SECOND-ORDER FUNCTIONAL-DIFFERENCE EQUATIONS. I: METHOD OF THE RIEMANN–HILBERT PROBLEM ON RIEMANN SURFACES

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Summary

An analytical method for scalar second-order functional-difference equations with meromorphic periodic coefficients is proposed. The technique involves reformulating the equation as a vector functional-difference equation of the first order and reducing it to a scalar Riemann–Hilbert problem for two finite segments on a hyperelliptic surface. The final step of the procedure is solution of the classical Jacobi's inversion problem. The method is illustrated by solving in closed form a second-order functional-difference equation when the corresponding surface is a torus. The solution is constructed in terms of elliptic functions.

1. Introduction

The analysis of canonical problems in acoustic and electromagnetic diffraction theory by half-planes and wedges is frequently carried out by using the Maliuzhinets technique (**1**). This method ultimately yields functional-difference equations which, in general, are equivalent to difference equations with periodic coefficients of nth order ($n \geqslant 2$). If the equation is of the second order and the shift coincides with the period of the coefficients, then it reduces to two difference equations of the first order (Jost (**2**), Demetrescu (**3**), Senior and Legault (**4**)). The coefficients of these equations are the roots of the characteristic quadratic equation with periodic coefficients and, in general, they are multi-valued functions. In the particular case when the coefficients are single-valued, a closed-form solution can be found either in terms of the Maliuzhinets functions (**1**), or by the method of the Riemann–Hilbert problem on a finite segment of a complex plane (Antipov and Silvestrov (**5**)). In spite of the importance of the second-order difference equations with periodic coefficients for the geometric theory of diffraction, a general exact method for their solution is still unavailable in the literature. For a survey of some results related to difference equations of the second order see (**6**, **7**).

Demetrescu *et al.* (**6**, **8**) have analysed some difference equations of the second order appearing in diffraction by a right-angled resistive wedge. They replace the initial equation by a couple of the first-order equations with multi-valued coefficients and then apply the Fourier transform. By taking

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the logarithms they reduce the equations to a boundary-value problem for a strip whose solution is given by a singular integral with a periodic analog of the Cauchy kernel. Analysis of the results (see the Appendix) shows that their solution is multi-valued in the complex plane. It is multi-valued in the strip $-\pi \leqslant \operatorname{Re} s \leqslant \pi$ because of the choice of the system of the branch cuts (one of the cuts crosses the integration contour of the Cauchy integral involved). Although by a correct choice of the cuts it is possible to achieve the desired property of the solution in the strip $|\operatorname{Re} s| \leqslant \pi$ and make it single-valued, it is impossible to overcome another obstacle: the solution is multi-valued outside the strip $|\operatorname{Re} s| \leqslant \pi$ regardless of the choice of the cuts. We emphasize that the spectral functions have to be single-valued in an extended strip and therefore the solution of the functional-difference equation has to be a meromorphic single-valued function in the complex plane.

Senior and Legault have generalized the equation deduced in (**6**) by adding another parameter into the coefficients of the second-order equation (**4**) and also by doubling the shift (**7**). The shift is equal to $2n\pi$, where π is the period of the coefficients and $n = 1$ in (**4**) and $n = 2$ in (**7**). They also reduce the second-order equation to two first-order equations with multi-valued coefficients. Then by taking the logarithmic derivatives they transform the problem into the equations of the form

$$\frac{d}{d\alpha} \log w(\alpha + n\pi) - \frac{d}{d\alpha} \log w(\alpha - n\pi) = g(\alpha), \tag{1.1}$$

where $g(\alpha)$ is a multi-valued 2π-periodic function with prescribed poles and branch points. The solution of the initial difference equation is expressed through solutions of equation (1.1). These solutions are multi-valued functions on the complex plane and have to be single-valued on a Riemann surface $\mathcal{R}$ defined by the branch points of the function $g(\alpha)$. The construction of a partial solution of equation (1.1) that is single-valued on $\mathcal{R}$, is the key step in the procedure (**4**, **7**). The property of the solution to be single-valued is satisfied by adding to the solution abelian integrals of the first kind with unknown coefficients and abelian integrals of the third kind with unknown logarithmic singularities. Eliminating the polar and cyclic periods of the function $\log w(\alpha)$ leads to systems of equations which are linear with respect to the unknown coefficients of the abelian integrals of the first kind and nonlinear with respect to the unknown singularities of the abelian integrals of the third kind. The number of equations is equal to the genus of the corresponding Riemann surface that is defined by the number of the zeros (they are assumed to be simple) of the determinant of the characteristic equation. In particular cases, when the genus of the surface is one or when the surface has a special symmetry, it is possible to simplify the above nonlinear system and to find its exact solution. In the case (**4**) the genus of the surface is one, and the solution is found in terms of elliptic functions. In the case (**7**) the corresponding surface is of genus three. By exploring the symmetry of the surface, the Cauchy theorem and the Riemann bilinear relations for abelian integrals, the nonlinear system reduces to inversion of the elliptic integrals. In general, however, the solvability of the corresponding nonlinear system has not been studied and methods for solution (exact or approximate) are unknown.

A new constructive method for a vector functional-difference equation of the first order

$$\mathbf{\Phi}(\sigma) = \mathbf{G}(\sigma)\mathbf{\Phi}(\sigma - h) + \mathbf{g}(\sigma), \quad \sigma \in \Omega = \{\operatorname{Re} s = \omega\}, \tag{1.2}$$

has been proposed by Antipov and Silvestrov (**9**). The authors have found that if the matrix $\mathbf{G}(\sigma)$ has the following structure:

$$\mathbf{G}(\sigma) = \begin{pmatrix} a_1(\sigma) + a_2(\sigma) f_1(\sigma) & a_2(\sigma) \\ a_2(\sigma) f_2(\sigma) & a_1(\sigma) - a_2(\sigma) f_1(\sigma) \end{pmatrix}, \quad \sigma \in \Omega, \tag{1.3}$$

then equation (1.2) reduces to a scalar Riemann–Hilbert boundary-value problem on two finite segments of a Riemann surface $\mathcal{R}$ of an algebraic function defined by the characteristic function $f(s) = f_1^2(s) + f_2(s)$. Here $a_1(\sigma), a_2(\sigma)$ are arbitrary Hölder functions on every finite segment of the contour Ω, $f_1(\sigma)$, $f_2(\sigma)$ are arbitrary single-valued meromorphic functions in the strip $\Pi = \{\omega - h < \mathrm{Re}(s) < \omega\}$ such that $f_j(\sigma) = f_j(\sigma - h), \sigma \in \Omega, j = 1, 2\cdot$ It is assumed that the functions $f_1(s)$ and $f(s)$ have finite numbers of poles in the strip Π. The number of zeros of the function $f(s)$ in the strip Π is also finite. The solution of equation (1.2) is constructed in terms of the Weierstrass integrals and the Riemann θ-function on the surface $\mathcal{R}$. This method is applied for solving in closed form a problem of electromagnetic scattering of a plane wave obliquely incident on an anisotropic impedance half-plane with four different impedances.

The main objectives of the present paper are

(i) to propose an efficient method for scalar functional-difference equations of the second order

$$a(s)f(s+h) + b(s)f(s) + c(s)f(s-h) = d(s), \quad s \in \mathbb{C}, \tag{1.4}$$

with h-periodic entire coefficients based on the theory of the Riemann–Hilbert problem on two finite segments of a hyperelliptic surface of any finite genus;

(ii) to illustrate the technique by studying the solvability and finding a closed-form solution of a class of second-order difference equations when the relevant surface is a torus.

The method to be proposed in the present paper is different from the one by Senior and Legault (**4**, **7**). The first step of the procedure is to convert the scalar functional-difference equation of the second order into a vector difference equation of the first order and decouple it. In contrast to the approach of Jost (**2**), Demetrescu (**3**) and Senior and Legault (**4**, **7**) as a first stage we arrive not at two functional equations of the first order with two-valued coefficients, but at a scalar Riemann–Hilbert problem for finite segments on a hyperelliptic surface. To solve this problem on the Riemann surface we use the singular integrals with the Weierstrass kernel that has a pole at infinity. Its order coincides with the genus of the surface ρ. This solution is single-valued on the surface and therefore does not require elimination of the polar and cyclic periods, the bulk of the procedure (**4**, **7**). The constructed solution, however, has an essential singularity at infinity. The condition for eliminating this singularity is equivalent to the Jacobi inversion problem (Farkas and Kra (**10**), Zverovich (**11**)). It has been shown by Antipov and Silvestrov (**12**) that this problem reduces to an algebraic equation of degree ρ or can be solved numerically by the method based on the principle of the argument on a Riemann surface.

The present paper is organized as follows. We formulate the problem, describe a class of solutions in section 2.1. Then (section 2.2) we reduce an auxiliary problem for equation (1.4) in a strip to a scalar Riemann–Hilbert problem on the segments $[-1, 1]$ on two sheets of a hyperelliptic surface of an algebraic function. It is also shown that if the number of branch points of a function $\Delta^{1/2}(s)$ is equal to $2\rho + 2$ (this number is always even), then the surface has genus ρ. Here $\Delta(s)$ is the discriminant of equation (1.4): $\Delta(s) = b^2(s) - 4a(s)c(s)$. The general theory of the scalar Riemann–Hilbert problem on the segments $[-1, 1]$ on two sheets of a hyperelliptic surface and solution of the associated Jacobi inversion problem has been proposed by Antipov and Silvestrov (**9**, **12**). We write down the solution of the Riemann–Hilbert problem associated with the second-order difference equation in section 2.3. In section 2.4, we find a general solution to the functional-difference equation (1.4) in the whole complex plane. The elliptic case is thoroughly analysed in section 3. An exact solution is constructed in terms of elliptic functions. In the Appendix, we show that the solution (**6**, **8**) is multi-valued.

To illustrate the proposed technique we have solved a problem on electromagnetic scattering by a right-angled magnetically conductive wedge (**5**). The governing equation to be solved is the following one:

$$(\cos s - \sin\theta)[f(s+\pi) + f(s-\pi)] = \cos s f(s), \tag{1.5}$$

and the corresponding auxiliary equation is

$$(\cos^2 s - \sin^2\theta)[f(s+2\pi) + f(s-2\pi)] + (\cos^2 s - 2\sin^2\theta) f(s) = 0. \tag{1.6}$$

The last equation is equivalent to a Riemann–Hilbert problem on a torus and it is solved by the technique proposed in the present paper. For equation (1.5), the period of the coefficients of the functional-difference equation is 2π whereas the shift is equal to π. The case when the shift is less than the period is not the subject of the present paper. This issue is addressed in Antipov and Silvestrov (**5**).

2. Scalar functional-difference equation of the second order

2.1 *Formulation*

Let $s \in \mathbb{C}$. Consider the following problem.

Given entire functions $a(s)$, $b(s)$, $c(s)$ and $d(s)$ find a function $f(s)$ meromorphic in $\mathbb{C}$ such that

$$a(s) f(s+h) + b(s) f(s) + c(s) f(s-h) = d(s). \tag{2.1}$$

The functions $a(s)$, $b(s)$ and $c(s)$ are h-periodic and the expressions $b(s)/a(s)$, $c(s)/a(s)$ have certain finite limits as $|s| \to \infty$ (Re *s is finite*). *It is also assumed that the function $f(s)$ is analytic in a strip $\Pi^0 = \{\omega_1 \leqslant \operatorname{Re} s \leqslant \omega_2\}$ apart from a finite set of poles. At the ends of the strip, that is, as $|s| \to \infty$, $f(s) = O(e^{2\pi\nu^{\pm}|\operatorname{Im} s|/h})$ with $\nu^{\pm}$ being real, finite and prescribed.*

At any zero of order ν_0 of the coefficients $a(s-h)$, $b(s)$ and $c(s+h)$, the unknown function $f(s)$ may have a pole of the same order ν_0.

Let ω be a real number, and Π^* be a strip $\{s \in \mathbb{C} : \omega - h < \operatorname{Re} s < \omega + h\}$ such that $\Pi^* \subset \Pi^0$ if $2h < \omega_2 - \omega_1$, $\Pi^* = \Pi^0$ if $2h = \omega_2 - \omega_1$, and $\Pi^* \supset \Pi^0$ if $2h > \omega_2 - \omega_1$.

First, we state and analyse an auxiliary problem.

Find a function $\hat{f}(s)$ that

- *is meromorphic in the strip Π^* with prescribed poles and admits a continuous extension up to the boundary $\partial\Pi^*$,*
- *at infinity may grow* (*decay*) *exponentially:*

$$\hat{f}(s) = O(e^{2\pi\nu^{\pm}|\operatorname{Im} s|/h}), \quad \operatorname{Im} s \to \pm\infty, \quad s \in \Pi^*, \tag{2.2}$$

- *on the contour $\Omega = \{\sigma \in \Pi^* : \operatorname{Re}\sigma = \omega\}$ satisfies the equation*

$$a(\sigma)\hat{f}(\sigma+h) + b(\sigma)\hat{f}(\sigma) + c(\sigma)\hat{f}(\sigma-h) = d(\sigma)\cdot \tag{2.3}$$

Without loss of generality assume that $b(s)/a(s)$, $c(s)/a(s)$ do not have poles and zeros on Ω.

Note that if $\Pi^* \subset \Pi^0$, that is, $2h < \omega_2 - \omega_1$, then it might happen that some of the prescribed poles, say $\hat{a}_j \in \Pi^0$ of the function $f(s)$ in (2.1), will be outside the strip Π^*. Therefore, not to lose them, we need to seek the function $\hat{f}(s)$ in Π^* with the additional poles at the points $\hat{a}_j + nh \in \Pi^*$ (n is an integer).

Clearly, in either case ($\Pi^* \subseteq \Pi^0$ or $\Pi^* \supset \Pi^0$) in the strip Π^*, the function $f(s)$ is a solution of the auxiliary problem.

2.2 *Scalar Riemann–Hilbert problem on a hyperelliptic surface*

The auxiliary problem for equation (2.1) is equivalent to a vector equation of the first order. To show this, introduce two functions

$$\Phi_1(s) = \hat{f}(s), \quad \Phi_2(s) = \hat{f}(s+h), \quad s \in \bar{\Pi}, \tag{2.4}$$

where $\bar{\Pi} = \{s \in \mathbb{C} : \omega - h \leqslant \operatorname{Re} s \leqslant \omega\}$. Then on the contour Ω, $\Phi_1(\sigma) = \Phi_2(\sigma - h)$. At the same time, equation (2.3) becomes

$$a(\sigma)\Phi_2(\sigma) + b(\sigma)\Phi_2(\sigma - h) + c(\sigma)\Phi_1(\sigma - h) = d(\sigma), \quad \sigma \in \Omega. \tag{2.5}$$

Equivalently, (2.3) can be rewritten in the vector form

$$\mathbf{\Phi}(\sigma) = \mathbf{G}(\sigma)\mathbf{\Phi}(\sigma - h) + \mathbf{g}(\sigma), \quad \sigma \in \Omega, \tag{2.6}$$

where

$$\mathbf{\Phi}(s) = \begin{pmatrix} \Phi_1(s) \\ \Phi_2(s) \end{pmatrix}, \quad \mathbf{g}(s) = \begin{pmatrix} 0 \\ d(s)/a(s) \end{pmatrix},$$

$$\mathbf{G}(s) = \begin{pmatrix} 0 & 1 \\ -c(s)/a(s) & -b(s)/a(s) \end{pmatrix}. \tag{2.7}$$

Let $\Delta(s) = b^2(s) - 4a(s)c(s)$ be the discriminant of the quadratic equation

$$a(s)\lambda^2(s) + b(s)\lambda(s) + c(s) = 0. \tag{2.8}$$

Then the eigenvalues of the matrix $\mathbf{G}(s)$ are given by

$$\lambda_j(s) = \frac{-b(s) + (-1)^{j-1}\Delta^{1/2}(s)}{2a(s)}, \quad j = 1, 2. \tag{2.9}$$

Call $\Delta(s)$ the discriminant of the functional-difference equation (2.1). Zeros of odd order of the discriminant are branch points of the function $\Delta^{1/2}(s)$. Let the branch points in the strip Π be $s_0, s_1, \ldots, s_{2\rho+1}$ ($\rho \geqslant 0$). Their number is always even. To fix a branch of the function $\Delta^{1/2}(s)$, cut the strip Π by smooth curves $\Gamma_j \subset \Pi$ ($j = 0, 1, \ldots, \rho$) which do not intersect each other and join the branch points so that $\Gamma_j = s_{2j}s_{2j+1}$ ($j = 0, 1, \ldots, \rho$). The positive direction is chosen from s_{2j} to s_{2j+1}. For the limit values of the fixed branch on the left (+) and on the right (−) sides of the cuts, $[\Delta^{1/2}(\sigma)]^+ = -[\Delta^{1/2}(\sigma)]^-$, $\sigma \in \Gamma_j$.

Let

$$\mathbf{T}(s) = \begin{pmatrix} 1 & 1 \\ \lambda_1(s) & \lambda_2(s) \end{pmatrix}. \tag{2.10}$$

The elements of this matrix are h-periodic. Therefore

$$[\mathbf{T}(s)]^{-1}\mathbf{G}(s)\mathbf{T}(s-h) = \Lambda(s), \tag{2.11}$$

where $\Lambda(s) = \operatorname{diag}\{\lambda_1(s), \lambda_2(s)\}\cdot$ Next, introduce a new vector function $\phi(s) = [\mathbf{T}(s)]^{-1}\mathbf{\Phi}(s)$, $s \in \Pi$, with the components

$$\phi_1(s) = \frac{1}{2}\left(\frac{b(s)}{\Delta^{1/2}(s)} + 1\right)\Phi_1(s) + \frac{a(s)}{\Delta^{1/2}}\Phi_2(s),$$

$$\phi_2(s) = \frac{1}{2}\left(-\frac{b(s)}{\Delta^{1/2}(s)} + 1\right)\Phi_1(s) - \frac{a(s)}{\Delta^{1/2}(s)}\Phi_2(s). \tag{2.12}$$

Because of relation (2.11), the new functions $\phi_1(s)$, $\phi_2(s)$ satisfy the two separate equations

$$\phi_1(\sigma) = \lambda_1(\sigma)\phi_1(\sigma - h) + \Delta^{-1/2}(\sigma)d(\sigma),$$

$$\phi_2(\sigma) = \lambda_2(\sigma)\phi_2(\sigma - h) - \Delta^{-1/2}(\sigma)d(\sigma), \quad \sigma \in \Omega. \tag{2.13}$$

The vector function $\Phi(s)$ is single-valued in the strip Π provided the following boundary conditions on the cuts Γ_j $(j = 0, 1, \ldots, \rho)$ hold:

$$\mathbf{T}^+(\sigma)\phi^+(\sigma) = \mathbf{T}^-(\sigma)\phi^-(\sigma), \quad \sigma \in \Gamma_j, \quad j = 0, 1, \ldots, \rho. \tag{2.14}$$

Equivalently,

$$\phi_1^+(\sigma) = \phi_2^-(\sigma), \quad \phi_1^-(\sigma) = \phi_2^+(\sigma), \quad \sigma \in \Gamma_j, \quad j = 0, 1, \ldots, \rho. \tag{2.15}$$

Analysis of relation (2.12) shows that the poles of the functions $\Phi_1(s)$, $\Phi_2(s)$ may or may not give poles to the functions $\phi_1(s)$, $\phi_2(s)$. This depends on whether the corresponding point is a zero of the functions $1 \pm b(s)\Delta^{-1/2}(s)$ or $a(s)$. In addition, the functions $\phi_1(s)$, $\phi_2(s)$ have poles at the zeros of even order of the discriminant $\Delta(s)$ in the strip Π. Let all the poles in Π of the functions $\phi_1(s)$, $\phi_2(s)$ be $a_1, a_2, \ldots, a_m$, and their orders be $\nu_1, \nu_2, \ldots, \nu_m$, respectively.

At the branch points of the function $\Delta^{1/2}(s)$,

$$\phi_j(s) \sim A_j(s - s_k)^{-\mu_k/2}, \quad s \to s_k, \quad A_j = \text{const}, \quad j = 1, 2;\ k = 0, 1, \ldots, 2\rho + 1, \tag{2.16}$$

where $\mu_j \geqslant 1$, μ_j are odd.

It is possible to reduce the problem (2.13), (2.15) to a vector Riemann–Hilbert problem on a system of contours. Transform the strip Π into a z-plane cut along the segment $[-1, 1]$ by the mapping

$$z = -i \tan\frac{\pi}{h}(s - \omega), \quad s = \omega + \frac{ih}{2\pi}\log\frac{1+z}{1-z}, \tag{2.17}$$

where the single branch of the function s is chosen such that $s \to \omega - \frac{1}{2}h$ as $z \to \infty$. Then the system of equations (2.13), (2.15) is equivalent to the following vector Riemann–Hilbert problem for the functions $F_j(z) = \phi_j(s)$ $(j = 1, 2)$:

$$F_j^+(t) = l_j(t)F_j^-(t) + D_j(t), \quad t \in (-1, 1), \quad j = 1, 2,$$

$$F_j^+(t) = F_{3-j}^-(t), \quad t \in \gamma_m, \quad j = 1, 2; \quad m = 0, 1, \ldots, \rho, \tag{2.18}$$

where

$$F_j^{\pm}(t) = \phi_j^{\pm}(\sigma), \quad l_j(t) = \lambda_j(\sigma),$$

$$D_j(t) = (-1)^{j-1}\Delta^{-1/2}(\sigma)d(\sigma), \quad \sigma = \omega + \frac{ih}{2\pi}\log\frac{1+t}{1-t}, \quad j = 1, 2. \tag{2.19}$$

The curves γ_m $(m = 0, 1, \ldots, \rho)$ which are the images of the contours Γ_m do not intersect each other and the segment $[-1, 1]$. The contour Ω is mapped onto the upper side of the cut $[-1, 1]$, and the left boundary of the strip Π, the contour Ω_{-1}, is mapped onto the lower bank of the cut $[-1, 1]$.

To solve the new problem (2.18), convert it into a scalar Riemann–Hilbert problem on a two-sheeted Riemann surface. Let $\mathcal{R}$ be the hyperelliptic surface of the algebraic function

$$w^2 = q(z), \quad q(z) = (z - z_0)(z - z_1)\ldots(z - z_{2\rho+1}), \tag{2.20}$$

formed by gluing two copies $\mathbb{C}_1$ and $\mathbb{C}_2$ of the extended complex plane $\mathbb{C} \cup \infty$ cut along the system of the curves γ_m $(m = 0, 1, \ldots, \rho)$. The positive (left) sides of the cuts γ_m on $\mathbb{C}_1$ are glued with the negative (right) sides of the curves γ_m on $\mathbb{C}_2$ and vice versa. Here z_j $(j = 0, 1, \ldots, 2\rho + 1)$ are the images of the branch points $s_0, s_1, \ldots, s_{2\rho+1}$:

$$z_j = -i\tan\frac{\pi}{h}(s_j - \omega), \quad j = 0, 1, \ldots, 2\rho + 1. \tag{2.21}$$

The constructed surface has genus ρ. Let $q^{1/2}(z)$ be the branch chosen such that $q^{1/2}(z) \sim z^{\rho+1}$, $z \to \infty$. Then the function w defined by (2.20) is single-valued on the surface $\mathcal{R}$:

$$w = \begin{cases} q^{1/2}(z), & z \in \mathbb{C}_1, \\ -q^{1/2}(z), & z \in \mathbb{C}_2. \end{cases} \tag{2.22}$$

Introduce now the following function on the surface $\mathcal{R}$:

$$F(z, w) = \begin{cases} F_1(z), & (z, w) \in \mathbb{C}_1, \\ F_2(z), & (z, w) \in \mathbb{C}_2. \end{cases} \tag{2.23}$$

By the second condition in (2.18) it becomes evident that the function $F(z, w)$ is meromorphic everywhere on the surface $\mathcal{R}$ apart from the contour $\mathcal{L} = L_1 \cup L_2$, with $L_1 = [-1, 1] \subset \mathbb{C}_1$ and $L_2 = [-1, 1] \subset \mathbb{C}_2$. On the contour $\mathcal{L} \subset \mathcal{R}$, this function satisfies the boundary condition

$$F^+(t, \xi) = l(t, \xi)F^-(t, \xi) + D(t, \xi), \quad (t, \xi) \in \mathcal{L}, \tag{2.24}$$

where

$$l(t,\xi)=\begin{cases} l_1(t), & (t,\xi)\in L_1,\\ l_2(t), & (t,\xi)\in L_2,\end{cases}\qquad D(t,\xi)=\begin{cases} D_1(t), & (t,\xi)\in L_1,\\ D_2(t), & (t,\xi)\in L_2,\end{cases}\qquad \xi=w(t). \quad (2.25)$$

At the ends of the contour $\mathcal{L}$, the behaviour of the function $F(z,w)$ is defined by the asymptotics of the functions $\phi_1(s)$, $\phi_2(s)$ as $\operatorname{Im} s\to\pm\infty$ and therefore by the asymptotics of the functions $f(s)$, $a(s)\Delta^{-1/2}(s)$, $1+b(s)\Delta^{-1/2}(s)$ and $1-b(s)\Delta^{-1/2}(s)$ as $\operatorname{Im} s\to\pm\infty$. For definiteness, assume that

$$|F(z,w)|\leqslant A_{\pm}^{(\mu)}|z\mp 1|^{-\nu_\mu^{\pm}},\quad (z,w)\in\mathbb{C}_\mu,\quad z\to\pm 1,\quad A_{\pm}^{(\mu)}=\text{const},\quad \mu=1,2, \quad (2.26)$$

where $\nu_\mu^{\pm}$ are definite real numbers. In particular, if the functions $a(s)\Delta^{-1/2}(s)$, $1+b(s)\Delta^{-1/2}(s)$ and $1-b(s)\Delta^{-1/2}(s)$ have finite non-zero limits, then $\nu_\mu^{\pm}=\nu^{\pm}$, $\mu=1,2$.

2.3 *General solution to the Riemann–Hilbert problem*

A general procedure for solution of the scalar Riemann–Hilbert problem for an open contour on a hyperelliptic surface of any finite genus is presented by Antipov and Silvestrov (**9**). In this section we write down the final formulae for the solution to the problem (2.24). The general representation for the function $F(z,w)$ has the form

$$F(z,w)=X(z,w)[\Psi(z,w)+R(z,w)], \quad (2.27)$$

where

$$\Psi(z,w)=\psi_1(z)+w(z)\psi_2(z),$$

$$\psi_1(z)=\frac{1}{4\pi i}\int_{\mathcal{L}}\frac{D(t,\xi)}{X^+(t,\xi)}\frac{dt}{t-z},\quad \psi_2(z)=\frac{1}{4\pi i}\int_{\mathcal{L}}\frac{D(t,\xi)}{\xi(t)X^+(t,\xi)}\frac{dt}{t-z}. \quad (2.28)$$

The meromorphic function $R(z,w)=R_1(z)+w(z)R_2(z)$ is expressed through the rational functions $R_1(z)$ and $R_2(z)$ with specified poles and arbitrary coefficients. The function $X(z,w)$ is a solution of the following problem.

Find a function $X(z,w)$ meromorphic on $\mathcal{R}\subset\mathcal{L}$ which has a finite number of poles and zeros, has non-zero boundary values $X^{\pm}(t,\xi)$ and satisfies the boundary condition

$$X^+(t,\xi)=l(t,\xi)X^-(t,\xi),\quad (t,\xi)\in\mathcal{L}\subset\mathcal{R}. \quad (2.29)$$

At the ends of the segments L_μ,

$$B_0|z\mp 1|^{-\nu_\mu^{\pm}+1}<|X(z,w)|\leqslant B_1|z\mp 1|^{-\nu_\mu^{\pm}},\quad (z,w)\in\mathbb{C}_\mu,\quad z\to\pm 1,\quad \mu=1,2, \quad (2.30)$$

where B_0, B_1 are positive constants.

Such a solution is called a canonical function of the problem (2.24) and it is given by $X(z,w)=\exp\{\Xi(z,w)\}$, $(z,w)\in\mathcal{R}$, where

$$\Xi(z,w)=\frac{1}{2\pi i}\int_{\mathcal{L}}\log l(t,\xi)dW+\sum_{\mu=1}^{2}\operatorname{sgn}\kappa_\mu\sum_{j=1}^{|\kappa_\mu|}\int_{p_{\mu 0}}^{p_{\mu j}}dW$$
$$+\sum_{j=1}^{\rho}\left(\int_{e_j}^{r_j}dW+m_j\oint_{\mathbf{a}_j}dW+n_j\oint_{\mathbf{b}_j}dW\right). \quad (2.31)$$

Here

$$dW = \frac{w+\xi}{2\xi}\frac{dt}{t-z}, \quad w = w(z), \quad \xi = w(t), \tag{2.32}$$

is the Weierstrass kernel (Zverovich (**11**)), an analogue of the Cauchy kernel on the surface $\mathcal{R}$. Under $\log l(t, \xi)$ on the segments $L_\mu \subset \mathbb{C}_\mu$ ($\mu = 1, 2$) we understand the branches of the functions $\log l_\mu(t)$ on L_μ fixed by the conditions

$$2\pi(\nu_\mu^- - 1) < \arg l_\mu(-1) \leqslant 2\pi\nu_\mu^-, \quad \mu = 1, 2. \tag{2.33}$$

The integers κ_μ in (2.31) are determined by

$$\kappa_\mu = \nu_\mu^+ + \left[\frac{1}{2\pi}\arg l_\mu(1)\right], \quad \mu = 1, 2. \tag{2.34}$$

Here $[a]$ is the integer part of the number a.

The first integral in (2.31) is discontinuous on $\mathcal{L}$, and the function $\exp\{\Xi(z, w)\}$ satisfies the boundary condition (2.29). In general, the function $\exp\{\Xi(z, w)\}$ grows exponentially at infinity and might not meet the inequalities (2.30). To achieve the prescribed behaviour of the solution at the ends of the contour $\mathcal{L}$, the second group of the line integrals along smooth curves is added in (2.31). The starting points $p_{\mu 0} \in \mathbb{C}_\mu$ ($\mu = 1, 2$) of the lines of integration coincide with the end $t = 1$ of the contours L_μ, respectively: $p_{10} = (1, q^{1/2}(1))$, $p_{20} = (1, -q^{1/2}(1))$. The upper limits

$$p_{\mu j} = (\gamma_{\mu j}, (-1)^{\mu-1}u_{\mu j}) \in \mathbb{C}_\mu, \quad u_{\mu j} = q^{1/2}(\gamma_{\mu j}), \quad j = 1, 2, \dots, |\kappa_\mu|, \quad \mu = 1, 2, \tag{2.35}$$

of the integrals are arbitrary fixed distinct points of the surface $\mathcal{R}$. The exponents of these integrals are continuous through the contours of integration. The last group of the line integrals in (2.31) is taken to remove the exponential growth of the solution at infinity. They do not violate the condition (2.29). The contours $\mathbf{a}_j$ and $\mathbf{b}_j$ form a system of canonical cross-sections of the surface $\mathcal{R}$. The points $e_j = (\delta_j, v_j) \in \mathbb{C}_1$, $v_j = q^{1/2}(\delta_j)$, $j = 1, 2, \dots, \rho$, are arbitrary fixed distinct points of the surface $\mathcal{R}$. We note that the arbitrary points e_j and $p_{\mu j}$ do not lie on the contour $\mathcal{L}$ and the canonical cross-sections. Also, they do not coincide with the branch points of the surface $\mathcal{R}$ and the poles of the function $F(z, w)$. The final formulae for the solution do not depend upon the choice of the points e_j and $p_{\mu j}$. The points $r_j = (\sigma_j, w_j)$ ($w_j = w(\sigma_j)$, $j = 1, 2, \dots, \rho$) may lie on either sheet of the surface. These points and the integers m_j, n_j have to satisfy the Jacobi inversion problem (Springer (**13**), Farkas and Kra (**10**), Zverovich (**11**))

$$\sum_{j=1}^{\rho}[\omega_\nu(\sigma_j, w_j) + m_j A_{\nu j} + n_j B_{\nu j}] = d_\nu^*, \quad \nu = 1, 2, \dots, \rho, \tag{2.36}$$

where

$$A_{\nu j} = \oint_{\mathbf{a}_j} \frac{t^{\nu-1}dt}{\xi(t)}, \quad B_{\nu j} = \oint_{\mathbf{b}_j} \frac{t^{\nu-1}dt}{\xi(t)} \tag{2.37}$$

are the A- and B-periods of the abelian integrals

$$\omega_\nu = \omega_\nu(z, w) = \int_{(z_0,0)}^{(z,w)} \frac{t^{\nu-1}dt}{\xi(t)}, \quad \nu = 1, 2, \dots, \rho, \tag{2.38}$$

and the constants d_ν^* are defined by

$$d_\nu^* = \sum_{j=1}^{\rho} \omega_\nu(\delta_j, v_j) - \frac{1}{2\pi i}\int_{-1}^{1}[\log l_1(t) - \log l_2(t)]\frac{t^{\nu-1}dt}{q^{1/2}(t)} + \sum_{\mu=1}^{2}(-1)^\mu \operatorname{sgn}\kappa_\mu \sum_{j=1}^{|\kappa_\mu|}\int_1^{\gamma_{\mu j}} \frac{t^{\nu-1}dt}{q^{1/2}(t)}. \tag{2.39}$$

For genus $\rho = 1$, the inversion problem (2.36) is solvable in terms of elliptic functions. This case is analysed in section 3. For genus $\rho > 1$, the Jacobi problem is equivalent to the identification of the zeros of the associated Riemann θ-function. This issue is discussed in (**12**).

We now summarize the procedure for finding the canonical function. First one needs to fix branches of the functions $\log l_1(t)$ and $\log l_2(t)$ on the segments $[-1, 1]$ by the inequality (2.33). Next, the integers κ_μ should be identified to meet the conditions (2.30). The last step of the procedure is to find the integers m_j, n_j and the unknown points $r_j = (\sigma_j, w_j) \in \mathcal{R}$ by solving the Jacobi inversion problem (2.36). The canonical function $X(z, w) = \exp\{\Xi(z, w)\}$ is given by (2.31). It is bounded at infinity, satisfies the boundary condition (2.29) and the inequality (2.30). The function $X(z, w)$ is analytic and non-zero everywhere in $\mathcal{R} \subset \mathcal{L}$ apart from the simple poles e_j and simple zeros r_j $(j = 1, 2, \dots, \rho)$ and, possibly, the points $p_{\mu j}$ $(j = 1, 2, \dots, |\kappa_\mu|,\ \mu = 1, 2)$. If $\kappa_\mu > 0$, then the points $p_{\mu j}$ are simple zeros. For negative κ_μ, the function $X(z, w)$ has simple poles at these points. For $\kappa_\mu = 0$, the points $p_{\mu j}$ are neither poles nor zeros.

Not all the constants in the rational functions $R_1(z)$ and $R_2(z)$ are arbitrary. They have to be chosen such that

$$\lim_{z\to\infty} z^k[\psi_2(z) + R_2(z)] = 0, \quad k = 1, 2, \dots, \rho,$$

$$\Psi(\delta_k, v_k) + R(\delta_k, v_k) = 0, \quad k = 1, 2, \dots, \rho,$$

$$\Psi(\gamma_{\mu j}, (-1)^{\mu-1}u_{\mu j}) + R(\gamma_{\mu j}, (-1)^{\mu-1}u_{\mu j}) = 0, \quad j = 1, 2, \dots, -\kappa_\mu, \quad \mu = 1, 2. \tag{2.40}$$

The last group of the conditions is required if $\kappa_\mu < 0$. The conditions (2.40) guarantee the boundedness of the function $F(z, w)$ at infinity, eliminate the poles at the points (δ_k, v_k) and assure the boundedness of the solution at the points $(\gamma_{\mu j}, (-1)^{\mu-1}u_{\mu j})$ when $\kappa_\mu < 0$.

2.4 *General solution to the functional-difference equation*

According to formulae (2.12), (2.4) and the h-periodicity of the functions $b(s)/a(s)$, $c(s)/a(s)$, the general solution to the auxiliary problem (2.3) has the form

$$\hat f(s) = \phi_1(s) + \phi_2(s), \quad \omega - h \leqslant \operatorname{Re} s \leqslant \omega,$$

$$\hat f(s) = -\frac{b(s)}{2a(s)}[\phi_1(s-h) + \phi_2(s-h)] + \frac{\Delta^{1/2}(s)}{2a(s)}[\phi_1(s-h) - \phi_2(s-h)], \quad \omega \leqslant \operatorname{Re} s \leqslant \omega + h, \tag{2.41}$$

where $\phi_j(s) = F_j(z(s))$, $z(s) = -i\tan(\pi/h)(s-\omega)\cdot$

In view of $\phi_j(s-h) = F_j(z(s-h)) = F_j(z(s))$, $j = 1, 2$, express the function $\hat f(s)$ in terms of the solution to the Riemann–Hilbert problem on the surface $\mathcal{R}$:

$$\hat f(s) = F(z, w) + F(z, -w), \quad \omega - h \leqslant \operatorname{Re} s \leqslant \omega,$$

$$\hat{f}(s) = -\frac{b(s)}{2a(s)}[F(z,w)+F(z,-w)] + \frac{\Delta^{1/2}(s)}{2a(s)}[F(z,w)-F(z,-w)], \quad \omega \leqslant \operatorname{Re} s \leqslant \omega + h, \tag{2.42}$$

where $w = q^{1/2}(z)$. To define the general solution to equation (2.1) in the whole complex plane, continue analytically the function $f(s)$ from the strip Π^* to the left and to the right

$$f(s) = \frac{1}{a(s)}[-b(s)\hat{f}(s-h) - c(s)\hat{f}(s-2h) + d(s-h)], \quad \omega + h \leqslant \operatorname{Re} s \leqslant \omega + 2h,$$

$$f(s) = \frac{1}{c(s)}[-a(s)\hat{f}(s+2h) - b(s)\hat{f}(s+h) + d(s+h)], \quad \omega - 2h \leqslant \operatorname{Re} s \leqslant \omega - h. \tag{2.43}$$

It is convenient to use the following notation:

$$f_k(s) = f(s), \quad s \in \Pi_k,$$

$$\Pi_k = \{s \in \mathbb{C} : \omega + (k-2)h \leqslant \operatorname{Re} s \leqslant \omega + (k-1)h\}, \quad k = 0, \pm 1, \dots. \tag{2.44}$$

The functions $f_1(s)$ and $f_2(s)$ defined in the strips Π_1 and Π_2 are given by (2.42): $f_1(s) = \hat{f}(s)$, $s \in \Pi_1$, and $f_2(s) = \hat{f}(s)$, $s \in \Pi_2$. As for the other functions, they can be obtained by the analytical continuation of relations (2.42) into the strips Π_k:

$$f_k(s) = \frac{1}{a(s)}[-b(s)f_{k-1}(s-h) - c(s)f_{k-2}(s-2h) + d(s-h)], \quad k = 3, 4, 5, \dots,$$

$$f_k(s) = \frac{1}{c(s)}[-a(s)f_{k+2}(s+2h) - b(s)f_{k+1}(s+h) + d(s+h)], \quad k = 0, -1, -2, \dots. \tag{2.45}$$

The function $f(s) = f_k(s)$, $s \in \Pi_k$, is continuous through the contours $\operatorname{Re} s = \omega + kh$ ($k = 0, \pm 1, \pm 2, \dots$), and is meromorphic and single-valued in the whole plane.

In the case $\Pi^* \subset \Pi^0$, the analytical continuation may cause undesired poles in the strip Π^0 of the function $f(s)$. They should be removed.

3. Elliptic case

In this section we aim to find an explicit solution to equation (2.1) when the scalar Riemann–Hilbert problem is set on a torus, a Riemann surface of genus $\rho = 1\cdot$

3.1 *Riemann–Hilbert problem on two arcs of a torus*

In the case under consideration the discriminant of equation (2.1), the function $\Delta(s)$, has four zeros $s_j \in \Pi$ ($j = 0, 1, 2, 3$) of odd order. The conformal mapping (2.17) that transforms the strip Π into the complex plane with a cut, is not unique. In the elliptic case instead of the function (2.17) it is useful to take the function which maps the strip Π onto a complex plane such that the branch points s_0, s_1, s_2 and s_3 are mapped into the points $-1/k$, -1, 1 and $1/k$, respectively, where k is a complex parameter to be determined. It will be done into two steps. First, map the strip Π onto the ζ-plane by the function $\zeta = \exp(2\pi i s/h)$. Obviously, the contours $\Omega = \{s \in \mathbb{C} : \operatorname{Re} s = \omega\}$ and $\Omega_{-1} = \{s \in \mathbb{C} : \operatorname{Re} s = \omega - h\}$ are transformed into the lower and the upper banks of the semi-infinite cut $\{\arg \zeta = 2\pi\omega/h\}$, respectively. Let

$$\zeta_j = e^{2\pi i s_j/h}, \quad j = 0, 1, 2, 3. \tag{3.1}$$

Secondly, we use the inverse linear rational function $\mathbb{C} \to \mathbb{C}$ that maps the points $-1/k$, -1, 1, $1/k$ into the points ζ_j ($j = 0, 1, 2, 3$) (see Hancock (**14**))

$$\zeta = \frac{\zeta_1 + \zeta_2}{2} + \frac{\zeta_1 - \zeta_2}{2}\frac{z - \mu}{\mu z - 1}, \tag{3.2}$$

where the parameters μ and k should be found from

$$\frac{1 + \mu}{1 - \mu} = \frac{\zeta_0 - \zeta_2}{\zeta_0 - \zeta_1}\frac{1 - k}{1 + k},$$

$$\left(\frac{1 - k}{1 + k}\right)^2 = \frac{\zeta_0 - \zeta_1}{\zeta_0 - \zeta_2}\frac{\zeta_3 - \zeta_2}{\zeta_3 - \zeta_1}. \tag{3.3}$$

By solving equation (3.2) with respect to z, find the mapping function

$$z = u(s), \quad u(s) = \frac{2e^{2\pi i s/h} - (\zeta_1 + \zeta_2) - \mu(\zeta_1 - \zeta_2)}{2\mu e^{2\pi i s/h} - \mu(\zeta_1 + \zeta_2) - (\zeta_1 - \zeta_2)}. \tag{3.4}$$

This function maps the branch points s_0, s_1, s_2 and s_3 into the points $-1/k$, -1, 1, $1/k$, respectively. The cuts Γ_0 and Γ_1 become smooth curves γ_0 and γ_1 joining the points $-1/k$, -1 and 1, $1/k$.

The contour Ω transforms into a circle arc $L = t_1 t_2 t_3$ defined by three points t_1, t_2 and t_3 with t_1 being a starting point (Fig.1):

$$t_1 = \frac{1}{\mu},$$

$$t_2 = \frac{2e^{2\pi i\omega/h} - (\zeta_1 + \zeta_2) - \mu(\zeta_1 - \zeta_2)}{2\mu e^{2\pi i\omega/h} - \mu(\zeta_1 + \zeta_2) - (\zeta_1 - \zeta_2)},$$

$$t_3 = \frac{\zeta_1 + \zeta_2 + \mu(\zeta_1 - \zeta_2)}{\mu(\zeta_1 + \zeta_2) + \zeta_1 - \zeta_2}. \tag{3.5}$$

It follows immediately from (3.2) that the inverse function is

$$s = v(z), \quad v(z) = \frac{h}{2\pi i}\log\left(\frac{\zeta_1 + \zeta_2}{2} + \frac{\zeta_1 - \zeta_2}{2}\frac{z - \mu}{\mu z - 1}\right), \tag{3.6}$$

where a single branch of the above function is chosen such that $\operatorname{Re} v(z) = \omega$, $z \in L^+$, with L^+ being the left bank of the cut L.

To find the functions

$$F_j(z) = \phi_j(v(z)), \quad z \in \mathbb{C}, \quad j = 1, 2, \tag{3.7}$$

we have to solve the vector Riemann–Hilbert boundary-value problem (2.18), where

$$l_j(t) = \lambda_j(v(t)), \quad D_j(t) = (-1)^{j-1}\Delta^{-1/2}(v(t))d(v(t)). \tag{3.8}$$

Following the procedure described in section 2 reduces the problem to the scalar Riemann–Hilbert problem (2.24) on the Riemann surface of genus $\rho = 1$ of the algebraic function

$$w^2 = q(z), \quad q(z) = (1 - z^2)(1 - k^2 z^2). \tag{3.9}$$

The contour $\mathcal{L}$ consists of two circular arcs $L_1 = L \subset \mathbb{C}_1$ and $L_2 = L \subset \mathbb{C}_2$. Because of the prescribed asymptotics (2.2) of the function $f(s)$ as $\operatorname{Im} s \to \pm\infty$, $s \in \Pi^*$, and formulae (2.4), (2.12) the function $F(z, w)$ has power singularities in the vicinities of the end points $z = t_1$, $z = t_3$:

$$F(z, w) = O(|z - t_1|^{-\nu_\mu^-}), \quad z \to t_1, \quad (z, w) \in \mathbb{C}_\mu, \quad \mu = 1, 2,$$

$$F(z, w) = O(|z - t_3|^{-\nu_\mu^+}), \quad z \to t_3, \quad (z, w) \in \mathbb{C}_\mu, \quad \mu = 1, 2, \tag{3.10}$$

where the parameters $\nu_\mu^\pm$ are defined by the numbers $\nu^\pm$ and the behaviour of the functions $a(s)\Delta^{-1/2}(s)$, $1 + b(s)\Delta^{-1/2}(s)$ and $1 - b(s)\Delta^{-1/2}(s)$ as $\operatorname{Im} s \to \pm\infty$. At the branch points of the surface, $z = -1/k$, -1, 1 and $1/k$, the function $F(z, w)$ possesses poles of orders μ_0, μ_1, μ_2 and μ_3, respectively. We emphasize that the poles are understood in the sense of Riemann surfaces (Springer (**13**)). Here the numbers μ_0, μ_1, μ_2 and μ_3 denote the orders of the zeros of the discriminant $\Delta(s)$ at the corresponding points s_j, $j = 0, 1, 2, 3$. To complete the description of the class of solutions, we indicate that the unknown function $F(z, w)$ may have poles of orders $\nu_1, \nu_2, \ldots, \nu_m$ on both sheets $\mathbb{C}_1$, $\mathbb{C}_2$ at the points with affixes $\alpha_1, \alpha_2, \ldots, \alpha_m$. Here α_j are the images of the poles a_j of the functions $\phi_1(s), \phi_2(s)$: $\alpha_j = u(a_j)$, $j = 1, 2, \ldots, m$.

3.2 *Canonical function of the Riemann–Hilbert problem*

To solve the Riemann–Hilbert problem (2.24) one needs to factorize the coefficient $l(t, \xi)$. This means constructing a canonical function of the problem (2.24). At the end points $z = t_1$ and $z = t_3$ this function may have power singularities

$$B_0^-|z - t_1|^{-\nu_\mu^- + 1} < |X(z, w)| \leqslant B_1^-|z - t_1|^{-\nu_\mu^-}, \quad z \to t_1, \quad (z, w) \in \mathbb{C}_\mu, \quad \mu = 1, 2,$$

$$B_0^+|z - t_3|^{-\nu_\mu^+ + 1} < |X(z, w)| \leqslant B_1^+|z - t_3|^{-\nu_\mu^+}, \quad z \to t_3, \quad (z, w) \in \mathbb{C}_\mu, \quad \mu = 1, 2. \tag{3.11}$$

First we construct **a**- and **b**-canonical cross-sections of the surface $\mathcal{R}$. The cross-section **a** consists of the banks of the cut γ_1 (Fig. 1) which simultaneously belong to $\mathbb{C}_1$ and $\mathbb{C}_2$. The positive direction on **a** is chosen such that the first sheet $\mathbb{C}_1$ is always on the left. The cross-section **b** is a smooth closed curve that consists of two parts. The first part is a curve of $\mathbb{C}_1$ joining the points $1/k$ and $-1/k$ and passing through infinity. The second part lies on the sheet $\mathbb{C}_2$ and joins the points $-1/k$ and $1/k$ through infinity. The starting point is $1/k$ and the first sheet is traced first. Both parts of the cross-section **b** are symmetric with respect to the origin. The function $w = w(z)$ on the surface $\mathcal{R}$ is defined by (2.22), where $q^{1/2}(z) = \sqrt{(1 - z^2)(1 - k^2 z^2)}$ is the branch single-valued in the z-plane cut along γ_0 and γ_1 and satisfying the relation $q^{1/2}(0) = 1\cdot$

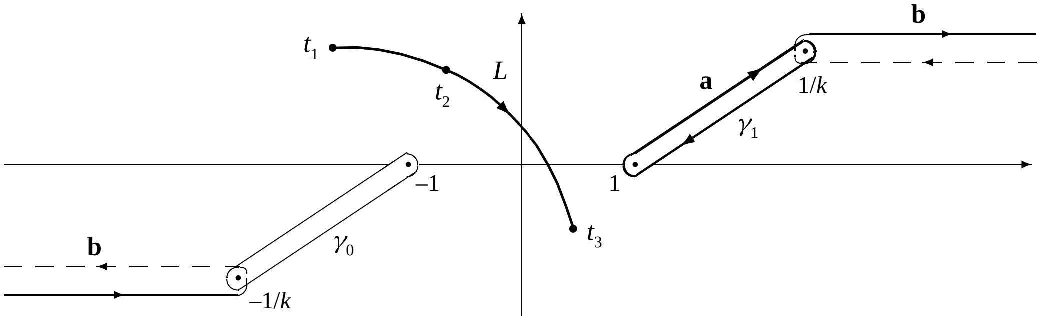

Fig. 1 Canonical cross-sections **a**, **b** and the contour L. The elliptic case

A canonical function of the problem (2.24) is given by

$$X(z,w)=\exp\{\Xi(z,w)\},\quad (z,w)\in\mathcal{R},$$

$$\Xi(z,w)=\frac{1}{2\pi i}\int_{\mathcal{L}}\log l(t,\xi)dW+\sum_{\mu=1}^{2}\operatorname{sgn}\kappa_\mu\sum_{j=1}^{|\kappa_\mu|}\int_{p_{\mu0}}^{p_{\mu j}}dW$$
$$+\int_{(\delta_0,v_0)}^{(\sigma_0,w_0)}dW+m_0\oint_{\mathbf{a}}dW+n_0\oint_{\mathbf{b}}dW. \quad (3.12)$$

A single branch of the logarithmic function $\log l(t,\xi)$ on each arc L_1, L_2 is fixed by the inequalities $2\pi(\nu_\mu^- - 1) < \arg l_\mu(t_1) \leqslant 2\pi\nu_\mu^-$, $\mu=1,2,$ and the integers κ_1,κ_2 are chosen as follows:

$$\kappa_\mu=\nu_\mu^+ + \left[\frac{1}{2\pi}\arg l_\mu(t_3)\right],\quad \mu=1,2. \quad (3.13)$$

Then analysis of the Weierstrass integrals in (3.12) implies

$$X(z,w)=O\{(z-t_1)^{\beta_\mu^-}\},\quad (z,w)\in\mathbb{C}_\mu,\quad z\to t_1,\quad \mu=1,2,$$

$$X(z,w)=O\{(z-t_3)^{\beta_\mu^+}\},\quad (z,w)\in\mathbb{C}_\mu,\quad z\to t_3,\quad \mu=1,2, \quad (3.14)$$

where

$$\beta_\mu^-=-\frac{1}{2\pi}\arg l_\mu(t_1),\quad \beta_\mu^+=\frac{1}{2\pi}\arg l_\mu(t_3)-\kappa_\mu,\quad \mu=1,2,$$

$$-\nu_\mu^\pm\leqslant\beta_\mu^\pm<1-\nu_\mu^\pm. \quad (3.15)$$

Therefore the function (3.12) is within the class of solutions (3.11).

The points $p_{\mu0}$ are chosen to be $p_{\mu0}=(t_3,(-1)^{\mu-1}q^{1/2}(t_3))\in\mathbb{C}_\mu$, $\mu=1,2\cdot$ As for the points (δ_0,v_0) and $p_{\mu j}$,

$$(\delta_0,v_0)\in\mathbb{C}_1,\quad v_0=q^{1/2}(\delta_0),$$

$$p_{\mu j} = (\gamma_{\mu j}, (-1)^{\mu-1}u_{\mu j}) \in \mathbb{C}_\mu, \quad u_{\mu j} = q^{1/2}(\gamma_{\mu j}), \quad j = 1, 2, \ldots, |\kappa_\mu|, \quad \mu = 1, 2, \tag{3.16}$$

they are arbitrary fixed distinct points of the surface $\mathcal{R}$ which fall neither on the contour $\mathcal{L}$ nor on the canonical cross-sections. In addition, it is required for the above points not to coincide with the branch points of the surface $\mathcal{R}$ and the poles of the function $F(z, w)\cdot$ The point (σ_0, w_0) $(w_0 = w(\sigma_0))$ and the integers m_0, n_0 are not arbitrary. They will be fixed later.

The integrals in (3.12) apart from the integrals over $\mathcal{L}$ and **a**, **b** are taken over smooth curves which join the end points and which do not cross the cross-sections **a**, **b** and the contour $\mathcal{L}$. These integrals are independent of the shape of the path of integration. The first integral in (3.12) is discontinuous through the contour $\mathcal{L}$ with the jump $\log l(t, \xi)$. The other integrals are also discontinuous through the contours of integration. The corresponding jumps are equal to $2\pi i m$ (m is an integer). Hence the function $X(z, w)$ satisfies the homogeneous boundary condition (2.29).

In general, for an arbitrary point (σ_0, w_0) and arbitrary integers m_0, n_0, the function $X(z, w)$ in (3.12) has an essential singularity at infinity. This is because $w(z) \sim (-1)^j kz^2$, $z \to \infty$, $z \in \mathbb{C}_j$, and the Weierstrass kernel (2.32) has a pole at infinity. To eliminate the essential singularity we evaluate the principal terms of the expansions of the function $\Xi(z, w)$ at infinity on both sheets of the surface:

$$\frac{k}{2}(-1)^{j-1}\left\{\frac{1}{2\pi i}\int_L [\log l_1(t) - \log l_2(t)]\frac{dt}{q^{1/2}(t)} - \sum_{\mu=1}^{2}(-1)^\mu \operatorname{sgn}\kappa_\mu \sum_{j=1}^{|\kappa_\mu|}\int_{t_3}^{\gamma_{\mu j}} \frac{dt}{q^{1/2}(t)} \right.$$
$$\left. + \int_{(\delta_0, v_0)}^{(\sigma_0, w_0)} \frac{dt}{\xi(t)} + m_0 \int_{\mathbf{a}} \frac{dt}{\xi(t)} + n_0 \int_{\mathbf{b}} \frac{dt}{\xi(t)}\right\} z. \tag{3.17}$$

Thus in order that the function $X(x, w)$ is bounded at infinity it is necessary and sufficient that

$$\int_{(\delta_0, v_0)}^{(\sigma_0, w_0)} \frac{dt}{\xi(t)} + m_0 \oint_{\mathbf{a}} \frac{dt}{\xi(t)} + n_0 \oint_{\mathbf{b}} \frac{dt}{\xi(t)} = d^0, \tag{3.18}$$

where

$$d^0 = \frac{1}{2\pi i}\int_L [\log l_2(t) - \log l_1(t)]\frac{dt}{q^{1/2}(t)} + \sum_{\mu=1}^{2}(-1)^\mu \operatorname{sgn}\kappa_\mu \sum_{j=1}^{|\kappa_\mu|}\int_{t_3}^{\gamma_{\mu j}} \frac{dt}{q^{1/2}(t)}. \tag{3.19}$$

This nonlinear equation is the Jacobi inversion problem for the surface $\mathcal{R}$ of genus $\rho = 1$. We next solve this problem in closed form. Since the single branch of the function $q^{1/2}(t)$ has already been fixed by the condition $q^{1/2}(0) = 1$ the A- and B-periods of the elliptic integrals in (3.18) become

$$\oint_{\mathbf{a}} \frac{dt}{\xi(t)} = 2i\mathbf{K}', \quad \oint_{\mathbf{b}} \frac{dt}{\xi(t)} = -4\mathbf{K}, \tag{3.20}$$

where $\mathbf{K} = \mathbf{K}(k)$, $\mathbf{K}' = \mathbf{K}(\sqrt{1-k^2})$ are complete elliptic integrals of the first order. Hence equation (3.18) reduces to

$$\int_{(0,1)}^{(\sigma_0, w_0)} \frac{dt}{\xi(t)} = d^* + 4n_0\mathbf{K} - 2im_0\mathbf{K}', \tag{3.21}$$

where

$$d^* = d^0 + \int_0^{\delta_0} \frac{dt}{q^{1/2}(t)}. \tag{3.22}$$

Assume, first, that the unknown point (σ_0, w_0) is in $\mathbb{C}_1$. Then $\xi(t) = q^{1/2}(t)$. By inversion of the elliptic integral we get $\sigma_0 = \operatorname{sn} d^*$. Next, find the numbers n_0, m_0 from (3.21):

$$m_0 = -\frac{\operatorname{Im}\{(I_0 - d^*)\overline{\mathbf{K}}\}}{2\operatorname{Re}\{\mathbf{K}\overline{\mathbf{K}'}\}}, \quad n_0 = \frac{\operatorname{Re}\{(I_0 - d^*)\overline{\mathbf{K}'}\}}{4\operatorname{Re}\{\mathbf{K}\overline{\mathbf{K}'}\}}, \tag{3.23}$$

where

$$I_0 = \int_0^{\sigma_0} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = F(\arcsin(\operatorname{sn} d^*), k), \tag{3.24}$$

and $F(x,k)$ is the elliptic integral of the first kind. If it turns out that the numbers m_0, n_0 given by (3.23) are integers, then the set $\{(\sigma_0, w_0) \in \mathbb{C}_1, m_0, n_0\}$ forms a solution of the problem (3.21). If, however, at least one of the numbers m_0, n_0 is not an integer, then, certainly, $(\sigma_0, w_0) \in \mathbb{C}_2$. In this case equation (3.21) can be rewritten as

$$-\int_0^{\sigma_0} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = d^* + 4\left(n_0 + \frac{1}{2}\right)\mathbf{K} - 2im_0\mathbf{K}'. \tag{3.25}$$

The above relation implies that $\sigma_0 = \operatorname{sn}(-d^* - 2\mathbf{K}) = \operatorname{sn} d^*$. The numbers m_0, n_0 are defined from the equation

$$I_0 + d^* + 4(n_0 + \tfrac{1}{2})\mathbf{K} - 2im_0\mathbf{K}' = 0 \tag{3.26}$$

by

$$m_0 = \frac{\operatorname{Im}\{(I_0 + d^*)\overline{\mathbf{K}}\}}{2\operatorname{Re}\{\mathbf{K}\overline{\mathbf{K}'}\}}, \quad n_0 = -\frac{1}{2} - \frac{\operatorname{Re}\{(I_0 + d^*)\overline{\mathbf{K}'}\}}{4\operatorname{Re}\{\mathbf{K}\overline{\mathbf{K}'}\}}. \tag{3.27}$$

Let $\varepsilon = 1$ if the numbers (3.23) are integers and $\varepsilon = -1$ if the numbers (3.27) are integers. Then the point (σ_0, w_0) is given by $(\operatorname{sn} d^*, \varepsilon q^{1/2}(\operatorname{sn} d^*))$.

We now turn to the canonical function. It is convenient to rewrite formula (3.12) in terms of two functions defined on the z-plane:

$$X(z,w) = \exp\{\Xi_1(z) + w(z)\Xi_2(z)\}, \tag{3.28}$$

where

$$\Xi_1(z) = \frac{1}{4\pi i}\int_L [\log l_1(t) + \log l_2(t)]\frac{dt}{t-z} + \frac{1}{2}\log\frac{z-\sigma_0}{z-\delta_0}$$
$$+\frac{1}{2}\sum_{\mu=1}^{2} \operatorname{sgn}\kappa_\mu \sum_{j=1}^{|\kappa_\mu|} \log\frac{z-\gamma_{\mu j}}{z-t_3},$$

$$\Xi_2(z) = \frac{1}{4\pi i}\int_L [\log l_1(t) - \log l_2(t)]\frac{dt}{q^{1/2}(t)(t-z)} - \frac{1}{2}\int_{-1}^{\delta_0}\frac{dt}{q^{1/2}(t)(t-z)}$$
$$+\frac{\varepsilon}{2}\int_{-1}^{\sigma_0}\frac{dt}{q^{1/2}(t)(t-z)} + m_0\int_1^{1/k}\frac{dt}{q^{1/2}(t^+)(t-z)} + n_0\int_{1/k}^{\infty}\frac{2z\,dt}{q^{1/2}(t)(t^2-z^2)}$$
$$+\frac{1}{2}\sum_{\mu=1}^{2}(-1)^{\mu-1}\operatorname{sgn}\kappa_\mu\sum_{j=1}^{|\kappa_\mu|}\int_{t_3}^{\gamma_{\mu j}}\frac{dt}{q^{1/2}(t)(t-z)}. \tag{3.29}$$

We assume that $\log\{(z-a)(z-b)^{-1}\} \to 0$, $z \to \infty$. This condition determines a single branch of the logarithmic function. The point $(\delta_0, v_0) \in \mathbb{C}_1$ is fixed in an arbitrary manner such that it does not fall on the contour L, cuts γ_0, γ_1 and does not coincide with the poles of the function $F(z,w)$. The final solution is independent of the choice of the point δ_0. The integral from 1 to $1/k$ in (3.29) is taken along the left bank of the cut γ_1.

Thus we have shown that the function (3.28) is a canonical function. This function satisfies the conditions (3.11) at the ends of the contour. The point $(\delta_0, v_0) \in \mathbb{C}_1$ is its simple pole. At the point $(\sigma_0, w_0) \in \mathcal{R}$ the function $X(z,w)$ has a simple zero. The behaviour of the function $X(z,w)$ at the points $p_{\mu j} = (\gamma_{\mu j}, (-1)^{\mu-1}u_{\mu j}) \in \mathbb{C}_\mu$ ($j = 1, 2, \dots, |\kappa_\mu|$; $\mu = 1, 2$) depends on the sign of the numbers κ_μ. If $\kappa_\mu > 0$, then all the points $p_{\mu j}$ are simple zeros of the function $X(z,w)$. Correspondingly, if $\kappa_\mu < 0$, then the points $p_{\mu j}$ are simple poles of the function $X(z,w)$. Finally, in the case $\kappa_\mu = 0$, the function $X(z,w)$ is bounded and is non-zero at the points $p_{\mu j}$.

3.3 *General solution*

For simplicity, assume that the function $d(s)/a(s)$ in (2.1) meets the requirement (2.2). Then from relations (2.25), (2.19) and (3.14) the function $D(t,\xi)[X^+(t,\xi)]^{-1}$ may have integrable singularities at the ends of the contour $\mathcal{L}$. Therefore, a partial solution of the non-homogeneous boundary-value problem (2.24) can be taken as $X(z,w)[\Psi_1(z) + w(z)\Psi_2(z)]$, where

$$\Psi_1(z) = \frac{1}{4\pi i}\int_{\mathcal{L}}\frac{D(t,\xi)}{X^+(t,\xi)}\frac{dt}{t-z}, \quad \Psi_2(z) = \frac{1}{2\pi i}\int_{\mathcal{L}}\frac{D(t,\xi)}{\xi(t)X^+(t,\xi)}\frac{dt}{t-z}. \tag{3.30}$$

The general solution of the Riemann–Hilbert problem (2.24) becomes (**5**)

$$F(z,w) = X(z,w)[\Psi_1(z) + R_1(z)] + w(z)X(z,w)[\Psi_2(z) + R_2(z)], \tag{3.31}$$

where

$$R_1(z) = C_0 + \frac{C_1 w_0}{z-\sigma_0} + \sum_{k=1}^{m}\sum_{j=1}^{\nu_k}\frac{D'_{kj}}{(z-\alpha_k)^j} + \sum_{k=0}^{3}\sum_{j=1}^{(\mu_k-1)/2}\frac{E'_{kj}}{(z-z_k)^j} - \sum_{\mu=1}^{2}(-1)^\mu\sum_{j=1}^{\tilde\kappa_\mu}\frac{H_{\mu j}u_{\mu j}}{z-\gamma_{\mu j}},$$

$$R_2(z) = \frac{C_1}{z-\sigma_0} + \sum_{k=1}^{m}\sum_{j=1}^{\nu_k}\frac{D''_{kj}}{(z-\alpha_k)^j} + \sum_{k=0}^{3}\sum_{j=1}^{(\mu_k+1)/2}\frac{E''_{kj}}{(z-z_k)^j} + \sum_{\mu=1}^{2}\sum_{j=1}^{\tilde\kappa_\mu}\frac{H_{\mu j}}{z-\gamma_{\mu j}}. \tag{3.32}$$

Here $\tilde\kappa_\mu = \max\{0, \kappa_\mu\}$, $\mu = 1, 2$; $u_{\mu j} = q^{1/2}(\gamma_{\mu j})$, $\sigma_0 = \operatorname{sn} d^*$, $w_0 = \varepsilon q^{1/2}(\sigma_0)$, $z_0 = -1/k$, $z_1 = -1$, $z_2 = 1$, $z_3 = 1/k$.

Assume that among the poles α_k ($k = 1, 2, \ldots, m$) there is no infinity. Then, for the function $F(z, w)$ to be bounded at infinity, it is necessary and sufficient that

$$\operatorname*{res}_{z=\infty} [\Psi_2(z) + R_2(z)] = 0. \tag{3.33}$$

If, however, the function $F(z, w)$ has poles of the same order ν_∞ at the points (∞, ∞) and $(\infty, -\infty)$, then the condition (3.33) should be disregarded and the terms

$$\sum_{j=1}^{\nu_\infty} G'_j z^j, \quad \sum_{j=0}^{\nu_\infty - 2} G''_j z^j \tag{3.34}$$

need to be added to the functions $R_1(z)$, $R_2(z)$, respectively. Note that the second term in (3.34) is added only if $\nu_\infty \geqslant 2$.

The procedure of solution of the Riemann–Hilbert problem will be accomplished if the following conditions hold:

$$\Psi_1(\delta_0) + R_1(\delta_0) + q^{1/2}(\delta_0)[\Psi_2(\delta_0) + R_2(\delta_0)] = 0,$$
$$\Psi_1(\gamma_{\mu j}) + R_1(\gamma_{\mu j}) + (-1)^{\mu-1} u_{\mu j}[\Psi_2(\gamma_{\mu j}) + R_2(\gamma_{\mu j})] = 0,$$
$$j = 1, 2, \ldots, -\kappa_\mu, \quad \mu = 1, 2. \tag{3.35}$$

The first condition removes the simple pole at the point $(\delta_0, v_0) \in \mathbb{C}_1$ of the function $F(z, w)$. The conditions at the points $\gamma_{\mu j}$ are required for $\kappa_\mu < 0$ and provide the boundedness of the solution at these points.

The general solution to the functional-difference equation (2.3) is given by (2.42). In addition to formula (2.42) we give another representation of the solution without functions on the Riemann surface. Let

$$Y_1(z) = \Psi_1(z) + R_1(z), \quad Y_2(z) = q^{1/2}(z)[\Psi_2(z) + R_2(z)], \tag{3.36}$$

with $z = u(s)$. In the strip $\omega - h \leqslant \operatorname{Re} s \leqslant \omega + h$, $f(s) = \hat{f}(s)$. Therefore

$$\begin{aligned}
f(s) &= 2e^{\Xi_1(z)}\{Y_1(z)\cosh[q^{1/2}(z)\Xi_2(z)] + Y_2(z)\sinh[q^{1/2}(z)\Xi_2(z)]\}, \quad \omega - h \leqslant \operatorname{Re} s \leqslant \omega,\\
f(s) &= -\frac{b(s)}{a(s)}e^{\Xi_1(z)}\{Y_1(z)\cosh[q^{1/2}(z)\Xi_2(z)] + Y_2(z)\sinh[q^{1/2}(z)\Xi_2(z)]\}\\
&\quad + \frac{\Delta^{1/2}(s)}{a(s)}e^{\Xi_1(z)}\{Y_1(z)\sinh[q^{1/2}(z)\Xi_2(z)]\\
&\quad + Y_2(z)\cosh[q^{1/2}(z)\Xi_2(z)]\}, \quad \omega \leqslant \operatorname{Re} s \leqslant \omega + h.
\end{aligned} \tag{3.37}$$

Formulae (2.43) to (2.45) define the solution $f(s)$ to equation (2.1) in the whole complex plane $\mathbb{C}$.

Assume first that $z = \infty$ is not a pole of the function $F(z, w)$. Then analysis of formulae (3.32) shows that the number of arbitrary constants in the general solution $f(s)$ is

$$2 + 2\sum_{k=1}^{m} \nu_k + \sum_{k=0}^{3} \mu_k + \tilde{\kappa}_1 + \tilde{\kappa}_2. \tag{3.38}$$

For their definition we have $2 + \tilde{\kappa}_1 + \tilde{\kappa}_2 - \kappa_1 - \kappa_2$ conditions (3.33) and (3.35). If $F(z, w)$ has poles at the infinite points on either sheet, then the difference between the number of the arbitrary constants and the number of the conditions is given by

$$2\sum_{k=1}^{m} \nu_k + \sum_{k=0}^{3} \mu_k + \kappa_1 + \kappa_2 + 2\nu_\infty. \tag{3.39}$$

The zeros of the function $a(s)$ in the strip $\omega \leqslant \operatorname{Re} s \leqslant \omega + h$ may bring inadmissible poles of the solution $f(s)$. Their number and multiplicity may increase the total number of additional conditions for the arbitrary constants.

4. Conclusion

In this paper we have proposed an analytical method for a scalar second-order functional-difference equation (2.1) whose coefficients are h-periodic and entire functions. The method is still applicable if only the functions $b(s)/a(s)$ and $c(s)/a(s)$ are h-periodic and if the coefficients are meromorphic functions. It has been shown that the solution of equation (2.1) can be constructed by analytical continuation of the general solution of an auxiliary boundary-value problem (2.2) in a strip $\Pi = \{s \in \mathbb{C} : \omega - h < \operatorname{Re} s < \omega + h\}$, where ω is a real number. The auxiliary problem has been reduced to a vector functional-difference equation of the first order (2.6) in the strip $\Pi = \{s \in \mathbb{C} : \omega - h < \operatorname{Re} s < \omega\}$. We have shown how to transform the above problem to a scalar Riemann–Hilbert boundary-value problem (2.24) on two finite segments of a hyperelliptic surface of genus ρ, where $2\rho + 2$ is the number of branch points of the function $\Delta^{1/2}(s) = [b^2(s) - 4a(s)c(s)]^{1/2}$ in the strip Π. A general technique for solution of such problems for arbitrary finite genus ρ (**9**) requires solution of the corresponding Jacobi inversion problems. This nonlinear problem is always solvable; it is equivalent to an algebraic equation of degree ρ (**12**) and therefore can be solved effectively. To illustrate the technique proposed, we have analysed in detail the elliptic case ($\rho = 1$) and constructed a closed-form solution of equation (2.1) in terms of elliptic functions.

The case when the shift is less than the period of the coefficients and a physical example analysed by the method of this paper are presented in (**5**).

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APPENDIX

We aim to show that the solution (**6**, **8**), the function $\Phi_1(\alpha) + \Phi_2(\alpha)$, is a multi-valued function with branch points $\pi + 2\pi m_1$, $m_1 \in \mathbb{Z}$, and $\pm\eta + \pi m_2$, m_2 is an integer such that $\pm\eta + \pi m_2 \notin \Pi$, $\Pi = \{\alpha \in \mathbb{C} : -\pi < \operatorname{Re}\alpha < \pi\}$. Here

$$\Phi_{1,2}(\alpha) = [F(\alpha)]^{\pm 1}, \quad |\operatorname{Re}\alpha| \leqslant \pi,$$

$$F(\alpha) = \exp\left\{\frac{1}{4\pi i}\int_{-i\infty}^{i\infty} \tan\frac{z-\alpha}{2}\log q(z)dz\right\}, \tag{A.1}$$

and we adopt the notation of (**6**):

$$q(\alpha) = \frac{u(\alpha) + \frac{1}{2}\sin\theta}{u(\alpha) - \frac{1}{2}\sin\theta},$$

$$u(\alpha) = \sqrt{\cos^2\alpha - \cos^2\eta}, \quad \cos\eta = \frac{\sqrt{3}}{2}\sin\theta. \tag{A.2}$$

A single-valued branch of the function $u(\alpha)$ is defined in the α-plane cut along the segments joining the pairs of the points $-\eta + m\pi$ and $\eta + m\pi$, $m \in \mathbb{Z}$. Because of this choice of the branch cuts, the point $z = 0$ lies on the cut joining the points $-\eta$ and η. Hence the density of the integral (A.1) is discontinuous at the point $z = 0$:

$$\log q(+i0) - \log q(-i0) = \delta_0, \tag{A.3}$$

where

$$\delta_0 = 2\log q(+i0) = 2\log\frac{\sqrt{4 - 3\sin^2\theta} + \sin\theta}{\sqrt{4 - 3\sin^2\theta} - \sin\theta}. \tag{A.4}$$

In a neighbourhood of the points $\alpha = \pm\pi$, the integral (A.1) behaves as the corresponding Cauchy integral with the discontinuous density $\log q(z)$. On using (**15**),

$$F(\beta \pm \pi) = \beta^{\delta_1}\Phi_*(\beta), \quad \beta \to 0, \quad \delta_1 = \frac{\delta_0}{2\pi i}. \tag{A.5}$$

The function $\Phi_*(\beta)$ has definite finite non-zero limits as $\beta \to 0$ and $\operatorname{Re}\beta \neq 0$. Therefore

$$\Phi_1(\alpha) + \Phi_2(\alpha) = (\alpha \mp \pi)^{\delta_1}\Phi_0(\alpha) + \frac{1}{(\alpha \mp \pi)^{\delta_1}\Phi_0(\alpha)}, \quad \alpha \to \pm\pi, \quad |\operatorname{Re}\alpha| \leqslant \pi, \tag{A.6}$$

and the function $\Phi_0(\alpha)$ is finite. Its limits as $\alpha \to \pm\pi$ exist and do not equal zero.

The functions $\Phi_{1,2}(\alpha)$ may be continued analytically into the strip $\pi \leqslant \operatorname{Re}\alpha \leqslant 3\pi$ as follows:

$$\Phi_{1,2}(\alpha) = q^{\mp 1}(\alpha - \pi)\Phi_{1,2}(\alpha - 2\pi). \tag{A.7}$$

Because of the relation $q(\alpha - \pi) = 1/q(\alpha)$ and the 2π-periodicity of the kernel $\tan\frac{1}{2}(z - \propto)$ we have

$$\Phi_{1,2}(\alpha) = [q(\alpha)F(\alpha)]^{\pm 1}, \quad \pi \leqslant \operatorname{Re}\alpha \leqslant 3\pi. \tag{A.8}$$

In the vicinities of the points $\alpha = \pi$, $\alpha = 3\pi$, bisected along the branch cuts, the function $q(\alpha)$ is single-valued. Therefore, if $\alpha \to \pi$ or $\alpha \to 3\pi$, then the sum $\Phi_1(\alpha) + \Phi_2(\alpha)$ may be written similarly to (A.6). Now it is clear that if the parameter δ_1 is not integer, then all the points $\pi + 2\pi m_1$ ($m_1 \in \mathbb{Z}$) are branch points of the function $\Phi_1(\alpha) + \Phi_2(\alpha)$. We notice that δ_1 is an integer if and only if

$$\frac{\sqrt{4 - 3\sin^2\theta} + \sin\theta}{\sqrt{4 - 3\sin^2\theta} - \sin\theta} = \pm 1 \tag{A.9}$$

or, equivalently, if either $\eta = 0$, or $\eta = \pi/2$. In both cases the function $u(s)$ does not have branch points.

Analyse next the points $\pm\eta + \pi m_2$, $m_2 \in \mathbb{Z}$. Those points which belong to the strip $-\pi \leqslant \operatorname{Re}\alpha \leqslant \pi$ are not branch points. Show that the next pair of the points are branch points. Let Γ_1 be a cut joining the points $2\pi - \eta$ and $2\pi + \eta$. On the banks of the cut,

$$\Phi_1(\alpha^\pm) + \Phi_2(\alpha^\pm) = q(\alpha^\pm)F(\alpha^\pm) + \frac{1}{q(\alpha^\pm)F(\alpha^\pm)}, \quad \alpha^\pm \in \Gamma_1^\pm. \tag{A.10}$$

Clearly, the function $F(\alpha)$ is continuous on the cut, $F(\alpha^+) = F(\alpha^-)$. As for the function $q(\alpha)$, it is discontinuous and $q(\alpha^-) = 1/q(\alpha^+)$. This means that the limit values of the function $\Phi_1(\alpha) + \Phi_2(\alpha)$ on the left and the right sides of the cut are not the same, and the points $\pm\eta + \pi m_2 \notin \Pi$ are branch points of the solution. In summary, the method presented in (**6**, **8**) yields a solution that is multi-valued in the exterior of the strip $-\pi \leqslant \operatorname{Re} s \leqslant \pi$ regardless of the choice of the system of the branch cuts.