_1(\gamma_{2j}) - v_{2j}\ [\psi_2(\gamma_{2j}) + R_2(\gamma_{2j})] = 0, \quad j = 1, 2, \ldots, -\kappa_2 \quad (\text{if } \kappa_2 < 0) \tag{7.34}$$

and the conditions (6.7), (6.11). Therefore, in total, we get $2\rho' + m - t + n^\circ + \hat{\kappa}_1 + \hat{\kappa}_2$ relations for arbitrary constants. Here $\hat{\kappa}_\mu = \max\{0, -\kappa_\mu\}$ ($\mu = 1, 2$), and $n^\circ$ is the number of equations (6.11).

## 7.4 *Odd solution*

Finally, we notice that the even canonical function can be used for finding the general solution of the problem (3.15) subject to the condition $F_\mu(z) = -F_\mu(-z)$ ($\mu = 1, 2$). We write down the solution in case such a problem might arise in other applications:

$$F(z, w) = z\chi(z, w) \{\psi_3(z) + R_1(z) + w(z)[\psi_4(z) + R_2(z)]\}, \tag{7.35}$$

where

$$\psi_3(z) = \frac{1}{2\pi i} \int_{\mathcal{L}^*} \frac{g^*(t,\xi)}{X^+(t,\xi)} \frac{dt}{t^2 - z^2},$$

$$\psi_4(z) = \frac{1}{2\pi i} \int_{\mathcal{L}^*} \frac{g^*(t,\xi)}{\xi(t) X^+(t,\xi)} \frac{dt}{t^2 - z^2}. \tag{7.36}$$

It should be pointed out that the odd solution of the problem (3.15) exists only under the conditions

$$l_\mu(t) l_\mu(-t) = 1, \quad g^*_\mu(t) - l_\mu(t) g^*_\mu(-t) = 0, \quad t \in (-1, 1), \quad \mu = 1, 2. \tag{7.37}$$

## 8. Diffraction by an anisotropic impedance half-plane

### 8.1 *Physical problem*

To illustrate the technique of the paper, we consider scattering of an electromagnetic wave at skew incidence by an anisotropic half-plane with different impedances. Let the primary source be a plane wave incident obliquely whose $z$-components are

$$E_z^i = e_z e^{ik\rho \sin\beta \cos(\theta - \theta_0) - ikz \cos\beta},$$

$$Z_0 H_z^i = h_z e^{ik\rho \sin\beta \cos(\theta - \theta_0) - ikz \cos\beta}, \tag{8.1}$$

where $(\rho, \theta, z)$ are cylindrical coordinates, $k$ is the wave number ($\mathrm{Im}(k) \leqslant 0$), $Z_0$ is the intrinsic impedance of free space, $\beta$ is the angle of incident ($0 < \beta < \pi/2$), and $e_z, h_z$ are prescribed parameters. In the most general case in which the impedance is anisotropic and differs on the upper and lower sides of the half-planes $\{0 < \rho < \infty, \theta = \pm\pi \mp 0, |z| < \infty\}$, the boundary conditions are (Senior, 1978)

$$E_\rho = \mp \eta_2^\pm Z_0 H_z, \quad \theta = \pm\pi \mp 0,$$

$$E_z = \pm \eta_1^\pm Z_0 H_\rho, \quad \theta = \pm\pi \mp 0, \tag{8.2}$$

where $\eta_1^\pm, \eta_2^\pm$ are the surface impedances of the upper ($\theta = \pi - 0$) and lower ($\theta = -\pi + 0$) half-planes, respectively. The surface impedances are assumed to be real. The $\rho$-components $E_\rho$ and $H_\rho$ are expressed in terms of $E_z$ and $H_z$ as follows:

$$E_\rho = \frac{1}{ik \sin^2\beta} \left[ \cos\beta \frac{\partial E_z}{\partial \rho} + \frac{1}{\rho} \frac{\partial (Z_0 H_z)}{\partial \theta} \right],$$

$$Z_0 H_\rho = \frac{1}{ik \sin^2\beta} \left[ \cos\beta \frac{\partial (Z_0 H_z)}{\partial \rho} - \frac{1}{\rho} \frac{\partial E_z}{\partial \theta} \right]. \tag{8.3}$$

Therefore, equivalently, the boundary conditions (8.2) can be written as

$$\frac{1}{\rho}\frac{\partial E_z}{\partial \theta} - \cos\beta \frac{\partial (Z_0 H_z)}{\partial \rho} \pm \frac{\mathrm{i}k \sin^2\beta E_z}{\eta_1^{\pm}} = 0, \quad \theta = \pm\pi \mp 0,$$

$$\frac{1}{\rho}\frac{\partial (Z_0 H_z)}{\partial \theta} + \cos\beta \frac{\partial E_z}{\partial \rho} \pm \mathrm{i}k\eta_2^{\pm} \sin^2\beta Z_0 H_z = 0, \quad \theta = \pm\pi \mp 0. \tag{8.4}$$

Represent the total field in the form of the Sommerfeld integral (Maliuzhinets, 1958)

$$E_z(\rho,\theta,z) = \frac{\mathrm{e}^{-\mathrm{i}kz\cos\beta}}{2\pi\mathrm{i}} \int_\gamma \mathrm{e}^{\mathrm{i}k\rho\sin\beta\cos\alpha} s_e(\alpha+\theta)\,\mathrm{d}\alpha,$$

$$Z_0 H_z(\rho,\theta,z) = \frac{\mathrm{e}^{-\mathrm{i}kz\cos\beta}}{2\pi\mathrm{i}} \int_\gamma \mathrm{e}^{\mathrm{i}k\rho\sin\beta\cos\alpha} s_h(\alpha+\theta)\,\mathrm{d}\alpha, \tag{8.5}$$

where $\gamma$ is the Sommerfeld contour, the functions $s_e(\alpha)$ and $s_h(\alpha)$ are analytic everywhere in the strip $-\pi < \mathrm{Re}(\alpha) < \pi$ apart from the point $\alpha = \theta_0$, where they have a simple pole with the residues defined by the incident field (8.1). At the infinite points $\alpha = x \pm \mathrm{i}\infty$ ($|x| < \infty$), the functions $s_e(\alpha)$ and $s_h(\alpha)$ are bounded. The boundary conditions (8.4) are satisfied if and only if (Maliuzhinets, 1958)

$$\left(\sin\alpha \pm \frac{1}{\eta_1^{\pm}}\sin\beta\right) s_e(\alpha\pm\pi) - \cos\alpha\cos\beta s_h(\alpha\pm\pi)$$

$$= \left(-\sin\alpha \pm \frac{1}{\eta_1^{\pm}}\sin\beta\right) s_e(-\alpha\pm\pi) - \cos\alpha\cos\beta s_h(-\alpha\pm\pi),$$

$$\left(\sin\alpha \pm \eta_2^{\pm}\sin\beta\right) s_h(\alpha\pm\pi) + \cos\alpha\cos\beta s_e(\alpha\pm\pi)$$

$$= \left(-\sin\alpha \pm \eta_2^{\pm}\sin\beta\right) s_h(-\alpha\pm\pi) + \cos\alpha\cos\beta s_e(-\alpha\pm\pi). \tag{8.6}$$

Next, following Senior & Legault (1998) introduce the two functions

$$\begin{aligned} \Phi_1(\alpha+\pi) &= \left(\sin\alpha + \frac{1}{\eta_1^{+}}\sin\beta\right) s_e(\alpha+\pi) - \cos\alpha\cos\beta s_h(\alpha+\pi), \\ \Phi_2(\alpha+\pi) &= \left(\sin\alpha + \eta_2^{+}\sin\beta\right) s_h(\alpha+\pi) + \cos\alpha\cos\beta s_e(\alpha+\pi). \end{aligned} \tag{8.7}$$

Inverting these relations gives

$$s_e(\alpha+\pi) = \frac{1}{\Gamma_\alpha(1/\eta_1^{+},\eta_2^{+})}[\left(\sin\alpha + \eta_2^{+}\sin\beta\right)\Phi_1(\alpha+\pi) + \cos\alpha\cos\beta\,\Phi_2(\alpha+\pi)],$$

$$s_h(\alpha+\pi) = \frac{1}{\Gamma_\alpha(1/\eta_1^{+},\eta_2^{+})}\left[\left(\sin\alpha + \frac{1}{\eta_1^{+}}\sin\beta\right)\Phi_2(\alpha+\pi) - \cos\alpha\cos\beta\,\Phi_1(\alpha+\pi)\right], \tag{8.8}$$

where

$$\Gamma_\alpha(a,b) = (\sin\alpha + a\sin\beta)(\sin\alpha + b\sin\beta) + \cos^2\alpha\cos^2\beta. \tag{8.9}$$

Because of the identities

$$\Phi_j(\alpha+\pi) = \Phi_j(-\alpha+\pi), \quad \Phi_j(-\alpha-\pi) = \Phi_j(\alpha+3\pi), \quad j=1,2, \tag{8.10}$$

the system of equations for the functions $s_e, s_h$ can be reduced to the system for the new functions $\Phi_1$, $\Phi_2$:

$$\begin{aligned}
&\frac{1}{\Gamma_\alpha(-1/\eta_1^+, -\eta_2^+)}\left[\Gamma_\alpha\left(\frac{1}{\eta_1^-}, -\eta_2^+\right)\Phi_1(\alpha+3\pi) - \frac{1}{\eta_1}\cos\alpha\sin 2\beta\,\Phi_2(\alpha+3\pi)\right]\\
&= \frac{1}{\Gamma_\alpha(1/\eta_1^+, \eta_2^+)}\left[\Gamma_\alpha\left(-\frac{1}{\eta_1^-}, \eta_2^+\right)\Phi_1(\alpha-\pi) - \frac{1}{\eta_1}\cos\alpha\sin 2\beta\,\Phi_2(\alpha-\pi)\right],\\
&\frac{1}{\Gamma_\alpha(-1/\eta_1^+, -\eta_2^+)}\left[\Gamma_\alpha\left(-\frac{1}{\eta_1^+}, \eta_2^-\right)\Phi_2(\alpha+3\pi) + \eta_2\cos\alpha\sin 2\beta\,\Phi_1(\alpha+3\pi)\right]\\
&= \frac{1}{\Gamma_\alpha(1/\eta_1^+, \eta_2^+)}\left[\Gamma_\alpha\left(\frac{1}{\eta_1^+}, -\eta_2^-\right)\Phi_2(\alpha-\pi) + \eta_2\cos\alpha\sin 2\beta\,\Phi_1(\alpha-\pi)\right],
\end{aligned} \tag{8.11}$$

where

$$\frac{1}{\eta_1} = \frac{1}{2}\left(\frac{1}{\eta_1^+} + \frac{1}{\eta_1^-}\right), \quad \eta_2 = \frac{\eta_2^+ + \eta_2^-}{2}. \tag{8.12}$$

If now express $\Phi_1(\alpha+3\pi)$, $\Phi_2(\alpha+3\pi)$ in terms of the values $\Phi_1(\alpha-\pi)$, $\Phi_2(\alpha-\pi)$ and put $\sigma = 3\pi+\alpha$, then, on $\Omega = \{\mathrm{Re}(\sigma) = 3\pi\}$,

$$\boldsymbol{\Phi}(\sigma) = \mathbf{G}(\sigma)\,\boldsymbol{\Phi}(\sigma-4\pi), \quad \sigma\in\Omega, \tag{8.13}$$

where

$$\boldsymbol{\Phi}(\sigma) = \begin{pmatrix}\Phi_1(\sigma)\\ \Phi_2(\sigma)\end{pmatrix}, \qquad \mathbf{G}(\sigma) = \begin{pmatrix}G_{11}(\sigma) & G_{12}(\sigma)\\ G_{21}(\sigma) & G_{22}(\sigma)\end{pmatrix}, \tag{8.14}$$

with

$$G_{11}(\sigma) = \frac{\Gamma_\sigma(1/\eta_1^-, -\eta_2^+)\Gamma_\sigma(1/\eta_1^+, -\eta_2^-) + \eta_2\eta_1^{-1}\cos^2\sigma\sin^2 2\beta}{D(\sigma)},$$

$$G_{22}(\sigma) = \frac{\Gamma_\sigma(-1/\eta_1^-, \eta_2^+)\Gamma_\sigma(-1/\eta_1^+, \eta_2^-) + \eta_2\eta_1^{-1}\cos^2\sigma\sin^2 2\beta}{D(\sigma)},$$

$$G_{12}(\sigma) = -\frac{\eta_0^-\sin\beta\sin 2\beta\sin 2\sigma}{\eta_1 D(\sigma)}, \quad G_{21}(\sigma) = \frac{\eta_0^+}{\eta_0^-}\eta_1\eta_2 G_{12}(\sigma), \tag{8.15}$$

$$D(\sigma) = \frac{\Gamma_\sigma(-1/\eta_1^+, -\eta_2^+)}{\Gamma_\sigma(1/\eta_1^+, \eta_2^+)}[\Gamma_\sigma(-1/\eta_1^-, \eta_2^+)\Gamma_\sigma(1/\eta_1^+, -\eta_2^-) + \eta_2\eta_1^{-1}\cos^2\sigma\sin^2 2\beta],$$

$$\eta_0^+ = \eta_2^+ - \frac{1}{\eta_1^-}, \quad \eta_0^- = \eta_2^- - \frac{1}{\eta_1^+}.$$

### 8.2 *Arbitrary impedances: a surface of genus $\rho' = 3$*

Equation (8.13) is a vector functional-difference equation of the first order with the shift $h = 4\pi$ subject to the additional condition of symmetry

$$\boldsymbol{\Phi}(\sigma) = \boldsymbol{\Phi}(2\pi - \sigma), \quad \sigma \in \Pi = \{-\pi < \mathrm{Re}(s) < 3\pi\}. \tag{8.16}$$

In this section we show how to reduce the problem (8.13), (8.16) to a particular case of the even Riemann–Hilbert problem (3.15), (7.1) analysed in Section 7, and also how to solve it.

#### 8.2.1 *Analysis of a Riemann–Hilbert problem on a surface.*

It is seen that the matrix (8.14) has the structure (2.10) required for the method to be applied. Indeed, in the notation of Section 2,

$$a_1(\sigma) = \tfrac{1}{2}[G_{11}(\sigma) + G_{22}(\sigma)], \quad a_2(\sigma) = G_{12}(\sigma),$$

$$f_1(s) = \frac{\eta_1(\eta_0^+ + \eta_0^-)}{2\eta_0^-\sin 2\beta\cos s}(\cos^2 s\cos^2\beta + \sin^2 s - e_0\sin^2\beta), \tag{8.17}$$

$$f_2(s) = \frac{\eta_0^+}{\eta_0^-}\eta_1\eta_2,$$

where

$$e_0 = \frac{1}{\eta_0^+ + \eta_0^-}\left(\eta_0^+\frac{\eta_2^-}{\eta_1^+} + \eta_0^-\frac{\eta_2^+}{\eta_1^-}\right). \tag{8.18}$$

Clearly, the functions (8.17) meet the conditions for $a_1(\sigma)$, $a_2(\sigma)$, $f_1(s)$ and $f_2(s)$ imposed in Section 2. The key function of the method is

$$f(s) = f_1^2(s) + f_2(s) = \left[\frac{\eta_1(\eta_0^+ + \eta_0^-)\tan\beta}{4\eta_0^- \cos s}\right]^2 f^*(s), \tag{8.19}$$

where

$$f^*(s) = \left(\cos^2 s - \frac{1 - e_0\sin^2\beta}{\sin^2\beta}\right)^2 + 16e_1\cos^2 s\cot^2\beta,$$
$$e_1 = \frac{\eta_2\eta_0^+\eta_0^-}{\eta_1(\eta_0^+ + \eta_0^-)^2}. \tag{8.20}$$

In the strip $\Pi = \{-\pi < \mathrm{Re}(s) < 3\pi\}$, the function $f(s)$ has four poles of the second order: $-\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{3}{2}\pi$ and $\frac{5}{2}\pi$. Define the branch points of the function $f^{1/2}(s)$. From (8.19), (8.20), they are the roots of the equations

$$\cos 2s = A_\nu, \quad s \in \Pi \quad (\nu = 1, 2), \tag{8.21}$$

where

$$A_\nu = -1 + \frac{2}{\sin^2\beta}\left[1 - e_0\sin^2\beta - 8e_1\cos^2\beta + 4i(-1)^\nu\cos\beta\sqrt{e_1(1 - e_0\sin^2\beta - 4e_1\cos^2\beta)}\right]. \tag{8.22}$$

In the above formula, $\sqrt{\ldots}$ is one of the branches of the square root. From the whole set of the roots

$$\pi j + \frac{\mathrm{i}}{2}\log\left(A_\nu \pm \sqrt{A_\nu^2 - 1}\right) \quad (\nu = 1, 2; \quad j = 0, \pm 1, \pm 2, \ldots), \tag{8.23}$$

one needs to choose those roots which lie in the strip $\Pi$. We note that the expression

$$d_0 = e_1(1 - e_0\sin^2\beta - 4e_1\cos^2\beta) \tag{8.24}$$

can be positive, negative and, also, equal to zero. If $d_0 = 0$, then, obviously, the function $f^{1/2}(s)$ does not have branch points at all. The roots of (8.21) become zeros of the function $f^{1/2}(s)$. This case is reported in Section 8.3. Henceforward we assume that $d_0 \neq 0$ and, therefore, in the strip $\Pi$, the function $f^{1/2}(s)$ has 16 branch points: $s_0, s_1, \ldots, s_{15}$.

For example, for $\beta = \frac{1}{4}\pi$, $\eta_2^+/\eta_1^+ = 2$, $\eta_1^-/\eta_1^+ = 3$ and $\eta_2^-/\eta_1^+ = 4$, the branch points are

$$s_0 = -1{\cdot}57080 - \mathrm{i}1{\cdot}70392, \quad s_1 = \overline{s_0},$$
$$s_2 = -1{\cdot}57080 - \mathrm{i}0{\cdot}05375, \quad s_3 = \overline{s_2}, \tag{8.25}$$
$$s_j = s_{j-4} + \pi \quad (j = 4, 5, \ldots, 15).$$

Analysis of formulae (8.17), (8.19) reveals that the functions $f_1(s)\,f^{-1/2}(s)$ and $f^{-1/2}(s)$ are free of poles in the strip $\Pi$ (all their singular points are the branch points $s_0, s_1, \dots s_{15}$). Since the functions $s_e(\alpha), s_h(\alpha)$ have the prescribed pole at the point $\alpha = \theta_0$ and because of the relation (8.16) the functions $\Phi_1(s)$, $\Phi_2(s)$ have simple poles at the points $s = \theta_0$, $s = 2\pi - \theta_0$. Therefore, the functions $\phi_1(s), \phi_2(s)$, defined by (3.3), have simple poles at the same points: $a_1 = \theta_0, a_2 = 2\pi - \theta_0$.

As for the behaviour at the ends of the strip, because of the presence of the functions $\sin\alpha$ and $\cos\alpha$ in (8.7), the functions $\Phi_1(s)$, $\Phi_2(s)$, grow exponentially as $s \to x \pm \mathrm{i}\infty$ $(-\pi \leqslant x \leqslant 3\pi)$: $\Phi_j(s) = O(\mathrm{e}^{|s|})$. The functions $f_1(s)\,f^{-1/2}(s) \pm 1$ and $f^{-1/2}(s)$ make different the principal term in the expansions of the functions $\phi_1(s)$ and $\phi_2(s)$ as $s \to \infty$, $s \in \Pi$. To show this, choose a branch of the function $f^{1/2}(s)$ such that

$$f^{1/2}(s) \sim -\frac{\eta_1}{2\sin 2\beta}\left|\frac{\eta_0^+}{\eta_0^-} + 1\right| \sin^2\beta \cos s, \quad s \to x \pm \mathrm{i}\infty, \quad -\pi \leqslant x \leqslant 3\pi. \quad (8.26)$$

Then

$$\frac{f_1(s)}{f^{1/2}(s)} = \operatorname{sgn}\left(\frac{\eta_0^+}{\eta_0^-} + 1\right) + O(\mathrm{e}^{-2|s|}), \quad s \to x \pm i\infty. \quad (8.27)$$

Formulae (3.3) and (8.26) indicate that one of the functions $\phi_1(s), \phi_2(s)$ grows at the ends of the strip, and the other is bounded:

$$\frac{\eta_0^+}{\eta_0^-} + 1 > 0: \quad \phi_1(s) = O(\mathrm{e}^{|s|}), \quad \phi_2(s) = O(1), \quad s \to x \pm \mathrm{i}\infty, \quad (8.28)$$

$$\frac{\eta_0^+}{\eta_0^-} + 1 < 0: \quad \phi_1(s) = O(1), \quad \phi_2(s) = O(\mathrm{e}^{|s|}), \quad s \to x \pm \mathrm{i}\infty. \quad (8.29)$$

From the relations (8.8), it is clear that the functions $s_e(\alpha), s_h(\alpha)$ have inadmissible poles at the zeros of the function $\Gamma_\alpha(-1/\eta_1^+, -\eta_2^+)$ which lie in the strip $-\pi < \mathrm{Re}(\alpha) < \pi$. Let these zeros be $\varepsilon_j$, $j = 1, 2, 3, 4$ ($\mathrm{Re}(\varepsilon_j) \in (-\pi, \pi)$). The points $\varepsilon_j$ become removable points of the functions $s_e(\alpha), s_h(\alpha)$, if the following conditions hold:

$$(-\sin s + \eta_2^+ \sin\beta)\,\Phi_1(s) - \cos s \cos\beta\,\Phi_2(s) = 0, \quad s = \varepsilon_j,$$

$$\left(-\sin s + \frac{1}{\eta_1^+}\sin\beta\right)\Phi_2(s) + \cos s\cos\beta\,\Phi_1(s) = 0, \quad s = \varepsilon_j, \quad j = 1, 2, 3, 4. \quad (8.30)$$

Since the determinant of this system $\Gamma_{\varepsilon_j}(-1/\eta_1^+, -1/\eta_2^+)$ is equal to 0, the above conditions are equivalent to the following four equations:

$$(-\sin\varepsilon_j + \eta_2^+ \sin\beta)\,\Phi_1(\varepsilon_j) - \cos\varepsilon_j \cos\beta\,\Phi_2(\varepsilon_j) = 0, \quad j = 1, 2, 3, 4. \quad (8.31)$$

Following the procedure of Section 3 we reduce the vector functional-difference equation (8.13) to the scalar Riemann–Hilbert problem (3.15) ($g^*(t,\xi)\equiv 0$) on the two-sheeted Riemann surface $\mathcal{R}$ of genus $\rho=7$ (the number of the branch points is 16). In the example (8.25) the branch points become

$$\begin{gathered} z_0=-0{\cdot}45822-0{\cdot}33792\mathrm{i},\quad z_2=-0{\cdot}01574-0{\cdot}41413\mathrm{i},\\ z_4=-1{\cdot}41358-1{\cdot}04246\mathrm{i},\quad z_6=-0{\cdot}09165-2{\cdot}41124\mathrm{i},\\ z_j=-\overline{z_{j-1}}\ \ (j=1,3,5,7),\quad z_j=\overline{z_{j-4}}\ \ (j=8,9,10,11),\\ z_j=\overline{z_{j-12}}\ \ (j=12,13,14,15). \end{gathered} \tag{8.32}$$

It turns out that in all possible cases the branch points $z_j$ are symmetric with respect to the origin. Since the function $z=-\mathrm{i}\tan\frac{s-3\pi}{4}$ maps the points $s$ and $2\pi-s$ into the points $z$ and $-z$, respectively, and because the functions $\Phi_1(s)$, $\Phi_2(s)$ and $f_1(s)$, $f^{1/2}(s)$ meet the relation (8.16), the functions $F_\mu(z)$, $\mu=1,2$ are even. It is also clear that they have simple poles at the points $\alpha_1=-\mathrm{i}\cot\frac{\pi-\theta_0}{4}$ and $\alpha_2=-\alpha_1$.

Define the behaviour of the functions $F_\mu(z)$ at the ends $z=\pm1$. Let, first, $\eta_0^+/\eta_0^-+1>0$. Because of formulae (8.28), the numbers $\nu_\mu^\pm$ in inequalities (4.1) become $\nu_1^\pm=2$, $\nu_2^\pm=0$. Indeed, for $F_1(z)$, for instance, we have

$$F_1(z)=\phi_1\left(3\pi+2\mathrm{i}\log\frac{1+z}{1-z}\right)\sim A_1^\circ e^{|s|}\sim A_1^\circ\exp\left\{2\left|\log\frac{1+z}{1-z}\right|\right\}\sim A_1^\circ|z\pm1|^{-2},$$

$$z\to\mp1\quad(s\to x\pm\mathrm{i}\infty,\ \ -\pi\leqslant x\leqslant 3\pi),\quad A_1^\circ=\text{const.} \tag{8.33}$$

For $\eta_0^+/\eta_0^-+1<0$, the same argument gives $\nu_1^\pm=0$, $\nu_2^\pm=2$.

Next, analysing formulae (8.15) as $\sigma\to3\pi\pm\mathrm{i}\infty$ and as $\sigma=3\pi$ we get

$$\begin{gathered} G_{jj}(\sigma)\sim1,\quad G_{jm}(\sigma)=O(e^{-2|\sigma|})\quad(j\neq m),\quad \sigma\to3\pi\pm\mathrm{i}\infty,\quad j,m=1,2,\\ G_{12}(3\pi)=0,\quad G_{11}(3\pi)=G_{22}(3\pi)=1, \end{gathered} \tag{8.34}$$

and therefore

$$\lambda_j(3\pi)=\lambda_j(3\pi\pm\mathrm{i}\infty)=1,\quad j=1,2. \tag{8.35}$$

8.2.2 *Even Riemann–Hilbert problem.* We have already shown that the poles $\alpha_1$, $\alpha_2$ and the branch points $z_j$ ($j=0,1,\ldots,15$) are simple and symmetric with respect to the origin. The end-points $z=\pm1$ are not branch points of the surface. In order that the functions $F_\mu(z)$ are even, it is necessary for the functions $l_\mu(t)$ to satisfy the condition (7.7), i.e. $l_\mu(t)l_\mu(-t)=1$, $t\in(-1,1)$. To check this relation, notice that for $\sigma=3\pi+\mathrm{i}\xi$ ($-\infty<\xi<\infty$)

$$\cos\bar\sigma=\cos\sigma,\quad \sin\bar\sigma=-\sin\sigma,\quad \Gamma_{\bar\sigma}(a,b)=\Gamma_\sigma(-a,-b). \tag{8.36}$$

Then from (8.15), (8.17), (8.19) and (2.8), (2.11)

$$G_{11}(\bar\sigma) = G_{22}(\sigma)\frac{D(\sigma)}{D(\bar\sigma)}, \quad G_{22}(\bar\sigma) = G_{11}(\sigma)\frac{D(\sigma)}{D(\bar\sigma)}, \quad G_{12}(\bar\sigma) = -G_{12}(\sigma)\frac{D(\sigma)}{D(\bar\sigma)},$$
$$a_1(\bar\sigma) = a_1(\sigma)\frac{D(\sigma)}{D(\bar\sigma)}, \quad a_2(\bar\sigma) = -a_2(\sigma)\frac{D(\sigma)}{D(\bar\sigma)}, \quad f^{1/2}(\bar\sigma) = f^{1/2}(\sigma), \quad \sigma \in \Omega. \tag{8.37}$$

So, for the characteristic functions $\lambda_1(\sigma)$, $\lambda_2(\sigma)$ we obtain

$$\lambda_\mu(\bar\sigma) = \left[a_1(\sigma) + (-1)^j a_2(\sigma) f^{1/2}(\sigma)\right]\frac{D(\sigma)}{D(\bar\sigma)}, \quad \mu = 1, 2. \tag{8.38}$$

Then

$$\lambda_\mu(\sigma)\lambda_\mu(\bar\sigma) = \frac{[G_{11}(\sigma)G_{22}(\sigma) - G_{12}(\sigma)G_{21}(\sigma)]D(\sigma)}{D(\bar\sigma)}, \quad \mu = 1, 2. \tag{8.39}$$

It is directly verified that

$$[(G_{11}(\sigma)G_{22}(\sigma) - G_{12}(\sigma)G_{21}(\sigma)]D(\sigma) - D(\bar\sigma)$$
$$= \frac{\eta_2 \cos^2\sigma \sin^2 2\beta}{\eta_1 D(\sigma)}\Big\{[\Gamma_\sigma(1/\eta_1^-, -\eta_2^+) - \Gamma_\sigma(-1/\eta_1^-, \eta_2^+)][\Gamma_\sigma(1/\eta_1^+, -\eta_2^-)$$
$$-\Gamma_\sigma(-1/\eta_1^+, \eta_2^-)] - 4\eta_0^+\eta_0^- \sin^2\beta \sin^2\sigma\Big\} = 0. \tag{8.40}$$

For this reason,

$$\lambda_\mu(\sigma)\lambda_\mu(\bar\sigma) = 1, \quad \sigma \in \Omega, \tag{8.41}$$

and

$$l_\mu(t)l_\mu(-t) = 1, \quad t \in (-1, 1), \quad \mu = 1, 2. \tag{8.42}$$

As for the quantities $\arg l_\mu(t)$, we get

$$l_\mu(-1) = l_\mu(0) = l_\mu(1) = 1. \tag{8.43}$$

Choose $\arg l_\mu(0) = 0$. Then, by formula (8.42), $\arg l_\mu(-1) = -\arg l_\mu(1)$. Numerical results for different sets of the parameters of the problem show that as the point $t$ traverses from 0 to 1, the point $\{\mathrm{Re}\, l_\mu(t), \mathrm{Im}\, l_\mu(t)\}$ always passes once round the origin in the negative direction (see Figs 3 and 4 for $\beta = \pi/4$, $\eta_2^+/\eta_1^+ = 0{\cdot}001$, $\eta_1^-/\eta_1^+ = 10$, $\eta_2^-/\eta_1^+ = 2$). This means that the increments $\Delta_\mu^*$ of the arguments of the functions $l_\mu(t)$, as the contours $L_\mu^* = [0, 1] \in \mathbb{C}_\mu$ are traversed by the point $t$ in the positive direction, are equal to $-2\pi$:

$$\frac{1}{2\pi}\Delta_\mu^* = -1. \tag{8.44}$$

We have verified all the conditions for the Riemann–Hilbert problem (3.15) to have an even solution. Thus, to construct it, we may follow the scheme of Section 7.

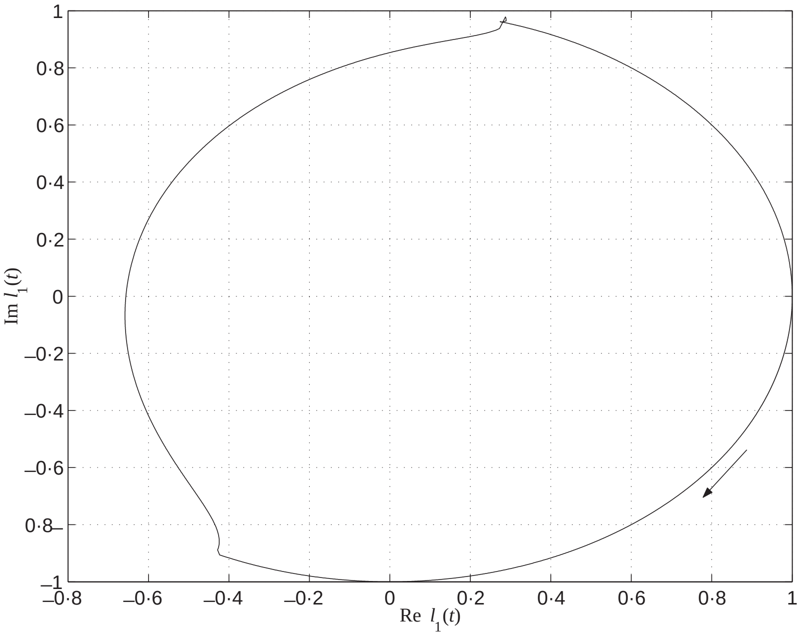


FIG. 3. The set $\{l_1(t) : 0 < t < 1\}$, for $\beta = \frac{1}{4}\pi$, $\eta_2^+/\eta_1^+ = 0{\cdot}001$, $\eta_1^-/\eta_1^+ = 10$, $\eta_2^-/\eta_1^+ = 2$.

8.2.3 *Closed-form solution.* We seek an even solution of the Riemann–Hilbert problem (3.15) in the class of functions

$$|F_\mu(z)| \leqslant A_\mu|z-1|^{-\nu_\mu}, \quad z \to 1, \quad \mu = 1, 2, \tag{8.45}$$

with

$$\nu_1 = \begin{cases} 2, & \eta_0^+/\eta_0^- > -1 \\ 0, & \eta_0^+/\eta_0^- < -1 \end{cases}, \quad \nu_2 = \begin{cases} 0, & \eta_0^+/\eta_0^- > -1 \\ 2, & \eta_0^+/\eta_0^- < -1 \end{cases}. \tag{8.46}$$

The integers $\kappa_1, \kappa_2$ are defined from (7.22)

$$\kappa_1 = \begin{cases} 1, & \eta_0^+/\eta_0^- > -1 \\ -1, & \eta_0^+/\eta_0^- < -1 \end{cases}, \quad \kappa_2 = \begin{cases} -1, & \eta_0^+/\eta_0^- > -1 \\ 1, & \eta_0^+/\eta_0^- < -1 \end{cases}. \tag{8.47}$$

The Riemann surface $\mathcal{R}'$ introduced in Section 7 becomes a surface of genus $\rho' = 3$. The even canonical function $\chi(z, w)$ has been constructed in Section 7.2, and it is defined by the relations (4.26), (7.27). The points $(\sigma_j^2, u_j) \in \mathcal{R}'$ and the integers $m_j, n_j$ $(j = 1, 2, 3)$ should be found by solving the Jacobi inversion problem (7.25).

We next specify formulae (7.27), (7.28) which describe the solution of the Riemann–

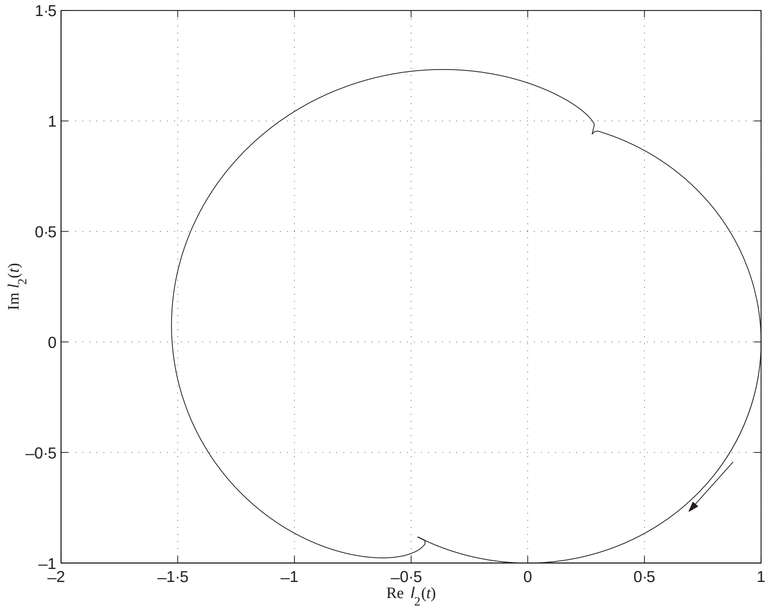


FIG. 4. The set $\{l_2(t) : 0 < t < 1\}$, for $\beta = \frac{1}{4}\pi$, $\eta_2^+/\eta_1^+ = 0{\cdot}001$, $\eta_1^-/\eta_1^+ = 10$, $\eta_2^-/\eta_1^+ = 2$.

Hilbert problem (3.15), (7.1) relevant to the physical problem under consideration:

$$F(z, w) = \chi(z, w)[R_1(z) + w(z)R_2(z)],$$

$$R_1(z) = C_0 + \sum_{j=1}^{3} \frac{C_j w_j}{z^2 - \sigma_j^2} + \frac{D_1}{z^2 - \alpha_1^2} + \frac{H_1 v_{11}\delta_{-\kappa_1,1}}{z^2 - \gamma_{11}^2} - \frac{H_2 v_{21}\delta_{-\kappa_2,1}}{z^2 - \gamma_{21}^2}, \tag{8.48}$$

$$R_2(z) = \sum_{j=1}^{3} \frac{C_j}{z^2 - \sigma_j^2} + \frac{D_2}{z^2 - \alpha_1^2} + \frac{H_1\delta_{-\kappa_1,1}}{z^2 - \gamma_{11}^2} + \frac{H_2\delta_{-\kappa_2,1}}{z^2 - \gamma_{21}^2} + \sum_{j=0}^{7} \frac{E_j}{z^2 - z_j^2},$$

with $C_j$ $(j = 0, 1, 2, 3)$, $D_1, D_2, H_1, H_2$ $E_j$ $(j = 0, 1, \ldots, 7)$, being arbitrary constants, $\delta_{m,n}$ being Kronecker's symbol, and

$$\alpha_1 = -\mathrm{i}\cot\frac{\pi - \theta_0}{4}, \quad z_j = -\mathrm{i}\tan\frac{s_j - 3\pi}{4} \quad (j = 0, 1, \ldots, 7). \tag{8.49}$$

Thus, the functions (8.48) possess 15 arbitrary constants. Define the number of additional conditions for them. Equations (7.31), (7.32) yield the first six conditions

$$\lim_{z\to\infty} z^{2j} R_2(z) = 0, \quad j = 1, 2, 3,$$

$$R_1(\delta_j) + v_j R_2(\delta_j) = 0, \quad j = 1, 2, 3. \tag{8.50}$$

From (7.33), (7.34) we get either

$$R_1(\gamma_{11}) + v_{11} R_2(\gamma_{11}) = 0 \quad \text{for} \quad \kappa_1 = -1, \quad \kappa_2 = 1, \tag{8.51}$$

or

$$R_1(\gamma_{21}) - v_{21} R_2(\gamma_{21}) = 0 \quad \text{for} \quad \kappa_1 = 1, \quad \kappa_2 = -1. \tag{8.52}$$

We also have the four equations (8.31) and the two regularity conditions (6.11) of the function $\Phi_2(s)$ at the points $-\frac{1}{2}\pi$, $\frac{1}{2}\pi$

$$\operatorname*{res}_{s=\pm\pi/2} \{[-f_1(s) + f^{1/2}(s)]\phi_1(s) - [f_1(s) + f^{1/2}(s)]\phi_2(s)\} = 0. \tag{8.53}$$

Note that then, because of the symmetry condition $\Phi_2(s) = \Phi_2(2\pi - s)$, the function $\Phi_2(s)$ will be regular at the points $s = \frac{3}{2}\pi$ and $s = \frac{5}{2}\pi$ automatically. These conditions follow from (3.2) and (2.13). Finally, to reproduce the incident field, the solution has to meet the two conditions

$$\operatorname*{res}_{\alpha=\theta_0} s_e(\alpha) = e_z, \qquad \operatorname*{res}_{\alpha=\theta_0} s_h(\alpha) = e_h. \tag{8.54}$$

The number of the constants is 15, and to fix them, we have the same number of linear equations.

The solution of the vector functional equation (8.13) is defined by (6.5). The constructed solution meets the symmetry condition (8.16). The closed-form solution of the scattering problem is given by formulae (8.5) and (8.8).

REMARK 1 If $\eta_\mu^+ = \eta_\mu^-$ $(\mu = 1, 2)$, then the initial vector functional-difference equation can be simplified to a new one with $h = 2\pi$ (see Senior, 1978, p.212). Following the above procedure reduces the problem to a Riemann–Hilbert problem on a surface of genus $\rho' = 1$ (a torus). The corresponding Jacobi inversion problem is solvable in terms of elliptic functions. This symmetric case was analysed by Hurd & Lüneberg (1985). They used the Wiener–Hopf formulation and the Daniele (1984) technique and found a closed-form solution in terms of elliptic functions.

REMARK 2 The above technique may be extended for the case of the complex impedances if the single branch of the function $f^{1/2}(s)$ is chosen such that

$$f^{1/2}(s) \sim -\frac{\eta_1}{2\sin 2\beta}\left(\frac{\eta_0^+}{\eta_0^-} + 1\right)\sin^2\beta\cos s, \quad s \to x \pm \mathrm{i}\infty, \quad -\pi \leqslant x \leqslant 3\pi. \tag{8.55}$$

We leave this interesting case and also physical and numerical analysis of the problem for future research.

### 8.3 *Case $\rho = -1$: no branch points*

By the convention of Section 3, if the function $f^{1/2}(s)$ does not have branch points in the strip $\Pi$, then $\rho = -1$. This is a very important case since the matrix of transformation

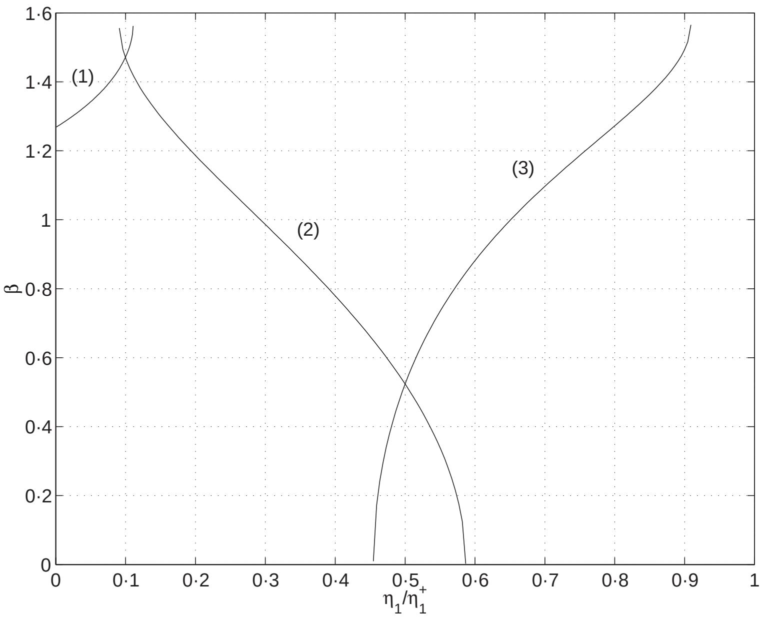


FIG. 5. Some cases when $\rho = -1$. The angle of incident $\beta$ versus $\eta_1^-/\eta_1^+$ for (1) $\eta_2^+/\eta_1^+ = 0{\cdot}01$, $\eta_2^-/\eta_1^+ = 10$; (2) $\eta_2^+/\eta_1^+ = 2$, $\eta_2^-/\eta_1^+ = 10$; (3) $\eta_2^+/\eta_1^+ = 2$, $\eta_2^-/\eta_1^+ = 0{\cdot}1$.

(2.13) becomes single-valued, and the solution of the Riemann–Hilbert problem on the Riemann surface $\mathcal{R}$ can be bypassed. In Fig. 5 we present those angles of incident $\beta$ for some values of the impedances when there are no branch points of $f^{1/2}(s)$ in the strip $\Pi$. Such cases can be used as a test for numerical computations for arbitrary values of the impedances. In this section we give a closed-form solution of the vector functional-difference equation (8.13) for $\rho = -1$. In addition, we show that for the isotropic case $\eta_1^\pm = \eta_2^\pm = \eta$, the integer $\rho$ is also equal to $-1$.

Instead of the Riemann–Hilbert problem on the surface $\mathcal{R}$, we get two separate problems on a plane:

$$\begin{aligned} F_1^+(t) &= l_1(t)F_1^-(t), \quad t \in (-1, 1), \\ F_2^+(t) &= l_2(t)F_2^-(t), \quad t \in (-1, 1). \end{aligned} \tag{8.56}$$

We are looking for a solution of the above problems subject to the conditions

$$F_\mu(z) = F_\mu(-z), \quad z \notin [-1, 1]. \tag{8.57}$$

Then the limit values of the functions $F_\mu(z)$ satisfy the relations $F_\mu^+(t) = F_\mu^-(-t), \quad t \in (-1, 1)$.

Let $\eta_0^+/\eta_0^- > -1$. In this case, as it was shown in Section 8.2,

$$F_1(z) = O(|z \pm 1|^{-2}), \quad F_2(z) = O(1), \quad z \to \mp 1. \tag{8.58}$$

We also get

$$\Delta_L^{(1)} = \Delta_L^{(2)} = -4\pi, \quad l_\mu(\pm 1) = 1. \tag{8.59}$$

Choose $\arg l_\mu(0) = 0$. Then, immediately, $\arg l_\mu(-1) = 2\pi$ and $\arg l_\mu(1) = -2\pi$. Factorize the functions $l_\mu(t)$:

$$l_\mu(t) = \frac{X_\mu^+(t)}{X_\mu^-(t)}, \quad t \in (-1, 1), \tag{8.60}$$

where

$$X_\mu(z) = (z^2 - 1)^{r_\mu} \exp\left\{ \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{\log l_\mu(t)}{t - z}\, \mathrm{d}t \right\} \tag{8.61}$$

with $r_\mu$ to be determined. Analysis of the Cauchy integral gives

$$\frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{\log l_\mu(t)}{t - z}\, \mathrm{d}t = -\log(z \pm 1) + \Omega_{\mu\mp}(z), \quad z \to \mp 1, \tag{8.62}$$

where the functions $\Omega_{\mu\pm}(z)$ are bounded as $z \to \pm 1$. Therefore,

$$X_\mu(z) \sim A_\mu^\pm (z \mp 1)^{r_\mu - 1}, \quad z \to \pm 1, \quad A_\mu^\pm = \text{const}. \tag{8.63}$$

The class of solutions (8.58) indicates that

$$r_1 = -1, \quad r_2 = 1. \tag{8.64}$$

The Sokhotski–Plemelj formulae and the identities (8.42) imply

$$X_\mu(z) = X_\mu(-z), \quad z \notin (-1, 1); \qquad X_\mu^+(t) = X_\mu^-(-t), \quad t \in (-1, 1). \tag{8.65}$$

The functions $F_1(z)$, $F_2(z)$ must be bounded at infinity (the point $z = \infty$ corresponds to the regular point $s = \pi$ of the functions $\Phi_1(s)$, $\Phi_2(s)$). They may have simple poles at the points $\pm z_j$ ($j = 0, 1, 2, 3$) and $\pm\alpha_1$, where

$$z_j = -\mathrm{i} \cot \frac{\pi - s_j^*}{4}, \quad \alpha_1 = -\mathrm{i} \cot \frac{\pi - \theta_0}{4}, \tag{8.66}$$

and $s_j^*$ ($j = 0, 1, 2, 3$) are the simple zeros in the strip $\{-\pi < \mathrm{Re}(s) < \pi\}$ of the function

$$f^{1/2}(s) = \frac{\eta_1(\eta_0^+ + \eta_0^-) \tan\beta}{4\eta_0^- \cos s} (\cos^2 s + 4e_1 \cot^2\beta). \tag{8.67}$$

The general solution of the Riemann–Hilbert problems (8.56), (8.57) becomes

$$\begin{aligned} F_1(z) &= X_1(z) \left[ d_1 + d_2 z^2 + \frac{D_1}{z^2 - \alpha_1^2} + \sum_{j=0}^{3} \frac{C_j}{(z^2 - z_j^2) X_1(z_j)} \right], \\ F_2(z) &= X_2(z) \left[ \frac{D_2}{z^2 - \alpha_1^2} - \sum_{j=0}^{3} \frac{C_j}{(z^2 - z_j^2) X_2(z_j)} \right], \end{aligned} \tag{8.68}$$

where $d_1, d_2, D_1, D_2, C_0, C_1, C_2, C_3$ are arbitrary constants. Here we used the relation

$$\operatorname*{res}_{z=\pm z_j} F_1(z) = -\operatorname*{res}_{z=\pm z_j} F_2(z), \quad j = 0, 1, 2, 3, \tag{8.69}$$

that follows from (3.3). Clearly, since $X_1(z) = O(z^{-2})$, $X_2(z) = O(z^2)$, $z \to \infty$, the functions $F_1(z)$, $F_2(z)$ are bounded at infinity.

To fix the eight constants in (8.68), we have the same number of equations (8.31), (8.53) and (8.54).

Finally, notice that in the isotropic case we get

$$\begin{gathered} \eta_1^\pm = \eta_2^\pm = \eta_1 = \eta_2 = \eta, \quad \eta_0^\pm = \eta - \frac{1}{\eta}, \\ e_0 = 1, \quad e_1 = \tfrac{1}{4}, \quad d_0 = 0, \end{gathered} \tag{8.70}$$

and the function $f^{1/2}(s)$ does not have branch points:

$$f^{1/2}(s) = \frac{\eta \tan\beta}{2\cos s}(\cos^2 s + \cot^2\beta). \tag{8.71}$$

Thus, this is a particular case of the above problem for $\rho = -1$.

## 9. Conclusion

In this paper we have analysed a class of vector functional-difference equations. It has been shown that a vector functional-difference equation of the first order, in a strip $\Pi$ of a complex plane subject to certain restrictions, is equivalent to a scalar Riemann–Hilbert boundary-value problem on a two-sheeted Riemann surface of genus $\rho$. The genus $\rho$ is defined through the number $N$ of the poles and zeros of odd order in the strip of a characteristic function of the matrix coefficient by the formula $\rho = (N-2)/2$ ($N$ is always even). In contrast with the Riemann–Hilbert problem on a union of two real axes of a hyper-elliptic surface considered by Antipov & Silvestrov (2002), in the present case, the corresponding Riemann–Hilbert problem is formulated on a union of two finite segments. We have constructed a closed-form solution of that new problem of the theory of analytic functions. The conditions quenching the pole of order $\rho$ at infinity of the Weierstrass kernel give rise to the classical Jacobi inversion problem.

Motivated by applications to diffraction theory, in addition to the general case, we have studied a special symmetric case of the vector functional-difference equation. It has been revealed that in this case the Riemann–Hilbert problem is reducible to a new problem on a surface of genus $\rho' = [\rho/2]$.

To convince the reader of the applicability and the viability of the technique proposed, we have solved a new model problem for an anisotropic half-plane with imperfect interfaces (the impedances $\eta_1^\pm, \eta_2^\pm$ are arbitrary) which are illuminated by a plane electromagnetic wave at oblique incidence. To solve this problem, we started with the Maliuzhinets formulation or, equivalently, with a vector functional-difference equation of the first order. It turns out that the matrix coefficient of the equation meets the restrictions for the method to be applied. The genus of the corresponding Riemann surface is equal to three. To complete the procedure of solution, one needs to solve the Jacobi inversion problem for a surface of genus three. A device for its exact solution has already been reported (Antipov & Silvestrov, 2002). We have also analysed a particular case when the characteristic function does not have poles and zeros of odd order, and the solution of the Jacobi inversion problem has been avoided. Numerical results will be reported elsewhere.

The proposed technique has a potential to be successfully applied to a variety of diffraction problems that have been considered insoluble. The complexity of the approach depends on the genus of the corresponding Riemann surface. From the numerical point view the only portion which becomes more complicated is the solution of the Jacobi inversion problem.

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