Math 7410 Graph Theory

Bogdan Oporowski

Department of Mathematics
Louisiana State University

January 17, 2023

Definition of a graph

Definition 1.1

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Example 1.2

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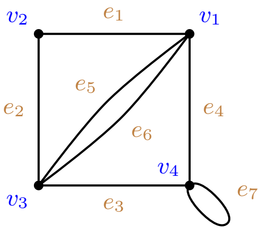

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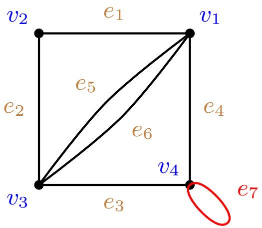

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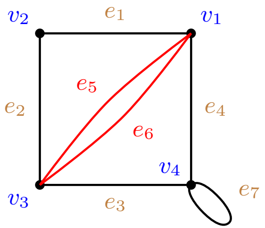

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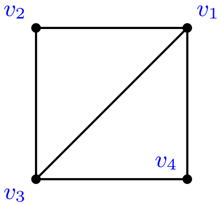

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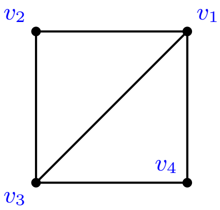

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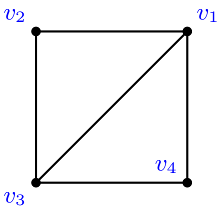

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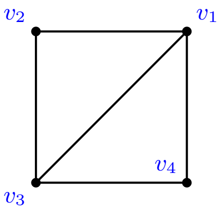

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Note 1.5

In some books, what we defined as a graph is called a multigraph and what we defined as a simple graph is called a graph.

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- The set of neighbors of a vertex v of G, other than v itself, is denoted by $N(v)$ or by $N_G(v)$.
- Similarly, if U is a subset of the vertex set of G, then $N(U)$ is the set of those vertices that are not in U, but are adjacent to a vertex in U.

Isomorphism

Definition 1.7

The graphs $G_1 = (V_1, E_1, \mathcal{I}_1)$ and $G_2 = (V_2, E_2, \mathcal{I}_2)$ are isomorphic, written $G_1 \cong G_2$, if there are bijections $\varphi : V_1 \to V_2$ and $\psi : E_1 \to E_2$ such that $(v, e) \in \mathcal{I}_1$ if and only if $(\varphi(v), \psi(e)) \in \mathcal{I}_2$. Such a pair of bijections is an isomorphism.

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Graph isomorphism problem can be solved in quasi-polynomial time.
There is a constant c and an algorithm that can decide whether two graphs on n vertices are isomorphic or not in at most $2^{\mathcal{O}((\log n)^c)}$ steps.

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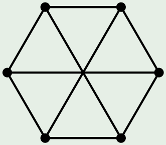
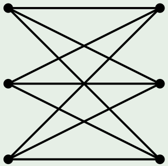
Example 1.10

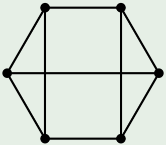
Which of the following graphs are isomorphic?



G_1



G_2



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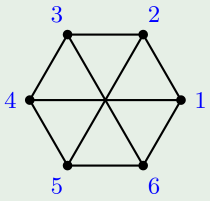

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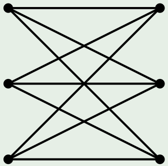

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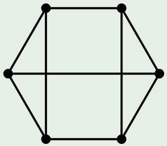

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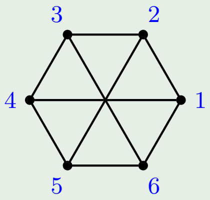

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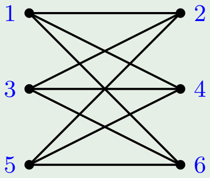

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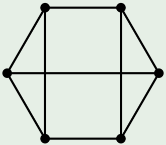

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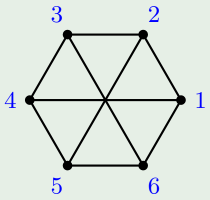

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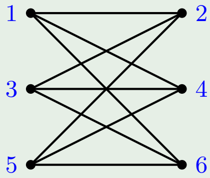

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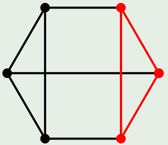

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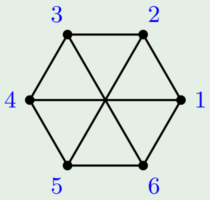

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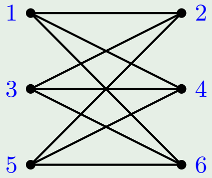

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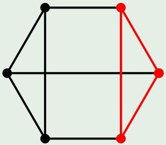

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Problem 1

For every positive integer n, construct a simple graph with exactly n automorphisms.

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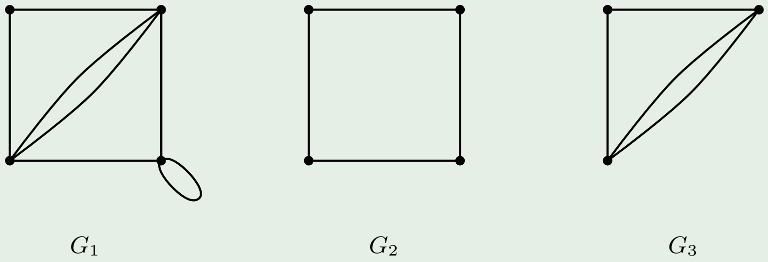
Subgraph Example

Example 1.16



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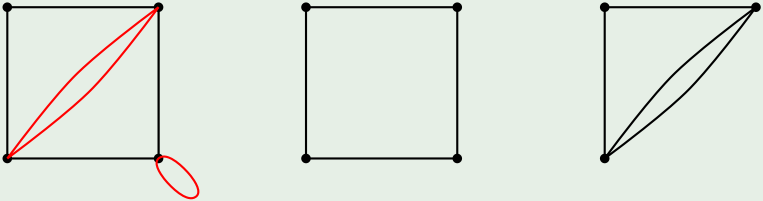

G_1 G_2 G_3

- G_2 is a subgraph of G_1, but it is not an induced subgraph.

Subgraph Example

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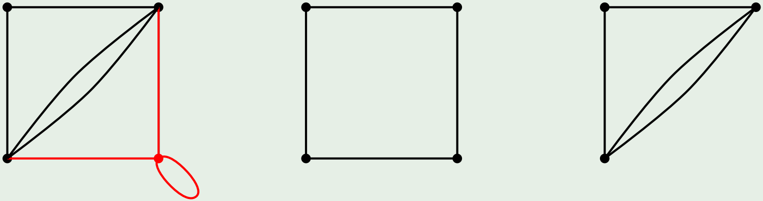

G_1 G_2 G_3

- G_2 is a subgraph of G_1, but it is not an induced subgraph.
- G_3 is an induced subgraph of G_1.

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Conjecture 1.18 (Edge-Reconstruction Conjecture)

Every simple graph on at least four edges is edge-reconstructible.

Walks, Trails, Paths, and Cycles

Definition 1.19

- A walk is a sequence v_0, e_1, v_1, e_2, v_2, ..., e_n, v_n, where each edge e_i is incident with vertices v_{i-1} and v_i.

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(b) *Construct a self-complementary graph of order n for every positive integer n congruent to 0 or 1 modulo 4.*

Hand-Shaking Lemma

Theorem 1.21 (Hand-Shaking Lemma)

$$\sum_{v \in V(G)} d(v) = 2\|G\|$$

Corollary 1.22

The number of vertices of odd degree is even.

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Theorem 2.2

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But P cannot contain v as $d_T(v) = 1$, and so it also lies in T'. $\square$

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The remainder of the proof is left as an exercise.

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If T and T' are two spanning trees of a connected graph G and $e \in E(T) \setminus E(T')$, then there is an edge $e' \in E(T') \setminus E(T)$ such that $T \setminus e \cup e'$ is a spanning tree of G.

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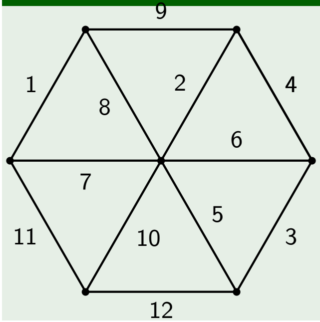

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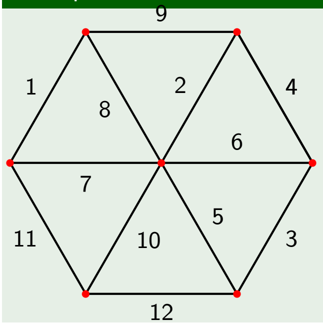

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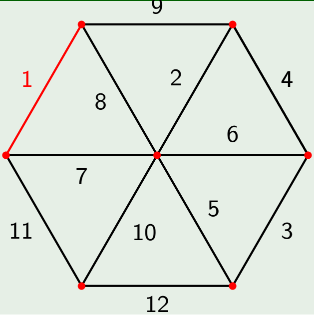

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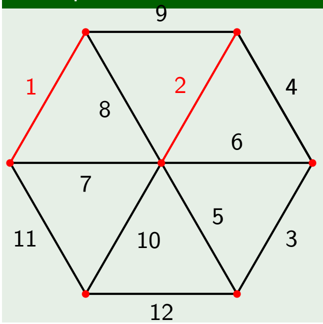

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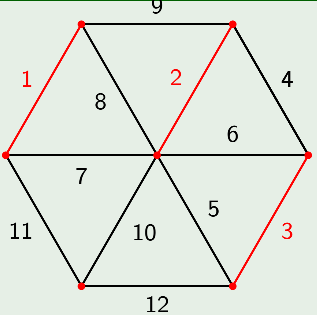

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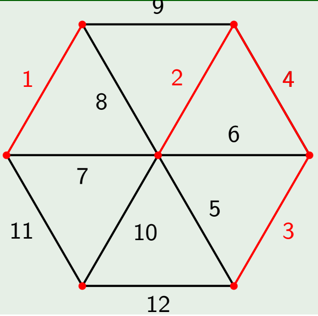

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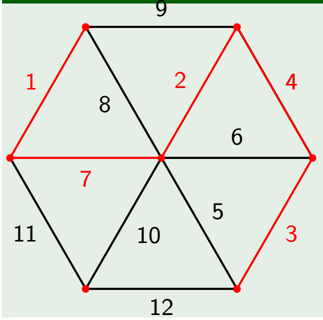

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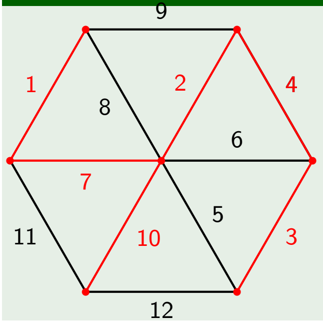

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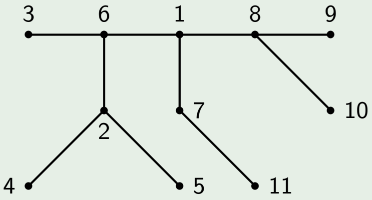

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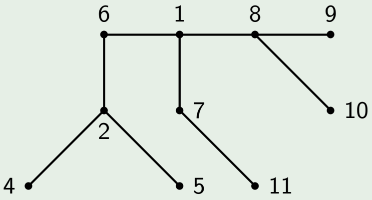

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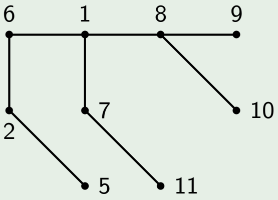

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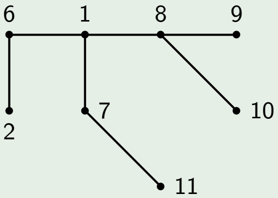

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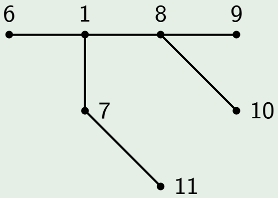

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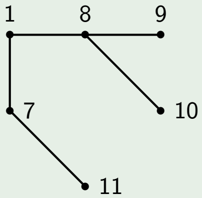

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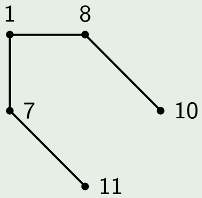

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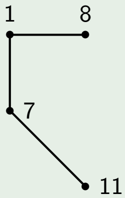

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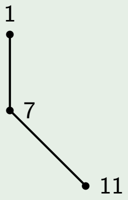

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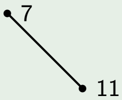

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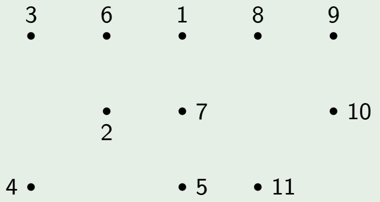

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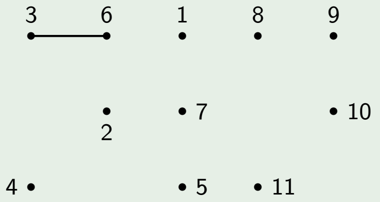

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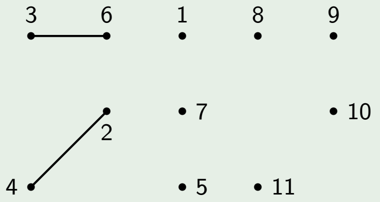

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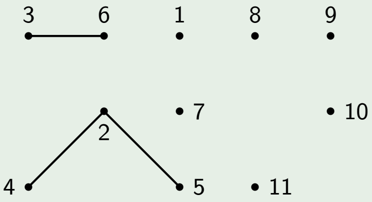

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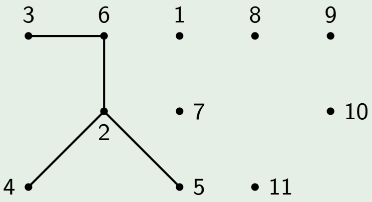

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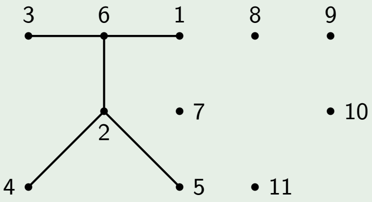

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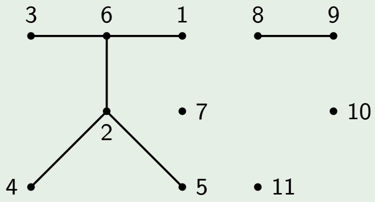
Sequence: 6, 2, 2, 6, 1, 8, 8, 1, 7

Finished: 3, 4, 5, 2, 6, 9



Trees from Sequences

Now we describe how to produce a tree from a Prüfer sequence.

- Begin with a forest having n isolated vertices labeled 1, 2, $\ldots$, n.
- Proceed with all $n-2$ elements of the sequence, and, at the ith step,
  - let x be the label in position i.
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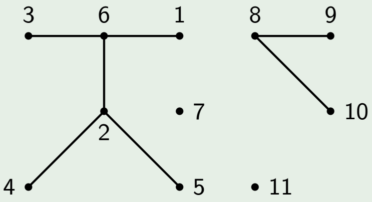
  - add the edge xy, and
  - mark y as finished.
- Join the two remaining unfinished vertices with an edge.

Example 2.10

Sequence: 6, 2, 2, 6, 1, 8, 8, 1, 7 Finished: 3, 4, 5, 2, 6, 9, 10



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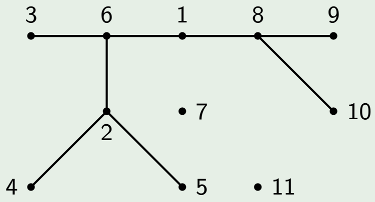
Example 2.10

Sequence: 6, 2, 2, 6, 1, 8, 8, 1, 7   Finished: 3, 4, 5, 2, 6, 9, 10, 8



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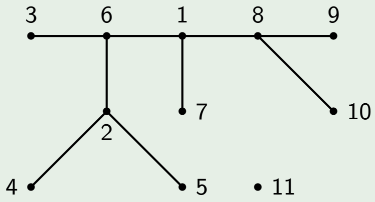
Example 2.10

Sequence: 6, 2, 2, 6, 1, 8, 8, 1, 7

Finished: 3, 4, 5, 2, 6, 9, 10, 8, 1



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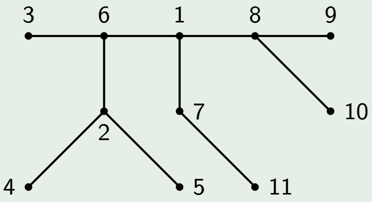

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Counting Trees with Prescribed Degrees

Corollary 2.11

The number of trees with vertex set $\{1, 2, \ldots, n\}$ *in which vertices* 1, 2, $\ldots$, n *have respective degrees* d_1, d_2, $\ldots$, d_n *is*

$$\frac{(n-2)!}{\prod (d_i - 1)!}.$$

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Therefore we count the trees by counting sequences of length $n - 2$ having $d_i - 1$ copies of i, for each i. If we distinguish between various copies of i, then there are $(n-2)!$ such sequences. Since we really cannot distinguish between the copies, we have over-counted by a factor of $(d_i - 1)!$ for each i. □

Minors

Definition 2.12

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Note 2.13

The order of operations of deleting and contracting to get a minor of a graph is irrelevant.

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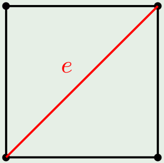

G

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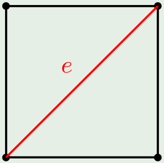
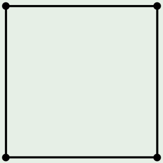


G



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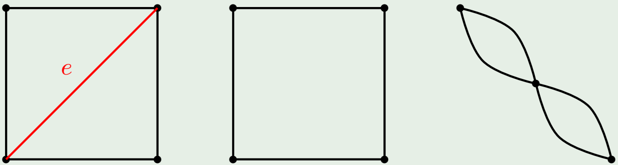

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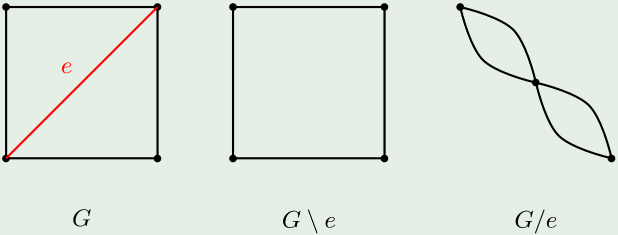

$$\tau(G) = \tau(G \setminus e) + \tau(G/e) = 4 + 4 = 8$$

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Using the deletion-contraction formula for calculating the number of spanning trees is inefficient. A much more efficient method is to construct a special matrix, called the Laplacian of the graph, and to compute its determinant.

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Definition 3.1

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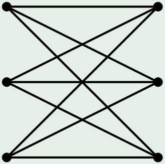

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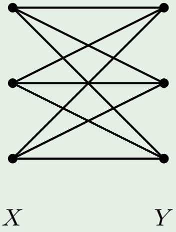

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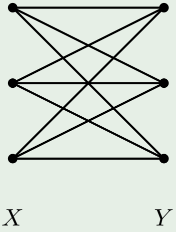

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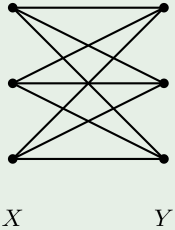

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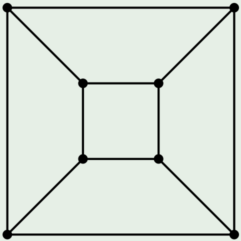

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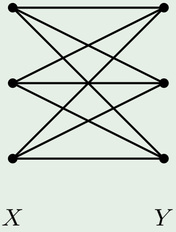

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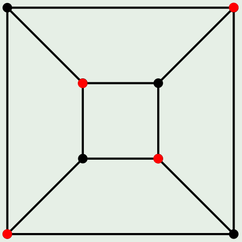

Characterization of Bipartite Graphs

Theorem 3.3

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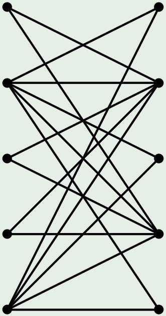

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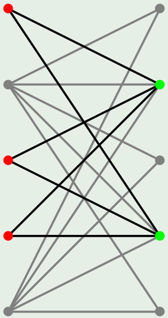

Does G have a matching that saturates all vertices on the left side?
No! Look at S, which has 3 elements, and $N(S)$, which has only 2 elements.

Hall's Marriage Theorem

Theorem 3.6 (Hall's Marriage Theorem, 1935)

Suppose G is a bipartite graph with bipartition $\{X, Y\}$. The graph G has a matching saturating X if and only if $|N(S)| \geq |S|$ for every subset S of X.

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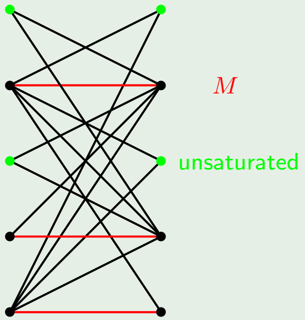

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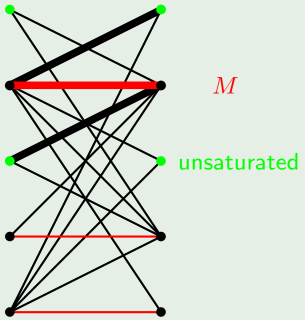

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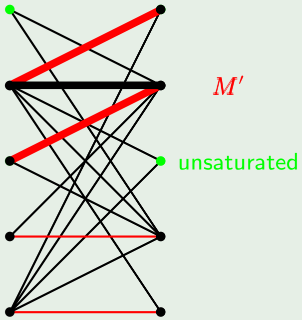

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Easy Direction.

Since distinct vertices must be used to cover the edges of a matching, we have $|U| \geq |M|$ whenever U is a vertex cover and M is a matching. $\square$

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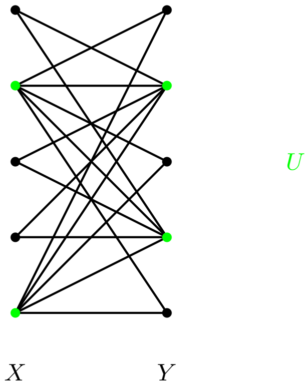

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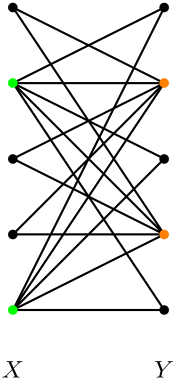

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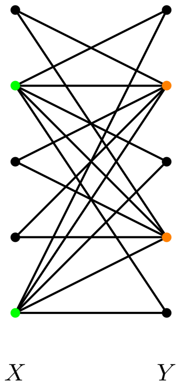

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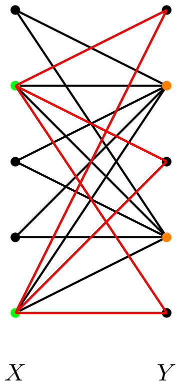

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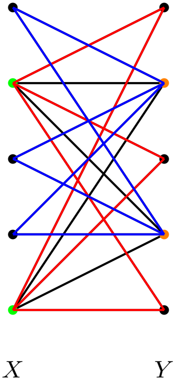

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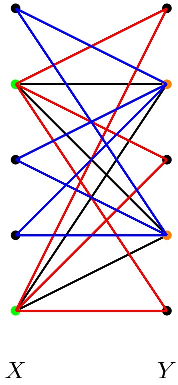

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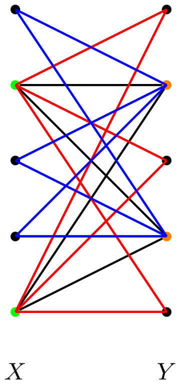

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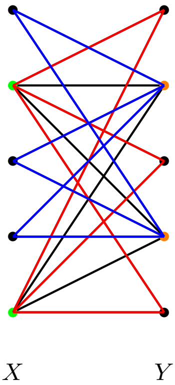

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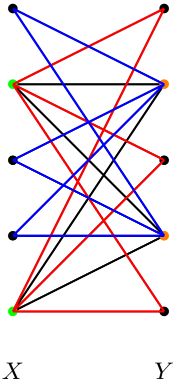

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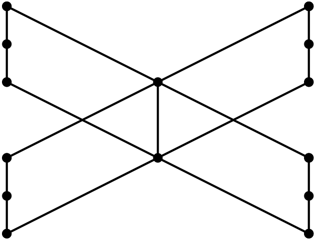

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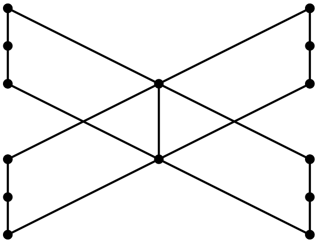

No, since removing the two vertices in the middle leaves more than two components of odd order.

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Note that after (1)–(3) are established, the proof is complete.

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Proof of Tutte's 1-Factor Theorem, Continued

Now we turn to (3), which states that G contains a set s_1v_1, s_2v_2, ..., s_mv_m of edges such that $S_0 = \{s_1, s_2, \ldots, s_m\}$ and $v_i \in V(Q_i)$ for all i.

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Homework Set 2

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Derive the sufficiency (the non-obvious direction) of the Hall's Marriage Theorem from the Tutte's 1*-Factor Theorem.*

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Problem 6

Prove that a tree T *has a perfect matching if and only if* $q(T - v) = 1$ *for every* $v \in V(T)$*. Do not invoke Tutte's* 1*-Factor Theorem.*

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We prove that G satisfies the Tutte Condition. Let $S \subseteq V(G)$, and count the edges between S and the odd components of $G - S$. Since G is 3-regular, every vertex in S is incident to at most three such edges.

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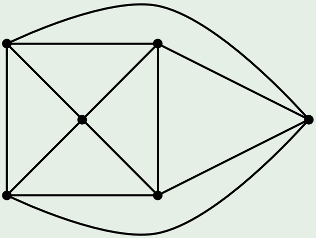

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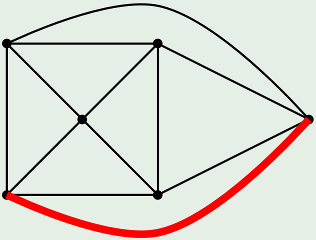

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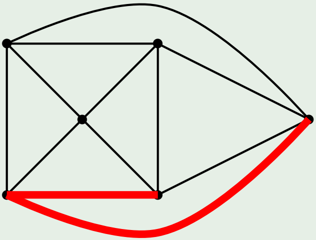

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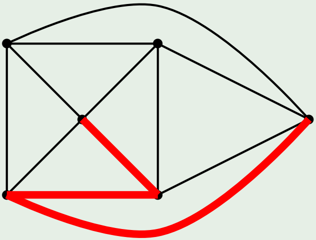

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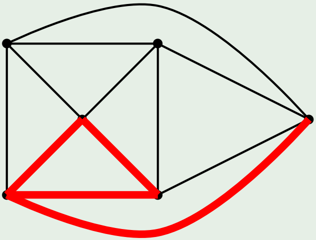

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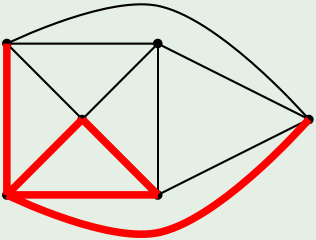

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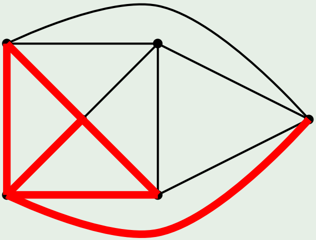

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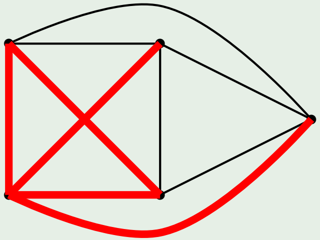

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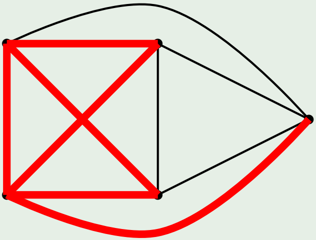

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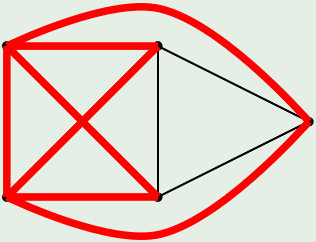

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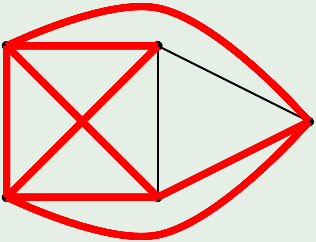

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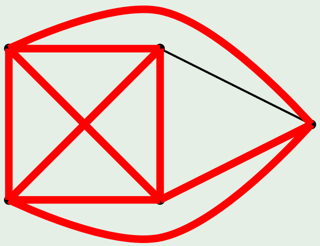

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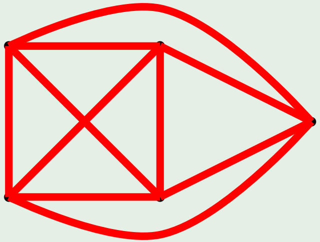

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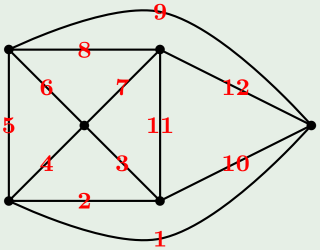

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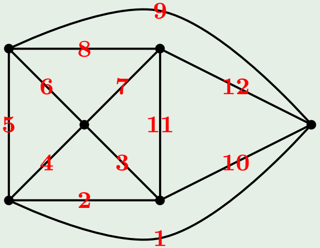

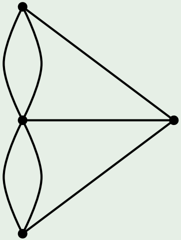

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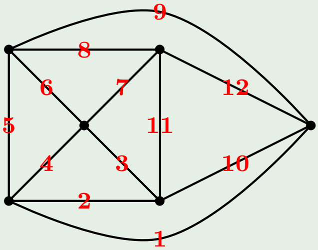

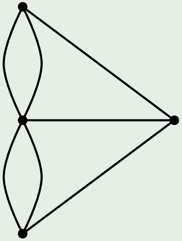

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Characterization of Eulerian Graphs

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Let T be a maximal non-trivial trail in some graph G with all degrees even. Since T is maximal, it includes all edges of G incident with its final vertex v. If T is not closed, then the degree of v must be odd, which is impossible. $\square$

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- An n-wheel W_n is obtained from C_n by adding a new vertex and joining it to all vertices of C_n. If $n \geq 3$, then $\kappa(W_n) = 3$.

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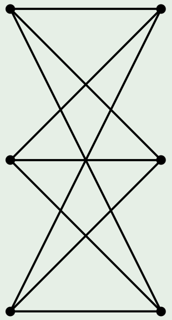

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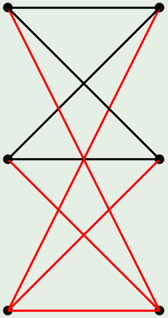

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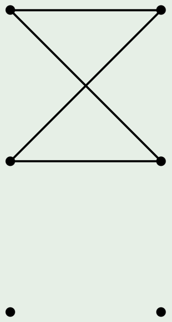

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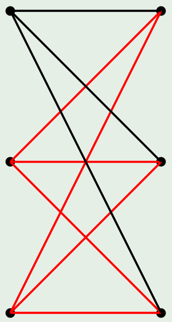

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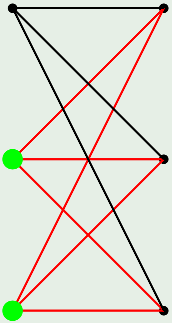

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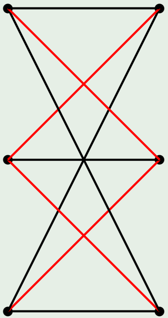

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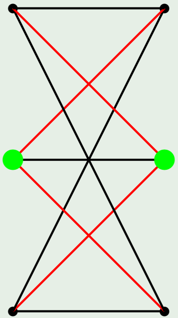

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The edges incident to a vertex form a disconnecting set, so $\kappa' \leq \delta$. Clearly, $\kappa(G) \leq |G| - 1$. Suppose $[S, \overline{S}]$ is a minimum edge cut of size $k' = \kappa'(G)$. If every vertex in S is adjacent to every vertex in $\overline{S}$, then $k' = |S||\overline{S}| \geq |G| - 1$, and the inequality follows. Hence we may assume that there are vertices $s \in S$ and $\overline{s} \in \overline{S}$ that are non-adjacent. Let T be the vertex set consisting of all neighbors of s in $\overline{S}$ and all vertices in $S - s$ that have neighbors in $\overline{S}$.

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Whitney's Theorem

Note 5.6

The edge connectivity of a graph is unaffected by adding or deleting loops, but is affected by adding and deleting edges in parallel.

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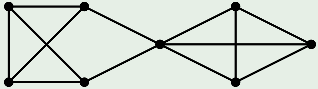
Example 5.8



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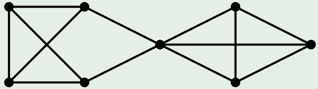

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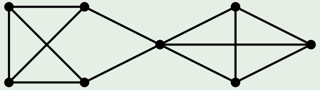

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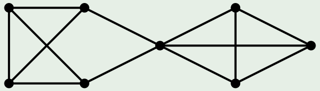

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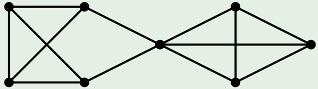

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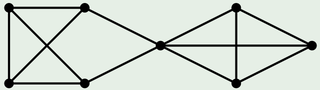

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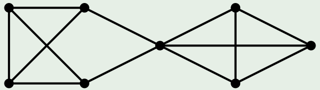

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If $G \setminus F$ has more than two components, then we may assume $S = A \cup B$ with no edges between A and B. Then $[A, \overline{A}]$ is an edge cut which is a proper subset of F; a contradiction. □

Tutte Connectivity

Definition 5.10

- A *k-separation* of a graph G is a pair of subgraphs $\{A, B\}$ of G such that each of A and B has size at least k, $A \neq G$, $B \neq G$, $A \cup B = G$, and $A \cap B$ is trivial of order at most k.

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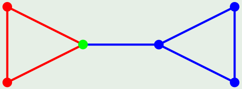

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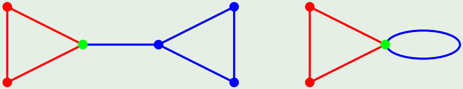

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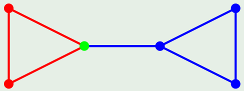

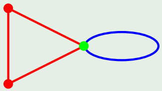

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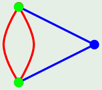

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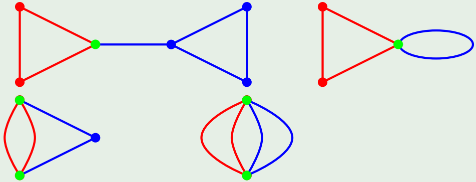

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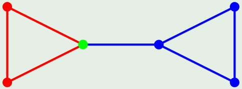

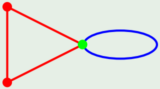

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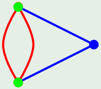

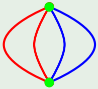

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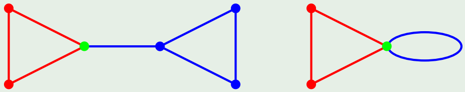

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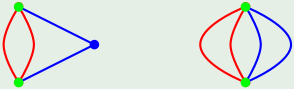

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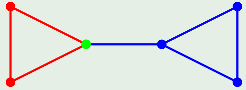
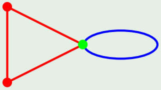
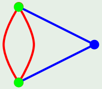
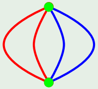
1-separations:  

2-separations:  

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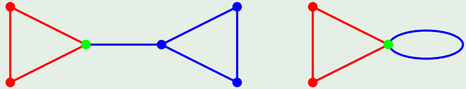

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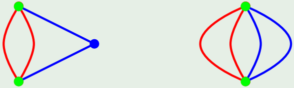

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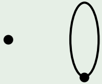

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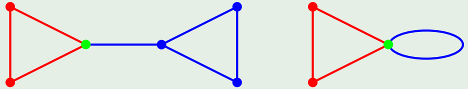

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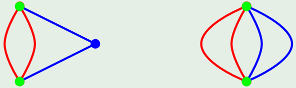

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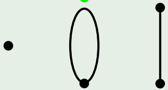

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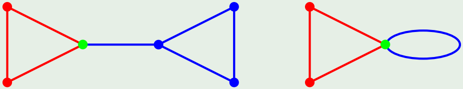

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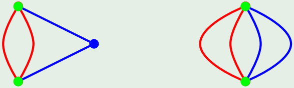

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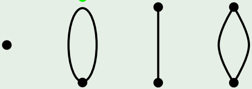

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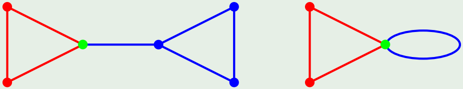

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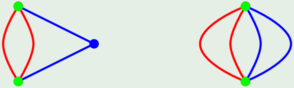

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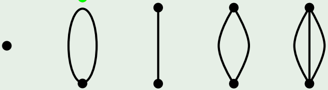

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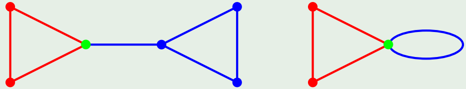

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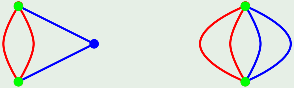

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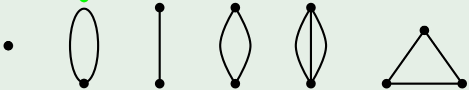

Tutte Connectivity vs. Vertex Connectivity

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If G is a graph on at least 3 vertices and $G \ncong K_3$, then the Tutte connectivity of G is $\min(\kappa(G), g(G))$, where $g(G)$ is the girth of G, that is, the length of a shortest cycle in G.

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- A block of a graph G is a maximal subgraph of G that has Tutte connectivity at least 2.

Tutte Connectivity vs. Vertex Connectivity

Theorem 5.12

If G is a graph on at least 3 vertices and $G \not\cong K_3$, then the Tutte connectivity of G is $\min(\kappa(G), g(G))$, where $g(G)$ is the girth of G, that is, the length of a shortest cycle in G.

Proof: Exercise.

Definition 5.13

- A component of a graph G is a maximal subgraph of G that has Tutte connectivity at least 1.
- A block of a graph G is a maximal subgraph of G that has Tutte connectivity at least 2.

Note 5.14

A block of a non-empty graph is an isolated vertex, a loop-graph, a graph on two vertices with a positive number of edges between those vertices, or is vertex-2-connected.

Block Tree

Note 5.15

Two distinct blocks in a graph share at most one vertex since otherwise their union would be Tutte-2-connected.

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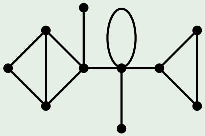

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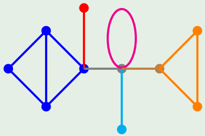

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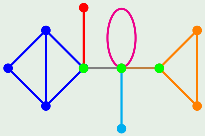

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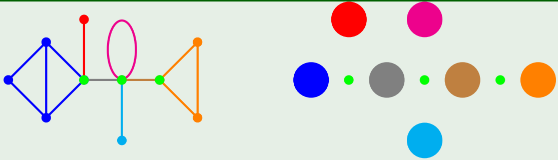

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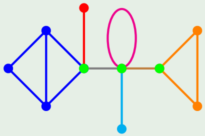

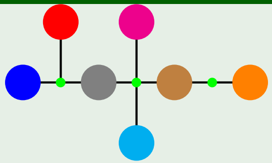

Whitney's Characterization of 2-Connected Graphs

Definition 5.18

Two paths are internally-disjoint if neither contains a non-endpoint of the other.

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If G has two internally-disjoint uv-paths, then deletion of one vertex cannot separate u from v. Hence G has no one-element vertex-cuts and so is 2-connected. $\square$

Proof of Sufficiency

For the converse, suppose that G is 2-connected.

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Lemma 5.20 (Expansion Lemma)

If G is a k-connected graph and G' is obtained from G by adding a new vertex y adjacent to at least k vertices of G, then G' is also k-connected.

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Whitney's Theorem 5.19 establishes the equivalence of (A) and (B). Clearly, (B) and (C) are equivalent. To see that (D) implies (C), apply (D) to edges incident to the desired x and y. $\square$

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Subdivisions and 2-Connectedness

Corollary 5.23

A subdivision of a 2-connected graph is also 2-connected.

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Suppose G' is formed by subdividing an edge uv of G with a new vertex w. By Theorem 5.21, it suffices to find a cycle through two arbitrary edges e and f of G'.

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Ears

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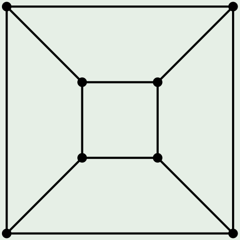

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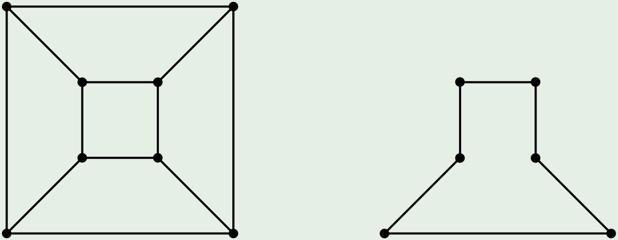

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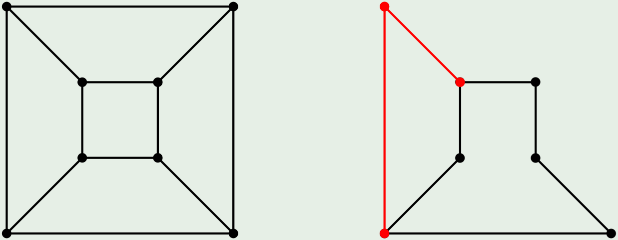

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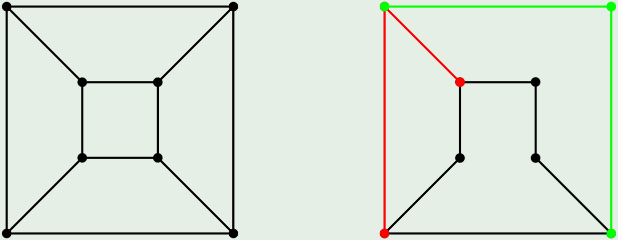

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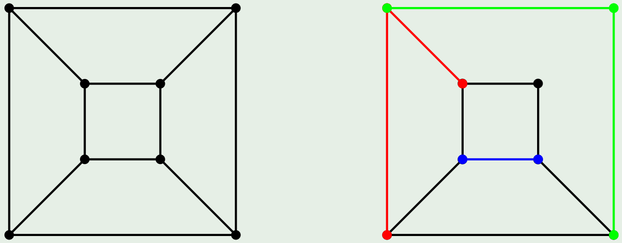

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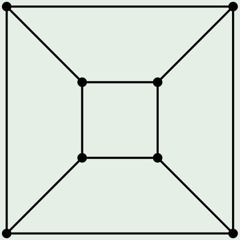

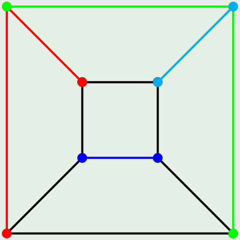

Whitney's Ear Decomposition

Theorem 5.26 (Whitney's Ear Decomposition)

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Closed-Ear Decomposition

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A simple graph is 2-edge-connected if and only if it has a closed-ear decomposition. Moreover, every cycle in a 2-edge-connected graph is the initial cycle in some closed-ear decomposition.

Proof omitted.

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Theorem 5.29 (Menger 1927)

If x and y are non-adjacent distinct vertices of a graph G, then the minimum size of a vertex-cut separating x from y equals the maximum number of pairwise internally-disjoint xy-paths.

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To prove the opposite inequality, we use induction on $|G|$. If $|G| = 2$, then $\kappa(x, y) = \lambda(x, y) = 0$. For the induction step, suppose $|G| > 2$ and let $k = \kappa(x, y)$; we construct k pairwise internally-disjoint xy-paths. $\square$

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If G has a vertex v outside $\{x, y\} \cup N(x) \cup N(y)$, then v is in no minimum xy-vertex-cut; hence $\kappa_{G-v}(x, y) = k$ and applying the induction hypothesis to $G - v$ yields the desired xy-paths in G.

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The Edge Version of Menger's Theorem

Theorem 5.30 (Edge Version of Menger's Theorem)

If x and y are distinct vertices of a graph, then the minimum size $\kappa'(x,y)$ of the set of edges that separate x from y equals the maximum number $\lambda'(x,y)$ of pairwise edge-disjoint xy-paths.

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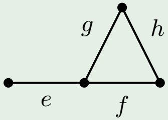

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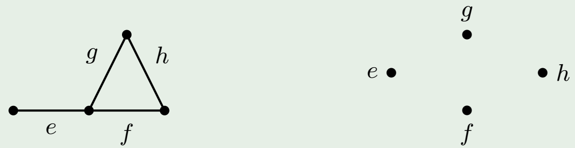

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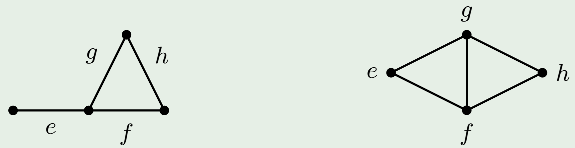

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The edge version follows immediately from Theorem 5.30 since $\kappa'(G) = \min_{x,y \in V(G)} \kappa'(x, y)$.

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Tutte's Wheel Theorem

Theorem 5.34 (Tutte's Wheel Theorem)

If G is a Tutte-3-connected graph on at least four vertices that is not a wheel, then there is an edge e of G such that at least one of G/e and $G \setminus e$ is also Tutte-3-connected.

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Lemma 5.35 (Thomassen 1980)

Every 3-connected graph G on at least five vertices has an edge e such that G/e is 3-connected.

Proof of the Lemma

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Clique Sums

Definition 5.36

- A clique-sum of two graphs G and H is obtained from the disjoint union of G and H by identifying a complete subgraph of G with a complete subgraph (of the same order) of H, and then deleting the edges of the identified subgraph.

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Decomposition of 2-Connected Graphs

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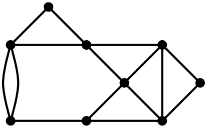

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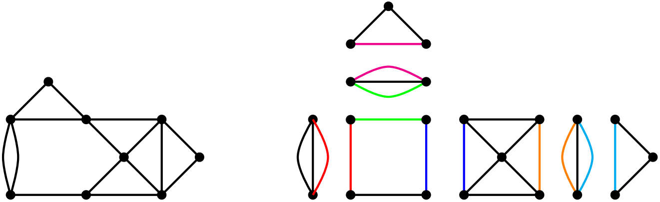

Homework Set 3

Problem 7

Suppose G is a graph that is non-trivial, connected, and such that every edge e is in some two cycles that meet only at e. What is the highest edge-connectivity of G that can be inferred from these properties?

Problem 8

Find all non-negative integers k for which the following statement is true: For every simple k-regular graph G on at least two vertices, $\kappa(G) = \kappa'(G)$.

Problem 9

Suppose G is a simple r-connected graph of even order with no $K_{1,r+1}$ as an induced subgraph for a positive integer r. Prove that G has a perfect matching.

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- *the incidences are reflected by the vertices (points) being in the closure of the edges.*

Definition 6.2

A polygonal curve in the plane is the union of finitely many line segments such that each segment starts at the end of the previous one and no point lies in more than one segment, except the end of one segment and the beginning of the next one coincide.
A simple open polygonal curve is one homeomorphic to a closed interval.
A simple closed polygonal curve is one homeomorphic to a unit circle.

Plane Graphs

Definition 6.3

- A drawing of a graph G is a function that maps each vertex $v \in V(G)$ to a point $f(v)$ in the plane, and each uv-edge to a simple polygonal $f(u)f(v)$-curve in the plane.

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Note 6.4

A plane embedding corresponds to an embedding of the graph in the sphere through a stereographic projection.

Faces

Theorem 6.5 (Jordan Curve Theorem)

If C is a simple closed polygonal curve in the plane, then the complement of C in the plane consists of two connected components each with C as the boundary.

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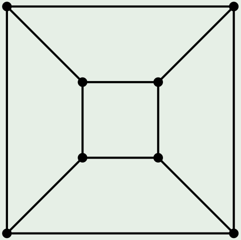
  - a vertex and an edge of G^* are incident if and only if the edge is the boundary of the corresponding face of G.

Example of a Dual Graph

Example 6.7



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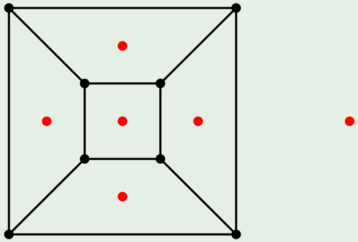

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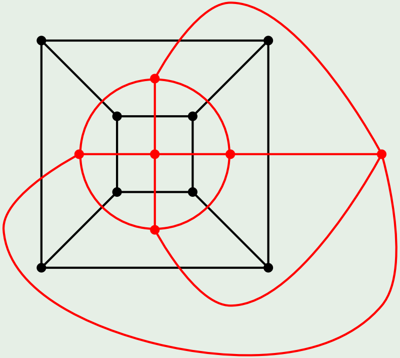

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Suppose D is a set of edges of G that contains a cycle.

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Suppose D is a set of edges of G that contains a cycle. By Jordan Curve Theorem 6.5, some face u^* of G lies inside this cycle, and some other v^* lies outside.

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Proof: Exercise.

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- *Deleting an edge or a vertex from a plane graph results in a plane graph.*
- *Contracting an edge in a plane graph can be visualized as sliding the two endvertices towards each other until they meet, pulling all incident edges along.*
- *Thus the class of planar graphs is minor-closed, that is, all minors of planar graphs are also planar.*

Euler's Formula

Theorem 6.13 (Euler's Formula)

If a connected non-empty plane graph has v vertices, e edges, and f faces, then $v - e + f = 2$.

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Suppose $v > 1$.

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If G is a planar graph whose order v is at least 3, whose size is e, and whose girth g is at least 3 but finite, then

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If G is simple, then $e \leq 3v - 6$.

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Is K_5 planar?

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Without loss of generality, we may assume that G is plane and connected. Let f_i denote the number of faces of G of length i. Since every edge appears in two faces or in the same face twice, we have $2e = \sum if_i \geq gf$. Substituting this into Euler's Formula gives $v - e + 2e/g \geq 2$.

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No, since $e = 10 > 3v - 6 = 9$.

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Is $K_{3,3}$ planar?

Proof

Without loss of generality, we may assume that G is plane and connected. Let f_i denote the number of faces of G of length i. Since every edge appears in two faces or in the same face twice, we have $2e = \sum if_i \geq gf$. Substituting this into Euler's Formula gives $v - e + 2e/g \geq 2$.

$$e(\frac{2}{g} - 1) \geq 2 - v$$

$$e \leq \frac{v-2}{\frac{2}{g} - \frac{g}{g}} = \frac{(v-2)g}{g-2}$$

Note that when G is simple, $g \geq 3$ and so $e \leq 3v - 6$.

Example 6.16

Is K_5 planar?
No, since $e = 10 > 3v - 6 = 9$.
Is $K_{3,3}$ planar?
No, since

$$e = 9 > \frac{(v-2)g}{g-2} = 8.$$

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We have $kv = 2e = lf$, and so the Euler's Formula 6.13 gives us $2e/k - e + 2e/l = 2$. Thus $e(2/k - 1 + 2/l) = 2$ and

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Then $-kl + 2l + 2k > 0$,

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Then $-kl + 2l + 2k > 0$, and so $-kl + 2l + 2k - 4 > -4$, And so $(k-2)(l-2) < 4$,

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$$e = \frac{2kl}{2k + 2l - kl}.$$

Then $-kl + 2l + 2k > 0$, and so $-kl + 2l + 2k - 4 > -4$, And so $(k-2)(l-2) < 4$, and so $k, l \geq 3$ and $k, l \leq 5$.

The Zoo of Platonic Solids

$$e = \frac{2kl}{2k + 2l - kl} \qquad v = \frac{2e}{k} \qquad f = \frac{2e}{l}$$

k	l	$(k-2)(l-2)$	e	v	f	name

The Zoo of Platonic Solids

$$e = \frac{2kl}{2k + 2l - kl} \qquad v = \frac{2e}{k} \qquad f = \frac{2e}{l}$$

k	l	$(k-2)(l-2)$	e	v	f	name
3	3					

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1				

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6			

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k	l	$(k-2)(l-2)$	e	v	f	name
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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	tetrahedron

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	tetrahedron
3	4					

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	tetrahedron
3	4	2				

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$$e = \frac{2kl}{2k + 2l - kl} \qquad v = \frac{2e}{k} \qquad f = \frac{2e}{l}$$

k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	tetrahedron
3	4	2	12			

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3	4	2	12	8		

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3	4	2	12	8	6	cube

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3	4	2	12	8	6	cube
4	3					

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5					

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5	3				

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3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5	3	30			

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3	3	1	6	4	4	tetrahedron
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3	5	3	30	20		

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3	5	3	30	20	12	dodecahedron

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5	3					

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4	3	2	12	6	8	octahedron
3	5	3	30	20	12	dodecahedron
5	3	3				

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5	3	30	20	12	dodecahedron
5	3	3	30			

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
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3	5	3	30	20	12	dodecahedron
5	3	3	30	12		

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k	l	$(k-2)(l-2)$	e	v	f	name
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4	3	2	12	6	8	octahedron
3	5	3	30	20	12	dodecahedron
5	3	3	30	12	20	

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3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5	3	30	20	12	dodecahedron
5	3	3	30	12	20	icosahedron

Statement of the Kuratowski Theorem

Theorem 6.17 (Kuratowski 1930)

A graph is planar if and only if it has neither K_5 *nor* $K_{3,3}$ *as a topological minor.*

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Lemma 6.19

If F is the edge-set of the boundary of a face of a plane graph G, then G has an plane embedding in which F is the boundary of the infinite face.

Proof.

Apply stereographic projection twice. □

Minimally No-Planar Graphs

Definition 6.20

A graph is minimally non-planar if it is non-planar, but every proper subgraph of it is planar.

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Every minimal non-planar graph is 2-connected.

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If G is disconnected, we can embed one component of G inside one face of the rest of G.

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Proof.

If G is disconnected, we can embed one component of G inside one face of the rest of G. Similarly, if G has a cut-vertex v, let G_1, G_2, $\ldots$, G_k be the subgraphs of G induced by v together with the components of $G - v$.

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Lemma 6.22

Suppose $G = H_1 \oplus_2 H_2$ is non-planar.

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Proof.

Let e be the common edge of H_1 and H_2.

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Proof.

Let e be the common edge of H_1 and H_2. Suppose both H_1 and H_2 are planar.

$\square$

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Let e be the common edge of H_1 and H_2. Suppose both H_1 and H_2 are planar. By Lemma 6.19, each of H_1 and H_2 can be embedded in the plane with e in the boundary of the infinite face.

□

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Kuratowski Graphs and Topological Minors

Lemma 6.22

Suppose $G = H_1 \oplus_2 H_2$ *is non-planar. Then at least one of* H_1 *and* H_2 *is non-planar.*

Proof.

Let e be the common edge of H_1 and H_2. Suppose both H_1 and H_2 are planar. By Lemma 6.19, each of H_1 and H_2 can be embedded in the plane with e in the boundary of the infinite face. It is now easy to put together the embeddings of H_1 and H_2 into a plane embedding of G. □

Definition 6.23

- A Kuratowski subgraph is a subgraph isomorphic to a subdivision of K_5 or of $K_{3,3}$.

Lemma 6.22

Suppose $G = H_1 \oplus_2 H_2$ is non-planar. Then at least one of H_1 and H_2 is non-planar.

Proof.

Let e be the common edge of H_1 and H_2. Suppose both H_1 and H_2 are planar. By Lemma 6.19, each of H_1 and H_2 can be embedded in the plane with e in the boundary of the infinite face. It is now easy to put together the embeddings of H_1 and H_2 into a plane embedding of G. □

Definition 6.23

- A Kuratowski subgraph is a subgraph isomorphic to a subdivision of K_5 or of $K_{3,3}$.
- A vertex of a graph G is a branch vertex of a Kuratowski subgraph H of G, if its degree in H exceeds two.

proof

Lemma 6.24

If G/e has a Kuratowski subgraph, then so does G.

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If G/e has a Kuratowski subgraph, then so does G.

Proof.

Let H be a Kuratowski subgraph of $G' = G/e$,

□

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If G/e has a Kuratowski subgraph, then so does G.

Proof.

Let H be a Kuratowski subgraph of $G' = G/e$, and let z be the vertex of G' obtained by contracting $e = xy$.

$\square$

proof

Lemma 6.24

If G/e has a Kuratowski subgraph, then so does G.

Proof.

Let H be a Kuratowski subgraph of $G' = G/e$, and let z be the vertex of G' obtained by contracting $e = xy$. If z is not a branch vertex of H, then G also has a Kuratowski subgraph obtained from H by lengthening a path through z if necessary.

$\square$

proof

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If G/e has a Kuratowski subgraph, then so does G.

Proof.

Let H be a Kuratowski subgraph of $G' = G/e$, and let z be the vertex of G' obtained by contracting $e = xy$. If z is not a branch vertex of H, then G also has a Kuratowski subgraph obtained from H by lengthening a path through z if necessary. If z is a branch vertex of H and at most one of the edges incident to z in H is incident to x in G,

□

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Lemma 6.24

If G/e has a Kuratowski subgraph, then so does G.

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Let H be a Kuratowski subgraph of $G' = G/e$, and let z be the vertex of G' obtained by contracting $e = xy$. If z is not a branch vertex of H, then G also has a Kuratowski subgraph obtained from H by lengthening a path through z if necessary. If z is a branch vertex of H and at most one of the edges incident to z in H is incident to x in G, then z can be expanded into xy to lengthen that path, and y becomes the corresponding branch vertex of a Kuratowski subgraph of G.

□

proof

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If G/e has a Kuratowski subgraph, then so does G.

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The only remaining case to consider is when H is a subdivision of K_5, z is a branch vertex of H, and each of x and y is incident in G to two of the four edges incident to z in H.

□

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Proof.

Let H be a Kuratowski subgraph of $G' = G/e$, and let z be the vertex of G' obtained by contracting $e = xy$. If z is not a branch vertex of H, then G also has a Kuratowski subgraph obtained from H by lengthening a path through z if necessary. If z is a branch vertex of H and at most one of the edges incident to z in H is incident to x in G, then z can be expanded into xy to lengthen that path, and y becomes the corresponding branch vertex of a Kuratowski subgraph of G.

The only remaining case to consider is when H is a subdivision of K_5, z is a branch vertex of H, and each of x and y is incident in G to two of the four edges incident to z in H. Let u_1, u_2 be the branch vertices of H that are at the other ends of paths leaving z on the edges incident with x,

□

proof

Lemma 6.24

If G/e has a Kuratowski subgraph, then so does G.

Proof.

Let H be a Kuratowski subgraph of $G' = G/e$, and let z be the vertex of G' obtained by contracting $e = xy$. If z is not a branch vertex of H, then G also has a Kuratowski subgraph obtained from H by lengthening a path through z if necessary. If z is a branch vertex of H and at most one of the edges incident to z in H is incident to x in G, then z can be expanded into xy to lengthen that path, and y becomes the corresponding branch vertex of a Kuratowski subgraph of G.

The only remaining case to consider is when H is a subdivision of K_5, z is a branch vertex of H, and each of x and y is incident in G to two of the four edges incident to z in H. Let u_1, u_2 be the branch vertices of H that are at the other ends of paths leaving z on the edges incident with x, and let v_1, v_2 be the other branch vertices of H.

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Tutte's Version of Kuratowski's Theorem

Definition 6.25

A plane embedding is convex if every face except the infinite one is a convex polygon.

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We proceed by induction on $|G|$. The only 3-connected simple graph on at most 4 vertices is K_4 and it has such an embedding. Let G be a graph on $n \geq 5$ vertices and suppose the theorem holds for all graphs on fewer than n vertices. $\square$

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Proof of Kuratowski Theorem

Recall: G is planar if and only if neither K_5 nor $K_{3,3}$ is a topological minor of G.

Proof.

Without loss of generality, we may asume that G is simple.

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Prove that every n-vertex plane graph isomorphic to its dual has $2n - 2$ edges. For each $n \geq 4$, construct a simple n-vertex plane graph isomorphic to its dual.

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Problem 12

A plane graph is outerplane if it has a face incident with all the vertices. A graph is outerplanar if it isomorphic to an outerplane graph. Prove that a graph is outerplanar if and only if it has neither K_4 nor $K_{2,3}$ as a topological minor.

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- $\chi(G) \leq \Delta(G) + 1$

Proof.

Color greedily:

□

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- $\chi(G)$ may exceed $\omega(G)$, for example, consider $C_{2r+1} \vee K_s$, that is, the graph formed from the disjoint union of C_{2r+1} and K_s by joining each vertex of C_{2r+1} to each vertex of K_s.
- There are triangle-free graphs of arbitrarily high chromatic number (Mycielski)
  - Let $M_2 = K_2$.
  - Construct M_{n+1} from M_n by:
    - Start with a copy M of M_n;
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- $\chi(G) \geq |G|/\alpha(G)$
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Proof.

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Facts About Chromatic Number

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Color greedily: Order the vertices arbitrarily as $v_1, v_2, \ldots, v_n$. Starting with $k = 1$, color each vertex v_k with the smallest color not used among the vertices $v_1, v_2, \ldots, v_{k-1}$ that are neighbors of v_k. □

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Theorem 7.2 (Brooks 1941)

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Proof of Brooks' Theorem, Continued

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Thus we may assume that G is 2-connected. Suppose that G has an induced 3-vertex path with vertices we label v_1, v_n, v_2, in order, such that $G - \{v_1, v_2\}$ is connected. We could then index the vertices of a spanning tree of $G - \{v_1, v_2\}$ as before, so that every v_i other than v_n has at least one neighbor among v_{i+1}, v_{i+2}, ..., v_n. Then every vertex v_i other than v_n would have at most $k-1$ neighbors among v_1, v_2, ..., v_{i-1}, while two of the neighbors of v_n, namely v_1 and v_2 would both receive color 1 by the greedy coloring. Hence, it suffices to show that every 2-connected k-regular graph G with $k \geq 3$, not a clique, not an odd cycle, has 3-vertex induced path v_1, v_n, v_2 such that $G - \{v_1, v_2\}$ is connected. Choose a vertex x. If $\kappa(G - x) \geq 2$, then let $v_1 = x$, and let v_2 be a vertex of distance two from x, which exists, since G is connected, regular and not a clique. If $\kappa(G - x) = 1$, then x has a neighbor in every block of $G - x$ that is a leaf in the block tree of $G - x$, since G has no cut-vertex. Neighbors v_1 and v_2 of x in two such blocks are non-adjacent. Furthermore, $G - \{x, v_1, v_2\}$ is connected, since blocks have no cut-vertices. Now, $G - \{v_1, v_2\}$ is also connected, since $k \geq 3$ and so x has a neighbor other than v_1 and v_2. Let $v_n = x$ to complete the proof.

5-Color Theorem

Theorem 7.3 (Heawood 1890)

Every loopless planar graph has a proper 5*-coloring.*

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Proof.

Suppose G is a plane graph that is a minimal counter-example.

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Suppose G is a plane graph that is a minimal counter-example. Then G is simple, and so $\|G\| \leq 3|G| - 6$ by Corollary 6.15.

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Suppose G is a plane graph that is a minimal counter-example. Then G is simple, and so $\|G\| \leq 3|G| - 6$ by Corollary 6.15. It follows that G has a vertex v of degree 5 or less.

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Suppose G is a plane graph that is a minimal counter-example. Then G is simple, and so $\|G\| \leq 3|G| - 6$ by Corollary 6.15. It follows that G has a vertex v of degree 5 or less. Then $G - v$ has a 5-coloring f by the minimality of G. Since G is not 5-colorable, each color appears at one of the neighbors of v (and so $d(v) = 5$). We may assume that the colors on the neighbors of v appear as 1, 2, 3, 4, and 5 as they are inspected clockwise. Let $G_{i,j}$ denote the subgraph of $G - v$ induced by the colors i and j. Note that we an exchange the two colors on any component of $G_{i,j}$ to obtain another proper coloring of $G - v$. If some two neighbors of v were in different components of $G_{i,j}$, then switching colors on one such component would result in two neighbors of G being colored the same,

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Discharging Method

Theorem 7.5

Suppose G is a plane triangulation with $\delta(G) \geq 5$. Then G contains two adjacent vertices one of which has degree 5, and the other has degree 5 or 6.

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$$\sum_{u \in V(G)} (6 - d(u)) = 6v - 2e = 6v - 2(3v - 6) = 12.$$

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Chromatic Polynomial

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Theorem 7.6 (Four-Color Theorem, restated)

If G is a planar loopless graph, then $P_G(4) > 0$.

Perfect Graphs

Definition 7.7

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Theorem 7.9 (Strong Graph Theorem (formerly Berge's Strong Graph Conjecture), Chudnovsky, Robertson, Seymour, Thomas 2002)

A graph is perfect if and only if it has no induced subgraph that is an odd cycle of length at least five or its complement.

Edge Colorings

Definition 7.10

- A *k-edge-coloring* of a graph G is a labeling $f : E(G) \to \{1, 2, \ldots, k\}$.

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Note 7.11

$\Delta(G) \le \chi'(G)$.

Edge-Coloring of Bipartite Graphs

Theorem 7.12 (König 1916)

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Vizing's Theorem

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If G is simple, then $\chi'(G) \leq \Delta(G) + 1$.

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Suppose uv is an edge left uncolored by a proper $(\Delta(G)+1)$-edge-coloring of a proper subgraph G' of G. After possibly re-coloring some edges, we extend this coloring to include uv; call this an augmentation. After an appropriate number of augmentations, we obtain a proper $(\Delta(G)+1)$-edge-coloring of G. Since the number of colors exceeds $\Delta(G)$, every vertex has some color not appearing on its incident edges. Let a_0 be a color missing at u and let a_1 be a color missing at v. We may assume that a_1 appears at u, or it could be used for the edge uv. Let v_1 be the neighbor of u along the edge colored a_1. At v_1 some color a_2 is missing. We may assume that a_2 appears at u, or we could re-color uv_1 from a_1 to a_2, and then use a_1 on uv to augment the coloring. □

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Hence we may assume that a_0 appears at v_l, but a_k does not. Let P be the (unique) maximal path of edges colored a_0 or a_k that begins at v_l. Switching on P means interchanging the colors a_0 and a_k on the edges of P.

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But there are graphs for which $\mathrm{ch}(G) \neq \chi(G)$. Consider $K_{3,3}$ where each side of the bipartition has lists $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$. The list-chromatic number of this graph is 3, while the chromatic number is 2.

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We prove this by induction. The claim is obvious for the smallest graph for which it makes sense, that is, a triangle. Suppose the claim is true for every graph on fewer than n vertices, and suppose that G is like described above, and $|G| = n$.

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Case 2

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List Colorings

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Conjecture 7.16

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- *If f is a circulation and $e = xy$ is a cut-edge, then $f(e, x, y) = 0$.*

Flow polynomial

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Clearly, $F_G(x)$ is a polynomial, and is called the flow polynomial of G. It follows:

Corollary 7.20

If $\mathbb{H}$ and $\mathbb{H}'$ are two finite abelian groups of equal order, then G has an $\mathbb{H}$-flow if and only if it has an $\mathbb{H}'$-flow.

Definition 7.21

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Proof of $\Rightarrow$ only.

Use the natural map $i \mapsto \bar{i}$ from $\mathbb{Z}$ to $\mathbb{Z}_k$. $\square$

k-Flows for Small k

Theorem 7.23

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By Corollary 7.22, a graph has a 2-flow if and only if it has a $\mathbb{Z}_2$-flow, that is, the constant map $\vec{E} \to \mathbb{Z}_2$ with value 1 satisfies (F2).

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Conversely, let G be bipartite with bipartition (X, Y).

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Let G be a cubic graph. Suppose first that G has a 3-flow, and thus a $\mathbb{Z}_3$-flow. We show that every cycle $C = x_0 x_1 \ldots x_l x_0$ has even length. Consider two consecutive edges of C: $e_{i-1} = x_{i-1} x_i$ and $e_i = x_1 x_{i+1}$. If $f(e_{i-1}, x_{i-1}, x_i) = f(e_i, x_i, x_{i+1})$, then f could not satisfy (F2) at x_i due to a non-zero value of the third edge at x_i. Therefore f assigns 1 and 2 to the edges of C alternately, and so C must be even.
Conversely, let G be bipartite with bipartition (X, Y). Since G is cubic, the map $\vec{E} \to \mathbb{Z}_3$ defined by $f(e, x, y) = 1$ and $(e, y, x) = 2$ for all edges xy with $x \in X$ and $y \in Y$ is a $\mathbb{Z}_3$-flow. □

Flow Number of Cliques

Theorem 7.25

$$\varphi(K_n) = \begin{cases} 2 & \textit{if } n \textit{ is odd;} \\ 4 & \textit{if } n = 4\textit{; and} \\ 3 & \textit{if } n \textit{ is even and exceeds } 4. \end{cases}$$

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Note that K_6 is the edge-disjoint union of G_1, G_2, and G_3 where $G_1 \cong G_2 \cong K_3$ and $G_3 \cong K_{3,3}$.

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4-Flows

Theorem 7.26

(i) *A graph has a* 4*-flow if and only if it is the union of two subgraphs whose vertices have all degrees even.*

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Proof.

Let $\mathbb{H} = \mathbb{Z}_2 \times \mathbb{Z}_2$.

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Proof.

Let $\mathbb{H} = \mathbb{Z}_2 \times \mathbb{Z}_2$. By Theorems 7.20 and 7.22, a graph has a 4-flow if and only if it as an $\mathbb{H}$-flow.

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Assume a cubic graph G has an $\mathbb{H}$-flow f. It is easy to check that f gives a 3-edge-coloring.

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Assume a cubic graph G has an $\mathbb{H}$-flow f. It is easy to check that f gives a 3-edge-coloring. Conversely, since the non-zero elements of $\mathbb{H}$ sum up to 0, every proper 3-edge-coloring of G using colors $\mathbb{H} \setminus 0$ defines an $\mathbb{H}$-flow on G. $\square$

Flow-Coloring Duality

Theorem 7.27 (Tait 1878)

A simple 2-edge-connected 3-regular plane graph G is 3-edge-colorable if and only if its dual is 4-colorable.

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Problem Set 5

Problem 13

Prove that every loopless planar graph on fewer than thirteen vertices admits a proper 4-coloring. In other words, prove the Four-Color Theorem for graphs on at most twelve vertices.

Problem 14

Show that every graph without a cut-edge admits a flow.

Problem 15

Show that if a graph has a spanning cycle, then it admits a 4-flow.

Tutte's Flow Conjectures

Conjecture 7.28 (Tutte)

- *(5-Flow Conjecture, 1954) Every graph with no cut-edge has a 5-flow.*

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Every cubic graph with no cut-edge and no Petersen graph minor has a 4-flow.

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Every planar graph with no edge-cuts of size 1 and 3 has a 3-flow.

Miscellany of Coloring Results

Theorem 7.32 (Tutte 1954)

$\chi(G) = \varphi(G^*)$

Proof.

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- *Unknown for $n \geq 7$.*

Hamilton Cycles

Definition 8.1

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- A spanning subgraph that is a cycle or a path is called a Hamilton cycle or a Hamilton path.
- A graph is Hamiltonian if it has a Hamilton cycle.

Sufficient Condition for Hamiltonicity

Theorem 8.2 (Dirac 1952)

Every graph of order $n \geq 3$ and $\delta \geq n/2$ is Hamiltonian.

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Proof.

Let G be a graph as described.

$\square$

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Every graph of order $n \geq 3$ *and* $\delta \geq n/2$ *is Hamiltonian.*

Proof.

Let G be a graph as described. Note that G is connected;

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Proof.

Let G be a graph as described. Note that G is connected; otherwise a vertex in a smallest component would have degree less than $n/2$.

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Proof.

Let G be a graph as described. Note that G is connected; otherwise a vertex in a smallest component would have degree less than $n/2$.
Let $P = x_0 x_1 \dots x_k$ be a longest path in G.

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Proof.

Let G be a graph as described. Note that G is connected; otherwise a vertex in a smallest component would have degree less than $n/2$.
Let $P = x_0x_1 \ldots x_k$ be a longest path in G. By the maximality of P, all neighbors of x_0 and all neighbors of x_k lie on P. Hence at least $n/2$ of the vertices x_0, x_1, ..., x_{k-1} are adjacent to x_k, and at least $n/2$ of the same $k < n$ vertices x_i are such that $x_0x_{i+1} \in E(G)$. By the Pigeon-Hole Principle, there is a vertex x_i that has both properties, that is, $x_0x_{i+1} \in E(G)$ and $x_ix_k \in E(G)$.

$\square$

Sufficient Condition for Hamiltonicity

Theorem 8.2 (Dirac 1952)

Every graph of order $n \geq 3$ and $\delta \geq n/2$ is Hamiltonian.

Proof.

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Note on Dirac's Theorem

Note 8.3

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Note on Dirac's Theorem

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Note that $n/2$ in Dirac's Theorem 8.2 is the best possible. We cannot replace it with $\lfloor n/2 \rfloor$ if n is odd, since then G which is a 1-sum of two copies of $K^{\lceil n/2 \rceil}$ would have $\delta = \lfloor n/2 \rfloor$, but no Hamilton cycle.

Another Sufficient Condition

Theorem 8.4

Every graph G with $|G| \geq 3$ and $\kappa(G) \geq \alpha(G)$ is Hamiltonian.

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Let $k = \kappa(G)$ and let C be a longest cycle in G.

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Proof.

Let $k = \kappa(G)$ and let C be a longest cycle in G. Enumerate the vertices of C cyclically so that $V(C) = \{v_i : i \in \mathbb{Z}_n\}$ with $v_i v_{i+1} \in E(C)$ for all $i \in \mathbb{Z}_n$.

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$\square$

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A Necessary Condition

Theorem 8.5

If G is a Hamiltonian graph, then for every set $\emptyset \neq S \subseteq V(G)$, the graph $G - S$ has at most S components.

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Proof.

When leaving a component of $G - S$, a Hamilton cycle can go only to S and the arrivals in S must occur at different vertices of S. Hence S must have at least as many vertices as $G - S$ has components. $\square$

Note that if we managed to prove that every 3-connected cubic plane graph is Hamiltonian, Note that if we managed to prove that every 3-connected cubic plane graph is Hamiltonian, then we woud have proved that every such graph has a 4-flow, Note that if we managed to prove that every 3-connected cubic plane graph is Hamiltonian, then we woud have proved that every such graph has a 4-flow, and so is 3-edge-colorable, Note that if we managed to prove that every 3-connected cubic plane graph is Hamiltonian, then we woud have proved that every such graph has a 4-flow, and so is 3-edge-colorable, and so is 4-face-colorable.

Note that if we managed to prove that every 3-connected cubic plane graph is Hamiltonian, then we woud have proved that every such graph has a 4-flow, and so is 3-edge-colorable, and so is 4-face-colorable. Unfortunately, there are 3-connected cubic plane graphs that are not Hamiltonian.

Grinberg's Theorem

Theorem 8.6 (Grinberg 1968)

If G is a loopless plane graph with a Hamilton cycle C, and G has f_i' faces of length i inside C and f_i'' faces of length i outside C, then $\sum_i (i-2)(f_i' - f_i'') = 0$.

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Corollary 8.7

The Tutte graph is not Hamiltonian.

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Theorem 8.9 (Thomas, Yu 1994)

Every 4*-connected projective graph is Hamiltonian.*

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The Tutte graph is not Hamiltonian.

Theorem 8.8 (Tutte 1956)

Every 4-connected planar graph is Hamiltonian.

Theorem 8.9 (Thomas, Yu 1994)

Every 4-connected projective graph is Hamiltonian.

Theorem 8.10 (Thomas, Yu 1997)

Every 5-connected toroidal graph is Hamiltonian.

Suppose we have a finite family $\mathcal{F}$ of pairwise disjoint convex polygons (with interiors) in the plane with all sides of length 1.

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Theorem 9.1

Every surface is homeomorphic to a triangulated surface.

Proof omitted.

Consider now two disjoint triangles T_1 and T_2 (such that all sides have same length) in a face F of a surface S with a 2-cell embedded graph G.

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Consider the sphere $\mathbb{S}_0$ (tetrahedron). If we add h handles to $\mathbb{S}_0$, then we obtain $\mathbb{S}_h$, which we call the orientable surface of genus h. If we add h crosscaps to $\mathbb{S}_0$, we obtain $\mathbb{N}_h$, the non-orientable surface of genus h.

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Theorem 9.2

Let S be the surface obtained from the sphere by adding h handles, t twisted handles, and c crosscaps. If $t = c = 0$, then $S = \mathbb{S}_h$. Otherwise, $S = \mathbb{N}_{2h+2t+c}$.

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Theorem 9.3

Let S be a surface and let G be a graph that is 2-cell embedded in S with v vertices, e edges and f faces. Then S is homeomorphic to either $\mathbb{S}_h$ or $\mathbb{N}_k$, where $v - e + f = 2 - 2h = 2 - k$.

Euler's Characteristic

Definition 9.4

The Euler characteristic $\chi(S)$ of a surface S is defined as

$$\chi(S) = \begin{cases} 2 - 2h, & \text{if } S = \mathbb{S}_h; \\ 2 - k, & \text{if } S = \mathbb{N}_k. \end{cases}$$

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Theorem 9.5

Every cellular embedding (an embedding where each face is homeomorphic to an open disk) into an orientable surface is determined by its rotation system.

Embedding Schemes

An embedding scheme is a pair $\Pi = (\pi, \lambda)$ where π is a rotation system, and $\lambda : E(G) \to \{-1, 1\}$ is a signature.

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Theorem 9.6

Every cellular embedding of a graph in some surface is uniquely determined, up to homeomorphism, by its embedding scheme.

Genus of a Graph

Definition 9.7

The genus $\gamma(G)$ and the non-orientable genus $\tilde{\gamma}(G)$ of a graph G are the minimum h and the minimum k, respectively, such that G has an embedding into the surface $\mathbb{S}_h$, respectively into $\mathbb{N}_k$.

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Let G be a connected graph. If $\tilde{\gamma}(G) < 2\gamma(G) + 1$, then every non-orientable minimum genus embedding of G is cellular. If $\tilde{\gamma}(G) = 2\gamma(G) + 1$ and G is not a tree, then G has both a cellular and a non-cellular embedding in $\mathbb{N}_{\tilde{\gamma}(G)}$.

Cycle Double-Cover Conjecture

Conjecture 9.10 (Cycle Double-Cover Conjecture)

Every 2-edge-connected graph G can be expressed as a union of cycles so that every edge of G appears in exactly two cycles.

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Every 2-edge-connected graph has an embedding in some surface so that every face with the boundary is homeomorphic to the closed unit disk.

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Every 2-edge-connected graph has an embedding in some surface so that every face with the boundary is homeomorphic to the closed unit disk. Holds for 4-connected graphs.

Comparisons of Graph Topological Parameters

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- Euler Genus: $g = 2 - \chi \Longrightarrow g = 2\gamma \quad \& \quad g = \tilde{\gamma}$

Bounds on Genus

Theorem 9.12

Let G be a simple connected graph with v vertices ($v \geq 3$) and e edges. Then

$$\gamma \geq \left\lceil \frac{e}{6} - \frac{v}{2} + 1 \right\rceil \qquad \textit{and} \qquad \tilde{\gamma} \geq \left\lceil \frac{e}{3} - v + 2 \right\rceil .$$

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Proof.

Let f denote the number of facial walks.

$\square$

Bounds on Genus

Theorem 9.12

Let G be a simple connected graph with v vertices ($v \geq 3$) and e edges. Then

$$\gamma \geq \left\lceil \frac{e}{6} - \frac{v}{2} + 1 \right\rceil \qquad \text{and} \qquad \tilde{\gamma} \geq \left\lceil \frac{e}{3} - v + 2 \right\rceil .$$

Proof.

Let f denote the number of facial walks. The sum of lengths of those walks is $2e$.

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Let f denote the number of facial walks. The sum of lengths of those walks is $2e$. Since the graph is simple, $2e \geq 3f$. By Euler's Formula, $3\chi = 3v - 3e + 3f \leq 3v - e$, and so $\chi \leq v - e/3$.

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$$\gamma \geq 1 - \frac{v - e/3}{2} = \frac{e}{6} - \frac{v}{2} + 1.$$

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Similarly, in the nonorientable case, we have

$$\tilde{\gamma} \geq 2 - (v - \frac{e}{3}) = \frac{e}{3} - v + 2.$$

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The theorem follows now from the fact that both γ and $\tilde{\gamma}$ are integers. $\square$

Genera of Complete Graphs

Corollary 9.13

$$\gamma(K_n) \geq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \qquad \textit{and} \qquad \tilde{\gamma}(K_n) \geq \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil$$

Heawood's Theorem

Theorem 9.14 (Ringel, Youngs)

If $n \geq 3$ and $n \neq 7$, then

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \qquad \textit{and} \qquad \tilde{\gamma}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil$$

Heawood's Formula

Theorem 9.15 (Heawood's Formula)

Let S be a surface with Euler genus $g = 2 - (v - e + f) > 0$ and let G be a loopless graph embedded in S. Then

$$\chi(G) \leq \left\lfloor \frac{7 + \sqrt{1 + 24g}}{2} \right\rfloor .$$

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Suppose the theorem fails and G is a minimal counter-example. Let $c = \chi(G)$, and note that since $g > 0$ and G is a counter-example, we have $c \geq 7$.

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Theorem 9.16 (Ringel–Youngs)

The bound in Heawood's Formula is the best possible, except that maximum chromatic number of graphs embedded in the Klein bottle is 6.

Homework

Problem 16

Prove that for every number n there is a bipartite graph whose choosability number is greater than n.

Problem 17

Find the (orientable) genus of the Petersen graph.

Problem 18

Does K_5 have cellular embeddings into two different orientable surfaces? Into two different non-orientable surfaces?

Well-Quasi-Ordering

Definition 10.1

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- An infinite sequence containing a good pair is good;

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- Then (x_i, x_j) is a good pair for the sequence.
- An infinite sequence containing a good pair is good; otherwise it is bad.

Antichains and Decreasing Sequences

Theorem 10.2

A quasi ordering $\preccurlyeq$ on X is a wqo if and only if X contains neither an infinite antichain nor an infinite strictly descending chain $x_0 \succ x_1 \succ \dots$.

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The forward implication is obvious.

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The forward implication is obvious. Conversely, let $\mathbf{x} = x_0, x_1, \dots$ be a bad sequence in X. We define infinite subsequences $\mathbf{y}$, $\mathbf{x}_0$, $\mathbf{x}_1 \dots$ of $\mathbf{x}$, and two sets A and B as follows.

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We start with $\mathbf{y}$ as the empty sequence, A and B as empty sets, and $\mathbf{x}_0 = \mathbf{x}$.

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We start with $\mathbf{y}$ as the empty sequence, A and B as empty sets, and $\mathbf{x}_0 = \mathbf{x}$. Suppose that for some $i = 0, 1, \dots$, the first i elements of $\mathbf{y}$ and the sequence $\mathbf{x}_i$ have been defined, and each of the first i elements of $\mathbf{y}$ has been placed in exactly one of A and B.

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- If infinitely many elements of $\mathbf{x}_i$ are incomparable with x, make $\mathbf{x}_{i+1}$ be an infinite subsequence of $\mathbf{x}_i$ consisting of the elements incomparable with x, and put x into A.

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- If infinitely many elements of $\mathbf{x}_i$ are incomparable with x, make $\mathbf{x}_{i+1}$ be an infinite subsequence of $\mathbf{x}_i$ consisting of the elements incomparable with x, and put x into A.
- If this doesn't happen, there are infinitely many elements of $\mathbf{x}_i$ smaller than x. In that case, let $\mathbf{x}_{i+1}$ be an infinite subsequence of $\mathbf{x}_i$ consisting of the elements x' such that $x \succ x'$ and put x into B.

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If X is a wqo, then every infinite sequence in X has an infinite increasing subsequence.

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Proof: Exercise.

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- The set of all finite sequences of elements of a set X will be denoted by $X^{<\omega}$.

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- The set of all finite subsets of elements of a set X will be denoted by $[X]^{<\omega}$.
- The relation $\preccurlyeq$ on X is extended to $X^{<\omega}$ as follows: If $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, we write $\mathbf{x} \preccurlyeq \mathbf{y}$ whenever there is a strictly increasing function $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ such that $x_i \preccurlyeq y_{f(i)}$ for all $i \in \{1, 2, \ldots, m\}$.

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- The relation $\preccurlyeq$ on X is extended to $[X]^{<\omega}$ by saying that two elements $\mathbf{x}$ and $\mathbf{y}$ of $[X]^{<\omega}$ satisfy $\mathbf{x} \preccurlyeq \mathbf{y}$ whenever there is a injection f from $\mathbf{x}$ to $\mathbf{y}$ such that $x \preccurlyeq f(x)$ for all $x \in \mathbf{x}$.

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- A quasi-order X is well-founded if it has no infinite strictly descending chains.

Higman's Theorem

Theorem 10.5 (Higman)

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Suppose $X^{<\omega}$ is not a wqo. Observe that $X^{<\omega}$ is well-founded. We construct a minimal bad sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $X^{<\omega}$.

$\square$

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If X is well-quasi-ordered by $\preccurlyeq$, then so is $X^{<\omega}$.

Proof.

Suppose $X^{<\omega}$ is not a wqo. Observe that $X^{<\omega}$ is well-founded. We construct a minimal bad sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $X^{<\omega}$. Given $n \in \mathbb{N}$, assume inductively that $\mathbf{x}_i$ has been defined for every $i < n$ and that there is a bad sequence in $X^{<\omega}$ starting with $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}$.

$\square$

Theorem 10.5 (Higman)

If X is well-quasi-ordered by $\preccurlyeq$, then so is $X^{<\omega}$.

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$\square$

Higman's Theorem

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$\square$

Proof of Higman's Theorem, Continued

Clearly, $(\mathbf{x}_n)_{n \in \mathbb{N}}$ is a bad sequence.

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Proof of Higman's Theorem, Continued

Clearly, $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a bad sequence. For each n, let $\mathbf{y}_n$ be $\mathbf{x}_n$ with the last element x_n deleted. By Theorem 10.3, the sequence $(x_n)_{n\in\mathbb{N}}$ has an infinite increasing subsequence $(x_{n_i})_{i\in\mathbb{N}}$. By the minimality of $\mathbf{x}_{n_0}$, the sequence $\mathbf{x}_0$, $\mathbf{x}_1, \ldots, \mathbf{x}_{n_0-1}, \mathbf{y}_{n_0}, \mathbf{y}_{n_1}, \ldots$ is good, so it has a good pair. This good pair must be of the form $(\mathbf{y}_i, \mathbf{y}_j)$. Extending the injection $\mathbf{y}_i \mapsto \mathbf{y}_j$ by $x_i \mapsto x_j$, we get a good pair $(\mathbf{x}_i, \mathbf{x}_j)$; a contradiction.

Corollary 10.6

If X is well-quasi-ordered by $\preccurlyeq$, then so is $[X]^{<\omega}$.

Proof of Higman's Theorem, Continued

Clearly, $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a bad sequence. For each n, let $\mathbf{y}_n$ be $\mathbf{x}_n$ with the last element x_n deleted. By Theorem 10.3, the sequence $(x_n)_{n\in\mathbb{N}}$ has an infinite increasing subsequence $(x_{n_i})_{i\in\mathbb{N}}$. By the minimality of $\mathbf{x}_{n_0}$, the sequence $\mathbf{x}_0$, $\mathbf{x}_1, \ldots, \mathbf{x}_{n_0-1}, \mathbf{y}_{n_0}, \mathbf{y}_{n_1}, \ldots$ is good, so it has a good pair. This good pair must be of the form $(\mathbf{y}_i, \mathbf{y}_j)$. Extending the injection $\mathbf{y}_i \mapsto \mathbf{y}_j$ by $x_i \mapsto x_j$, we get a good pair $(\mathbf{x}_i, \mathbf{x}_j)$; a contradiction.

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If X is well-quasi-ordered by $\preccurlyeq$, then so is $[X]^{<\omega}$.

Note 10.7 (Rado)

Higman's Theorem does not extend to infinite sequences.

Ordering Trees

Consider two trees T and T' with roots, respectively r and r'.

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Theorem 10.8 (Kruskal 1960)

Trees are well-quasi-ordered by the topological minors relation.

Proof of Kruskal's Theorem

We show that rooted trees are well-quasi-ordered by $\preccurlyeq$.

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Proof of Kruskal's Theorem, Continued

By Corollary 10.6, the sequence $(A_n)_{n \in \mathbb{N}}$ has a good pair (A_i, A_j).

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By Corollary 10.6, the sequence $(A_n)_{n \in \mathbb{N}}$ has a good pair (A_i, A_j). Let $f : A_i \to A_j$ be an injection such that $T \preccurlyeq f(T)$ for all $T \in A_i$.

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Tree-Decomposition

Let G be a graph, T be a tree, and let $\mathcal{V} = \{V_t\}_{t \in V(T)}$ be a family of vertex sets $V_t \subseteq V(G)$ (called bags).

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The width of the decomposition $(T, \mathcal{V})$ is the maximum of $|V_t| - 1$ taken over all $v \in V(T)$. The tree-width of G, denoted by $\mathrm{tw}(G)$ is the minimum width over all possible tree-decomposisions.

Properties of Tree-Decompositions

Theorem 10.10

If H is a subgraph of G, and $(T, \{V_t\}_{t \in V(T)})$ is a tree-decomposition of G, then $(T, \{V_t \cap V(H)\}_{t \in V(T)})$ is a tree-decomposition of H.

Proof is very easy.

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Lemma 10.11

Let t_1t_2 be an edge of T, and let T_1 and T_2 be the components of $T \setminus t_1t_2$, with $t_1 \in V(T_1)$ and $t_2 \in V(T_2)$.

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Proof.

Both t_1 and t_2 lie on every $s_1 s_2$-path in T with $s_1 \in V(T_1)$ and $s_2 \in V(T_2)$. $\square$

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We orient the edges of T as follows. For each edge t_1t_2 of T, define U_1 and U_2 as in Lemma 10.11;

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We orient the edges of T as follows. For each edge t_1t_2 of T, define U_1 and U_2 as in Lemma 10.11; then $V_{t_1} \cap V_{t_2}$ separates U_1 from U_2. If $V_{t_1} \cap V_{t_2}$ does not separate any two vertices of W, then $W \subseteq U_i$ for some $i \in \{1, 2\}$; we orient t_1t_2 towards that t_i.

□

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We orient the edges of T as follows. For each edge t_1t_2 of T, define U_1 and U_2 as in Lemma 10.11; then $V_{t_1} \cap V_{t_2}$ separates U_1 from U_2. If $V_{t_1} \cap V_{t_2}$ does not separate any two vertices of W, then $W \subseteq U_i$ for some $i \in \{1, 2\}$; we orient t_1t_2 towards that t_i.
Let t be the last vertex of a maximal directed path in T; we claim that $W \subseteq V_t$.

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Lemma 10.12

For every $W \subseteq V(G)$, one of the following holds:

(i) *there is a $t \in V(T)$ such that $W \subseteq V_t$;*

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If G is obtained from H by deleting an edge, then a tree-decomposition of H is also a tree-decomposition of G. If G is obtained from H by deleting a vertex, then a tree-decomposition of H may be modified by removing the vertex from all bags to form a tree-decomposition of G.

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For every positive integer k, the graphs of tree-width less than k are well-quasi-ordered by the minor relation.

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Brambles

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Since each set in a bramble is connected and meets both of the covers, it also meets any set separating these covers. □

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Theorem 10.21

The tree-width of an $n \times n$ *grid* ($n > 1$) *is* n.

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Planar graphs are well-quasi-ordered by the minor relation.

Representativity

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Theorem 10.27

For every surface S (orientable or not), the graphs embeddable in S are well-quasi-ordered by the minors relation.

Graphs Almost Embedded on Surfaces

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Theorem 10.28 (Robertson-Seymour)

The class $\mathcal{G}(r,s,t,u)$ is well-quasi-ordered by the minor relation.

Graph Minors Theorem

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Every minor-closed class of graphs other than the class of all graphs is a subclass of some $\mathcal{G}(r, s, t, u)$.

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Corollary 10.31

The class of all (finite) graphs is well-quasi-ordered by the minor relation.

Problem 19

For each integer n exceeding one, find a bramble of order $n+1$ in the $n \times n$ grid.

Problem 20

A tree T is a caterpillar if T contains a path P such that every vertex of T either lies on P or is adjacent to a vertex of P. A caterpillar forest is a disjoint union of caterpillars. Find the minor-minimal graphs that are not caterpillar forests.

Problem 21

What is the tree-width of the graph obtained from the Petersen graph by deleting one edge?

The Turán Graph

Question: Given a graph H, what is the greatest possible number of edges in a simple graph of order n that does not have H as a subgraph? We will answer this question when H is a complete graph.

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Theorem 11.2 (Turán 1941)

Given integers r and n exceeding 1, the unique simple graph of order n without K_r as a subgraph of maximum possible size is $T^{r-1}(n)$.

Proof of Turán's Theorem

First, observe that among all simple k-partite $(k < r)$ graphs on n vertices, $T^{r-1}(n)$ has the largest size.

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If not, then non-adjacency is not an equivalence relation on $V(G)$, that is, there are vertices y_1, x, and y_2 such that y_1x and xy_2 do not form edges of G, but y_1y_2 does. If $d(y_1) > d(x)$, then deleting x and duplicating y_1 yields another K_r-free graph with more edges than G.

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Erdős-Stone Theorem

Corollary 11.3

If G is a simple graph of order n and size more than $t_{r-1}(n)$, then G contains K_r as a subgraph.

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Definition 11.5

Given a simple graph H and an integer n, let $h_n(H)$ denote the maximum edge density that a simple H-free graph of order n can have;

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Given a simple graph H and an integer n, let $h_n(H)$ denote the maximum edge density that a simple H-free graph of order n can have; that is, the maximum number of edges that a simple H-free graph of order n can have divided by $\binom{n}{2}$.

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For every simple, non-trivial graph H,

$$\lim_{n \to \infty} h_n(H) = \frac{\chi(H) - 2}{\chi(H) - 1}.$$

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Let $r = \chi(H)$.

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On the other hand, if K_r^s denotes the complete r-partite graph on rs vertices with every part of cardinality s, then $h_n(H) \leq h_n(K_r^s)$ for all sufficiently large s.

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$$h_n(K_r^s) < h_n(K_r) + \frac{\epsilon n^2}{\binom{n}{2}}.$$

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Then Lemma 11.6 finishes the proof. $\square$

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Theorem 11.8 (Ramsey 1930)

For every natural number r there is a natural number n such that every simple graph of order at least n contains either K_r or $\overline{K_r}$ as an induced subgraph.

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Inductively, $|V_{i-1} \setminus \{v_{i-1}\}| = 2^{2r-1-i} - 1$, and $V_{i-1} \setminus \{v_{i-1}\}$ contains a subset V_i satisfying (i)–(iii); pick v_i arbitrarily in V_i.

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Infinite version of Ramsey's Theorem

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Theorem 11.9

Let k and c be positive integers, and let X be an infinite set. If $[X]^k$ is c-colored, then X has an infinite monochromatic subset.

Proof

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(i) $X_{i+1} \subseteq X_i \setminus \{x_i\}$; and

(ii) all k-element sets of the form $\{x_i\} \cup Z$ where $Z \subseteq [X]^{k-1}$ have the same color, which we associate with x_i.

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Start with $X_0 = X$ and pick $x_0 \in X_0$ arbitrarily. Having chosen X_i and $x_i \in X_i$, we c-color $[X_i \setminus \{x_i\}]^{k-1}$ by giving each set Z the color of $\{x_i\} \cup Z$ in our c-coloring of $[X]^k$.

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Since c is finite, one of the colors is associated with infinitely many x_i—they form an infinite monochromatic subset of X.

Theorem 11.10 (König Infinity Lemma)

Let V_0, V_1, ... be an infinite sequence of disjoint non-empty finite sets, and let G be an infinite graph on their union. Assume that every vertex v in V_n, for $n \geq 1$, has a neighbor $f(v)$ in V_{n-1}. Then G contains a ray, that is a one-way-infinite path, $v_0 v_1 \ldots$ with $v_n \in V_n$ for all n.

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Let $\mathcal{P}$ be the set of all finite paths of the form $v f(v) f(f(v)) \ldots$.

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Theorem 11.11

For all positive integers k, c, and r there is an integer $n \geq k$ such that every n-element set X has a monochromatic r-element subset with respect to any c-coloring of $[X]^k$.

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Then g is a bad coloring of $[\mathbb{N}]$ since every r-element subset S of $\mathbb{N}$ is contained in some sufficiently large $[n]$, and so S cannot be monochromatic since g coincides on $[n]^k$ with the bad coloring g_n.

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Then g is a bad coloring of $[\mathbb{N}]$ since every r-element subset S of $\mathbb{N}$ is contained in some sufficiently large $[n]$, and so S cannot be monochromatic since g coincides on $[n]^k$ with the bad coloring g_n. This contradicts 11.9.

Ramsey numbers

Definition 11.12

The least integer n associated with k, c, and r as in Theorem 11.11 is the Ramsey number for k, c and r, and is denoted by $R(k, c, r)$.

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We proved before that $R(2, 2, 3) = 6$, and that $R(K_3, K_3) = 6$. In most cases the exact Ramsey numbers are not known. Most known values and bounds are listed at http://mathworld.wolfram.com/RamseyNumber.html

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Let s and t be positive integers, and let T be a tree of order t. Then $R(T, K_s) = (s-1)(t-1)+1$.

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Theorem 11.14 (Chvatál, Rödl, Szemerédi and Trotter 1983)

For every positive integer Δ there is a constant c such that $R(H) \leq c|H|$ for all graphs H with $\Delta(H) \leq \Delta$.

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Proof omitted—uses the Regularity Lemma.

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Theorem 11.15 (Deuber; Erdős, Hajnal, Pósa; Rödl 1973)

Every graph has a Ramsey graph. For every graph H there is a graph G such that, for every partition $\{E_1, E_2\}$ of $E(G)$, has an induced subgraph H with $E(H) \subseteq E_1$ or $E(H) \subseteq E_2$.

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Given two graphs $G = (V, E)$ and H, and $U \subseteq V$, we write $G[U \to H]$ to denote the graph obtained from G by replacing each vertex u in U by a copy $H(u)$ of H,

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(*) For any two graphs H_1 and H_2, there is a graph $G = G(H_1, H_2)$ such that every edge-coloring of G with colors 1 and 2 yields either an induced $H_1 \subseteq G$ with all edges colored 1, or an induced $H_2 \subseteq G$ with all edges colored 2.

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We proceed by induction on $|H_1| + |H_2|$. If either H_1 or H_2 has no edges, in particular, $|H_1| + |H_2| \leq 1$, then (*) holds with $G = \overline{K_n}$ for sufficiently large n.

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We proceed by induction on $|H_1| + |H_2|$. If either H_1 or H_2 has no edges, in particular, $|H_1| + |H_2| \leq 1$, then (*) holds with $G = \overline{K_n}$ for sufficiently large n. For the induction step, assume that both H_1 and H_2 have at least one edge, and that (*) holds for all pairs (H_1', H_2') with smaller $|H_1'| + |H_2'|$.

For each $i \in \{1, 2\}$, pick a vertex $x_i \in H_i$ that is incident with an edge, let $H_i' = H_i - x_i$, and let H_i'' be the subgraph of H_i induced by the neighbors of x_i.

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By the induction hypothesis, there are Ramsey graphs $G_1 = G(H_1, H'_2)$ and $G_2 = G(H'_1, H_2)$.

For each $i \in \{1, 2\}$, pick a vertex $x_i \in H_i$ that is incident with an edge, let $H_i' = H_i - x_i$, and let H_i'' be the subgraph of H_i induced by the neighbors of x_i. We will construct a sequence G^0, G^1, ..., G^n of disjoint graphs with G^n being the desired Ramsey graph $G(H_1, H_2)$. We will also define subsets $V_i \subseteq V(G^i)$ and a map $f : V^1 \cup V^2 \cup \cdots \cup V^n \to V^0 \cup V^1 \cup \cdots \cup V^{n-1}$ such that $f(V^i) = V^{i-1}$ for all $i \geq 1$. We will also write f^i for composing f with itself i times, with f^0 understood as the identity map on $V(G^0)$. Note that $f^i(v) \in V^0$ for all $v \in V^i$; we call $f^i(v)$ the origin of v.

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If some $G_2(u)$ contains an induced H_2 colored 2, then the conclusion holds. If not, then every $G_2(u)$ has an induced subgraph $H'_1(u) \cong H'_1$ colored 1. Let F be the family of these graphs $H'_1(u)$, one for each $u \in U^{i-1}$ and let $x = x(F)$.

If, for some $u \in U^{i-1}$, all the $x - H_1''(u)$ edges in G^i are also colored 1, then we have an induced copy of H_1 in G^i and again the conclusion holds.

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to G^{i-1}, and so our edge-coloring of G^i induces an edge-coloring of G^{i-1}. If this coloring gives an induced $H_1 \subseteq G^{i-1}$ colored 1, or an induced $H_2 \subseteq G^{i-1}$ colored 2, we have these also in $\hat{G}^{i-1} \subseteq G^i$ and again the conclusion holds.
By (**) for $i-1$ we may then assume that G^{i-1} has an induced H' colored 2 with $V(H') \subseteq V^{i-1}$ and such that the restriction of f^{i-1} to $V(H')$ is an isomorphism from H' to $G^0[W_k'] \cong H_2'$ for some $k \in \{i-1, \ldots, n-1\}$. Let $\hat{H}'$ be the corresponding induced subgraph of $\hat{G}^{i-1} \subseteq G^i$ (also colored 2).

If, for some $u \in U^{i-1}$, all the $x - H_1''(u)$ edges in G^i are also colored 1, then we have an induced copy of H_1 in G^i and again the conclusion holds. So we may assume that each $H_1''(u)$ has a vertex y_u for which the edge xy_u is colored 2. The restriction $y_u \mapsto u$ of f to $\hat{U}^{i-1} = \{y_u | u \in U^{i-1}\} \subseteq V^i$ extends by $(v, \emptyset) \mapsto v$ to an isomorphism from

$$\hat{G}^{i-1} = G^i[\hat{U}^{i-1} \cup \{(v, \emptyset) | v \in V(G^{i-1}) \setminus U^{i-1}\}]$$

to G^{i-1}, and so our edge-coloring of G^i induces an edge-coloring of G^{i-1}. If this coloring gives an induced $H_1 \subseteq G^{i-1}$ colored 1, or an induced $H_2 \subseteq G^{i-1}$ colored 2, we have these also in $\hat{G}^{i-1} \subseteq G^i$ and again the conclusion holds.

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$$f^i(V(\hat{H}')) = f^{i-1}(V(H')) = W_k',$$

and $F^i : \hat{H}' \to G^0[W_k']$ is an isomorphism.

If, for some $u \in U^{i-1}$, all the $x - H_1''(u)$ edges in G^i are also colored 1, then we have an induced copy of H_1 in G^i and again the conclusion holds. So we may assume that each $H_1''(u)$ has a vertex y_u for which the edge xy_u is colored 2. The restriction $y_u \mapsto u$ of f to $\hat{U}^{i-1} = \{y_u | u \in U^{i-1}\} \subseteq V^i$ extends by $(v, \emptyset) \mapsto v$ to an isomorphism from

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By (**) for $i-1$ we may then assume that G^{i-1} has an induced H' colored 2 with $V(H') \subseteq V^{i-1}$ and such that the restriction of f^{i-1} to $V(H')$ is an isomorphism from H' to $G^0[W_k'] \cong H_2'$ for some $k \in \{i-1, \dots, n-1\}$. Let $\hat{H}'$ be the corresponding induced subgraph of $\hat{G}^{i-1} \subseteq G^i$ (also colored 2). then $V(\hat{H}') \subseteq V_i$,

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If $k \geq i$, then the proof of (**) is complete with $H = \hat{H}'$.

We thus assume that $k < i$, and so $k = i - 1$.

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Problem 22

Prove that for every positive integer k there is an integer N such that if G is a 2-connected graph of order at least N, then G has a subdivision of C_k or $K_{2,k}$. Find an upper bound on N it terms of k.

Problem 23

Find a Ramsey graph for C_4, that is, find a graph G such that if the edges of G are partitioned into $\{E_1, E_2\}$, then G has a induced subgraph isomorphic to C_4 all of whose edges belong to one of E_1 or E_2.

Ramsey Theorem for Connected Graphs

Theorem 11.16

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Parallel Minors

Recall that a graph is a topological minor of another if it can be obtained by

- deleting edges or isolated vertices
- contracting edges in series that are in series with another edge

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Topological minors are also called series minors
A graph is a parallel minor of another if it can be obtained by

- contracting edges
- deleting edges that are in parallel with other edges (simplifying)

Unavoidable Parallel Minors

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Theorem 11.21

For every positive integer k there is an integer N such that if G is a 2-connected graph of order at least N, then G contains as a parallel minor $K'_{2,k}$ ($K_{2,k}$ plus an edge) or

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For every positive integer k there is an integer N such that if G is a 4-connected graph of order at least N, then G contains as a parallel minor $K'_{4,k}$ or DW_k or DW'_k (double wheel with an axle) or TF_k (triple fan) or Z_k or M_k or K_k.

Conjecture 11.24 (Chartrand, Geller, Hedetniemi)

Every planar graph is a union of two outerplanar graphs.

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Theorem 11.25 (Ding, O., Sanders, Vertigan; Kedlaya)

Every planar has an edge-partition into two graphs of tree-width ≤ 2.

Partitioning Graphs on Surfaces

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Proofs

- Set $v \in V(G)$ and $V_k =$ set of vertices distance k from v.
- Vertex-partitions: graphs induced by $\bigcup_{k \text{ even}} V_k$ and $\bigcup_{k \text{ odd}} V_k$
- Edge-partitions: let $H_k =$ induced by edges $[V_k, V_k]$ and $[V_k, V_{k+1}]$ $\bigcup_{k \text{ even}} H_k$ and $\bigcup_{k \text{ odd}} H_k$

Minor-Closed Classes

Conjecture 11.30 (Thomas)

For every G there is an integer k such that every graph with no G-minor has a vertex-partition and edge-partition into two graphs of tw $\leq k$.

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- deleting edges and/or vertices
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Theorem 11.31 (DOSV, DeVos, Reed, Seymour)

For every minor-closed class of graphs other than the class of all graphs there is a number k such that every member of the class has a vertex-partition and edge-partition into two graphs of tw $\leq k$.

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All members of any minor-closed class of graphs other than the class of all graphs are clique-sums of graphs that can "almost" be embedded on surfaces of bounded genus.

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Theorem 11.33 (R&S)

Every minor-closed class of graphs can be characterized by excluding finitely many graphs as minors.

Partitions and Contractions

Question 11.34 (Oxley)

Can every co-graphic matroid be partitioned into two series-parallel matroids?

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Theorem 11.37 (Demaine, Hajiaghayi, Mohar)

The edges of a graph of genus g can be partitioned into E_1 and E_2 such that each of G/E_1 and G/E_2 has tw $\leq O(g^2)$.

Definition of $T(k, l, r)$

- Start with K_k, and assign all of its vertices level 0

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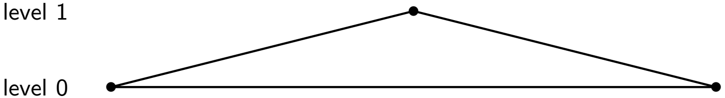
$T(2,4,1)$



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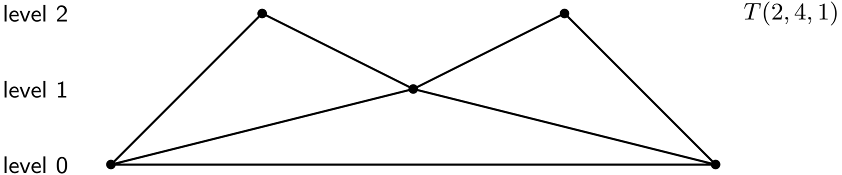

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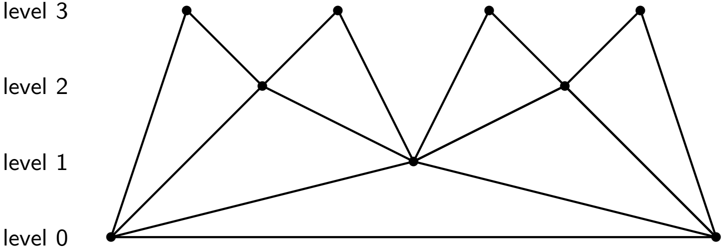

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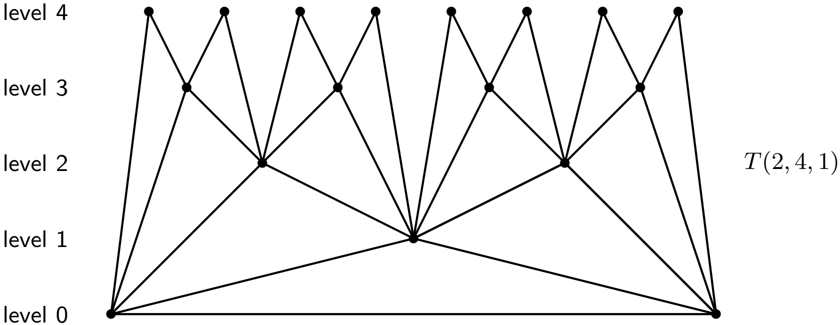

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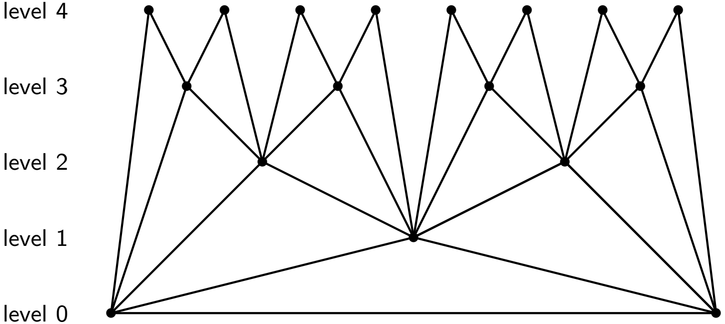
- Stop after having created all level-l subgraphs.



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Partitioning k-Trees

Definition 11.38

k-tree: $T(k, l, r)$ where l is arbitrary, and r can very arbitrarily at every stage.

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Every $(k_1 + k_2 + 1)$-tree has a vertex-partition into a k_1-tree and a k_2-tree.

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Ramsey-Type Results

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For every k_1, k_2, l, and r there is L such that

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For every k_1, k_2, l, *and* r *there is* L *such that*
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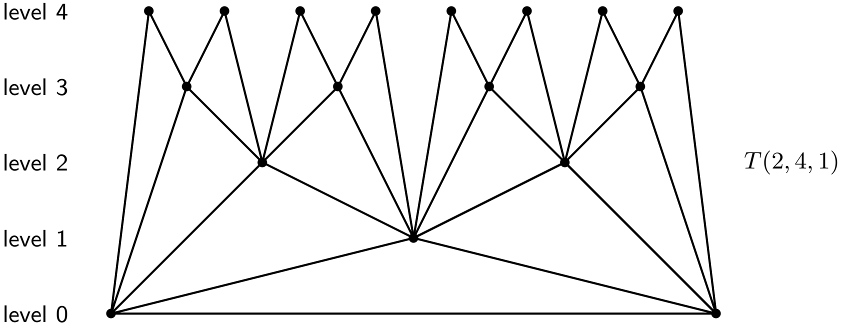
Conjecture 11.43 (DOSV)

For every k, l, and r there is L such that for every edge-partition $\{G_1, \ldots, G_k\}$ of $T(k, L, r)$: at least one G_i contains a subdivision of $T(1, l, r)$.

Large k-Trees and Edge-Partitions



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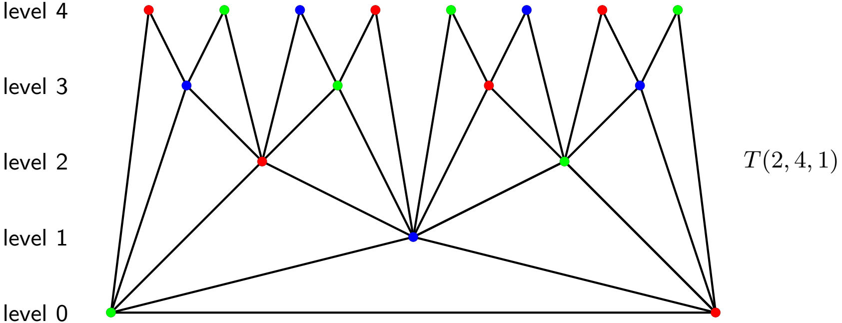

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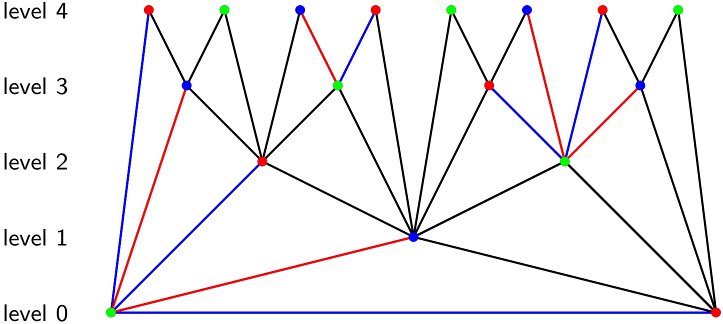

$T(2,4,1)$

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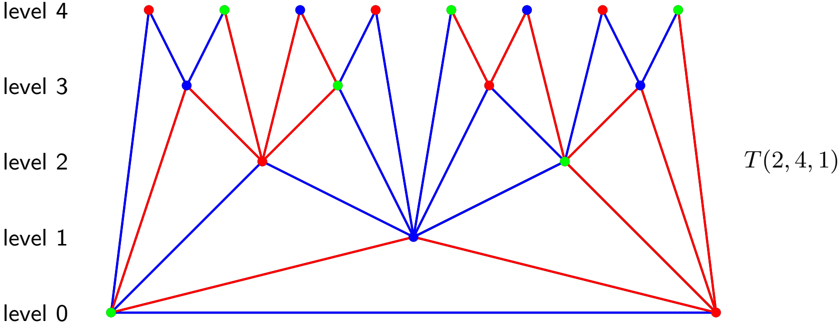

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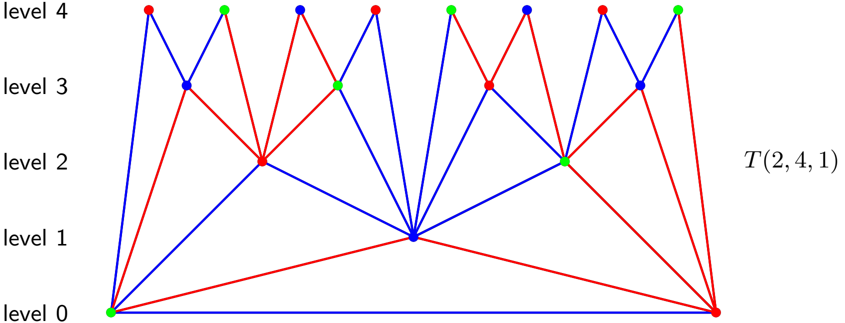

Theorem 11.44 (DOSV)

For every l and r there are L and R such that if $T(2, L, R)$ has its edges colored red and blue, then it contains a red $T(1, l, r)$ or a blue subdivision of $T(1, l, r)$.

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A: No, there are 4-regular graphs of arbitrarily large girth (Erdős, Sachs)

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- *there is a vertex-partition of G into 2 graphs with components on only $\leq 24k\Delta$ vertices;*
- *there is an edge-partition of G into 2 graphs with components on only $\leq 24k\Delta(\Delta+1)$ vertices.*

Q: Is it enough to bound just the tree-width?

A: No, consider

- large star for edge-partitions,
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Four-Color Theorem for Minor-Closed Classes of Graphs

Theorem 11.51 (ADOV+S)

Let $\mathcal{G}$ be a minor-closed class of graphs other than the class of all graphs, and pick Δ.
There is a number $c(\mathcal{G}, \Delta)$ such that every member of $\mathcal{G}$ whose max degree is $\leq \Delta$ can be vertex 4-colored so that all monochromatic components have at most c vertices.

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- This gives a 4-coloring of G with components on at most c vertices.

$\square$