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# MOTION PLANNING IN CONNECTED SUMS OF REAL PROJECTIVE SPACES

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Abstract. The topological complexity $\mathsf{TC}(X)$ is a homotopy invariant of a topological space $X$, motivated by robotics, and providing a measure of the navigational complexity of $X$. The topological complexity of a connected sum of real projective planes, that is, a high genus nonorientable surface, is known to be maximal. We use algebraic tools to show that the analogous result holds for connected sums of higher dimensional real projective spaces.

## 1. Introduction

Let $X$ be a finite, path-connected CW-complex. Viewing $X$ as the space of configurations of a mechanical system, the motion planning problem consists of constructing an algorithm which takes as input pairs of configurations $(x_0, x_1) \in X \times X$, and produces a continuous path $\gamma\colon [0,1] \to X$ from the initial configuration $x_0 = \gamma(0)$ to the terminal configuration $x_1 = \gamma(1)$. The motion planning problem is of significant interest in robotics; see, for example, Jean-Claude Latombe [15] and Micha Sharir [17].

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Michael Farber develops a topological approach to the motion planning problem in [9], [10], and [11]. Let $I = [0, 1]$ be the unit interval, and let $X^I$ be the space of continuous paths $\gamma\colon I \to X$ (with the compact-open topology). The map $\mathrm{ev}\colon X^I \to X \times X$, defined by sending a path to its endpoints, $\mathrm{ev}(\gamma) = (\gamma(0), \gamma(1))$, is a fibration, with fiber $\Omega(X)$, the based loop space of $X$. The motion planning problem requests a section of this fibration, a map $s\colon X \times X \to X^I$ satisfying $\mathrm{ev} \circ s = \mathrm{id}_{X\times X}$. It would be desirable for the motion planning algorithm to depend continuously on the input. However, there exists a globally continuous section $s\colon X \times X \to PX$ if and only if $X$ is contractible; see [9, Theorem 1]. This prompts the study of the discontinuities of such algorithms and leads to the following definition from [10].

**Definition 1.1.** A *motion planner* for $X$ is a collection of subsets $F_0, F_1, \ldots, F_m$ of $X \times X$ and continuous maps $s_i\colon F_i \to PX$ such that

(1) the sets $F_i$ are pairwise disjoint, $F_i \cap F_j = \emptyset$ if $i \neq j$, and cover $X \times X$,
$$X \times X = F_0 \cup F_1 \cup \cdots \cup F_m;$$
(2) $\mathrm{ev} \circ s_i = \mathrm{id}_{F_i}$ for each $i$; and
(3) each $F_i$ is a Euclidean neighborhood retract.

Refer to the sets $F_i$ as local domains of the motion planner and the maps $s_i$ as local rules. Call a motion planner optimal if it requires a minimal number of local domains (rules, respectively).

**Definition 1.2.** For a finite, path-connected CW-complex $X$, the (reduced) *topological complexity of* $X$, $\mathsf{TC}(X)$, is one less than the number of local domains in an optimal motion planner for $X$, $\mathsf{TC}(X) = m$ if there exists an optimal motion planner $F_0, F_1, \ldots, F_m$ for $X$.

### 1.1. Motion planning in cell complexes.

We briefly recall from [10, §3] a construction of a motion planner for a finite cell complex. Recall that $X$ is a finite, path-connected CW-complex, and let $X^k$ be the $k$-dimensional skeleton of $X$. Assume that $\dim(X) = n$, and for $k = 0, 1, \ldots, n$, let $V^k = X^k \setminus X^{k-1}$ be the union of the open $k$-cells of $X$. For $i = 0, 1, \ldots, 2n$, the sets $F_i = \bigcup_{k+l=i} V^k \times V^l \subset X \times X$ are homeomorphic to disjoint unions of balls, so are Euclidean neighborhood retracts. Note that $F_0 \cup F_1 \cup \cdots \cup F_{2n} = X \times X$.

To define a local rule $s_i\colon F_i \to X^I$, since $F_i$ is the union of disjoint sets $V^k \times V^l$ (which are both open and closed in $F_i$), it suffices to construct a continuous map $s_{k,l}\colon V^k \times V^l \to X^I$ satisfying $\mathrm{ev} \circ s_{k,l} = \mathrm{id}_{V^k\times V^l}$. Pick a point $v_k \in V^k$ for each $k$, and fix a path $\gamma_{k,l}$ in $X$ from $v_k$ to $v_l$ for each $k$ and $l$. Then, for any $(x, y) \in V^k \times V^l$, one can construct a path $s_{k,l}(x, y)$

from $x$ to $y$ by first moving from $x$ to $v_k$ in the cell $V^k$, then traversing the fixed path $\gamma_{k,l}$, and finally moving from $v_l$ to $y$ in $V^l$.

This construction exhibits a motion planner for $X$ with $2\dim(X)+1$ local domains. Consequently, we have the upper bound $\mathsf{TC}(X) \leq 2\dim(X)$ (for a finite, path-connected CW-complex $X$). This upper bound is achieved by many spaces of interest in topology and applications. For instance, it is well known that $\mathsf{TC}(\Sigma_g) = 4$ for an orientable surface $\Sigma_g$ of genus $g \geq 2$; see [9]. More recent work of Alexander Dranishnikov [6], [7] and the authors [3] shows that the same holds for nonorientable surfaces of high genus. Observe that the construction above provides an optimal motion planner in these instances.

### 1.2. Main result.

The objective of this note is to establish a higher dimensional analog of these last results. Let $\mathcal{P}_g^n = \mathbb{RP}^n \# \cdots \# \mathbb{RP}^n$ be the connected sum of $g$ copies of the real projective space $\mathbb{RP}^n$.

**Theorem 1.3.** *For $n \geq 2$ and $g \geq 2$, we have $\mathsf{TC}(\mathcal{P}_g^n) = 2n$.*

Thus, applying the construction in §1.1 above to a standard CW decomposition of the space $\mathcal{P}_g^n$ yields an optimal motion planner for this space.

When $n = 2$, $\mathcal{P}_g^2 = N_g$ is the nonorientable surface of genus $g$, and it has been established in [3] that $\mathsf{TC}(N_g) = 4$ for $g \geq 2$, completing results obtained by Dranishnikov [6], [7] in the case $g \geq 4$. So we focus on the case $n \geq 3$ below. As we will see, the methods developed in [3] admit extensions to this higher dimensional case.

**Remark 1.4.** The case $g = 1$, with $\mathcal{P}_1^n = \mathbb{RP}^n$, is significantly more subtle. As shown by Farber, Serge Tabachnikov, and Sergey Yuzvinsky [12], for $n \neq 1, 3, 7$, the topological complexity and immersion dimension of $\mathbb{RP}^n$ are equal, $\mathsf{TC}(\mathbb{RP}^n) = \mathrm{imm}(\mathbb{RP}^n)$.

**Remark 1.5.** For closed $n$-dimensional manifolds $M$ and $N$, techniques analogous to those presented here provide conditions under which $\mathsf{TC}(M \# N) = \mathsf{TC}(M) = 2n$ is maximal; see Remark 3.2.

## 2. Preliminaries

Let $p\colon E \to B$ be a fibration. The (reduced) sectional category, or Schwarz genus, of $p$, denoted by $\mathrm{secat}(p)$, is the smallest integer $m$ such that $B$ can be covered by $m + 1$ open subsets, over each of which $p$ has a continuous section. Classical references include A. S. Schwarz [16] and I. M. James [14]. The following result makes clear the topological nature of the motion planning problem.

**Theorem 2.1** (cf. [11, §4.2])**.** *If $X$ is a finite CW-complex, then the topological complexity of $X$ is equal to the sectional category of the path-space fibration* $\mathrm{ev}\colon X^I \to X \times X$, $\mathsf{TC}(X) = \mathrm{secat}(\mathrm{ev})$.

The equality $\mathsf{TC}(X) = \mathrm{secat}(\mathrm{ev}\colon X^I \to X \times X)$ yields the following estimates:

$$\max\{\mathrm{cat}(X), \mathrm{zcl}_{\Bbbk}(X)\} \leq \mathsf{TC}(X) \leq 2\,\mathrm{cat}(X) \leq 2\dim(X);\ \text{see [9]}.$$

Here, $\mathrm{cat}(X)$ is the reduced Lusternik–Schnirelmann (LS) category of $X$ and $\mathrm{zcl}_{\Bbbk}(X)$ is the zero-divisors cup-length of the cohomology of $X$ with coefficients in a field $\Bbbk$. More precisely, $\mathrm{zcl}_{\Bbbk}(X)$ is the nilpotency of the kernel of the cup product $H^*(X;\Bbbk) \otimes H^*(X;\Bbbk) \to H^*(X;\Bbbk)$, the smallest nonnegative integer $n$ such that any $(n+1)$-fold cup product in this kernel is trivial.

As noted in §1.1, the upper bound $\mathsf{TC}(X) \leq 2\dim(X)$ may also be obtained from an explicit motion planner construction. We will not make further use of the lower bounds $\mathrm{cat}(X)$ and $\mathrm{zcl}_{\Bbbk}(X)$, which are included here primarily for context and are both insufficient for our purposes. Indeed for $g \geq 2$, one can show that $\mathrm{cat}(\mathcal{P}_g^n) = n$ and $\mathrm{zcl}_{\mathbb{Z}_2}(\mathcal{P}_g^n) = 2n-1$. Following [3], we will instead utilize the topological complexity analog of the classical Berstein–Schwarz cohomology class, which informs on the LS category; see [4, Theorem 2.51].

Let $X$ be a space and $\pi = \pi_1(X)$ its fundamental group. Let $\mathbb{Z}[\pi]$ be the group ring of $\pi$, $\epsilon\colon \mathbb{Z}[\pi] \to \mathbb{Z}$ the augmentation map, and $I(\pi) = \ker(\varepsilon\colon \mathbb{Z}[\pi] \to \mathbb{Z})$ the augmentation ideal. Recall that $\mathbb{Z}[\pi]$ and $I(\pi)$ are both (left) $\mathbb{Z}[\pi \times \pi]$-modules through the action given by

$$(a,b) \cdot \sum n_i a_i = \sum n_i (a a_i \bar{b}).$$

Here, $n_i \in \mathbb{Z}$, $a, b, a_i \in \pi$, and $\bar{b}$ is the inverse of $b$. In general, (see [19, §6]), left $\mathbb{Z}[\pi \times \pi]$-modules correspond to local coefficient systems on $X \times X$, which we denote by the same symbols.

Let $\mathfrak{v} = \mathfrak{v}_X \in H^1(X \times X; I(\pi))$ be the Costa–Farber canonical class of $X$ introduced in [5], corresponding to the crossed homomorphism $\pi \times \pi \to I(\pi)$, $(a,b) \mapsto a\bar{b} - 1$. The significance of this cohomology class in the context of topological complexity is given by the following result.

**Theorem 2.2** ([5, Theorem 7])**.** *Suppose that $X$ is a CW-complex of dimension $n \geq 2$. Then $\mathsf{TC}(X) = 2n$ if and only if the $2n^{th}$ power of $\mathfrak{v}$ does not vanish:*

$$\mathsf{TC}(X) = 2n \iff \mathfrak{v}^{2n} \neq 0 \text{ in } H^{2n}(X \times X; I(\pi)^{\otimes 2n}).$$

Here $I(\pi)^{\otimes 2n} = I(\pi) \otimes_{\mathbb{Z}} I(\pi) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} I(\pi)$ is the tensor product of $2n$ copies of $I(\pi)$, with the diagonal action of $\pi \times \pi$.

## 3. Reduction to the Case $g = 2$

Let $\pi_g$ denote the fundamental group of the space $\mathcal{P}_g^n$. Since $n \geq 3$, we have $\pi_g = \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$ ($g$ copies). As in [3], we will prove that $\mathsf{TC}(\mathcal{P}_g^n) = 2n$ by proving that the evaluation of $\mathfrak{v}^{2n} \in H^{2n}(\mathcal{P}_g^n \times \mathcal{P}_g^n; I(\pi_g)^{\otimes 2n})$ on the $\mathbb{Z}_2$ top class $[\mathcal{P}_g^n \times \mathcal{P}_g^n]_{\mathbb{Z}_2} \in H_{2n}(\mathcal{P}_g^n \times \mathcal{P}_g^n; \mathbb{Z}_2)$ does not vanish and use the bar resolution to carry out the calculation. As noted in [5, Corollary 8], if $f : X \to Y = K(\pi, 1)$ induces an isomorphism of fundamental groups, we have $(f \times f)^* \mathfrak{v}_Y = \mathfrak{v}_X$. In general, for $f : X \to Y$ and $\rho = \pi_1(f) : \pi_1(X) \to \pi_1(Y)$, we have

$$(f \times f)^* \mathfrak{v}_Y = I(\rho)\mathfrak{v}_X \in H^1(X \times X; I(\pi_1(Y))) \tag{3.1}$$

Let $f_g \colon \mathcal{P}_g^n \to K(\pi_g, 1)$ denote the canonical map, the unique (up to homotopy) map such that $\pi_1(f_g) = \mathrm{id}$. We then analyze the cohomology class $\mathfrak{v}^{2n}$, its evaluation on the homology class $[\mathcal{P}_g^n \times \mathcal{P}_g^n]_{\mathbb{Z}_2}$ in particular, using the cap product diagram (cf. [1, Ch. V, §10])

$$\begin{array}{ccc} H_{2n}(\mathcal{P}_g^n \times \mathcal{P}_g^n; \mathbb{Z}_2) \otimes H^{2n}(\mathcal{P}_g^n \times \mathcal{P}_g^n; I(\pi_g)^{\otimes 2n}) & \xrightarrow{\ \cap\ } & I(\pi_g; \mathbb{Z}_2)^{\otimes 2n}_{\pi_g \times \pi_g} \\ \downarrow (f_g \times f_g)_* \qquad \uparrow (f_g \times f_g)^* & & \downarrow = \\ H_{2n}(\pi_g \times \pi_g; \mathbb{Z}_2) \otimes H^{2n}(\pi_g \times \pi_g; I(\pi_g)^{\otimes 2n}) & \xrightarrow{\ \cap\ } & I(\pi_g; \mathbb{Z}_2)^{\otimes 2n}_{\pi_g \times \pi_g}. \end{array}$$

Here $I(\pi_g; \mathbb{Z}_2) = I(\pi_g) \otimes \mathbb{Z}_2$, and $I(\pi_g; \mathbb{Z}_2)^{\otimes 2n}_{\pi_g \times \pi_g}$ denotes the coinvariants of $I(\pi_g; \mathbb{Z}_2)^{\otimes 2n}$ with respect to the diagonal action of $\pi_g \times \pi_g$, which coincides with $H_0(\mathcal{P}_g^n \times \mathcal{P}_g^n; I(\pi_g)^{\otimes 2n} \otimes \mathbb{Z}_2) = H_0(\pi_g \times \pi_g; I(\pi_g)^{\otimes 2n} \otimes \mathbb{Z}_2)$.

As in [3, Theorem 14], the study of the general case $g \geq 2$ can be reduced to the case $g = 2$. Consider the projection $\mathcal{P}_g^n \to \mathcal{P}_{g-1}^n$ that collapses the last $\mathbb{RP}^n$ connected summand of $\mathcal{P}_g^n$ and induces the projection $\pi_g \to \pi_{g-1}$ which sends the last $\mathbb{Z}_2$ to 1. We have a (homotopy) commutative diagram

$$\begin{array}{ccc} \mathcal{P}_g^n & \xrightarrow{\ f_g\ } & K(\pi_g, 1) \\ \downarrow & & \downarrow \\ \mathcal{P}_{g-1}^n & \xrightarrow{\ f_{g-1}\ } & K(\pi_{g-1}, 1) \end{array} \tag{3.2}$$

The space $\mathcal{P}_g^n$ admits CW-complex structure, based on the standard CW decomposition of $\mathbb{RP}^n$ with a single cell in each dimension. Identify the $(n-1)$-skeleton of the last $\mathbb{RP}^n$ connected summand of $\mathcal{P}_g^n$ with $\mathbb{RP}^{n-1}$, and note that $(\mathcal{P}_g^n, \mathbb{RP}^{n-1})$ is an NDR-pair. Identifying $H_*(\mathcal{P}_g^n, \mathbb{RP}^{n-1}; \mathbb{Z}_2)$ with the reduced homology of $\mathcal{P}_g^n / \mathbb{RP}^{n-1} \simeq \mathcal{P}_{g-1}^n$ in the long exact homology sequence of this pair (cf. [13, Theorem 2.13]), we conclude that the projection $\mathcal{P}_g^n \to \mathcal{P}_{g-1}^n$ induces an isomorphism $\mathbb{Z}_2 \cong H_{2n}(\mathcal{P}_g^n;\mathbb{Z}_2) \longrightarrow H_{2n}(\mathcal{P}_{g-1}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2$.

Write $F_k = (f_k \times f_k)_*$ and $\mathfrak{v}_k = \mathfrak{v}_{\pi_k}$. Considering the morphism $I(\pi_g;\mathbb{Z}_2) \to I(\pi_{g-1};\mathbb{Z}_2)$ induced by the projection $\pi_g \to \pi_{g-1}$, the naturality condition (3.1), and the diagram (3.2), we obtain the following commutative diagram

$$\begin{array}{ccccc} H_{2n}(\mathcal{P}_g^n \times \mathcal{P}_g^n) & \xrightarrow{F_g} & H_{2n}(\pi_g \times \pi_g) & \xrightarrow{\cap \mathfrak{v}_g^{2n}} & I(\pi_g;\mathbb{Z}_2)^{\otimes 2n}_{\pi_g \times \pi_g} \\ \cong \downarrow & & \downarrow & & \downarrow \\ H_{2n}(\mathcal{P}_{g-1}^n \times \mathcal{P}_{g-1}^n) & \xrightarrow{F_{g-1}} & H_{2n}(\pi_{g-1} \times \pi_{g-1}) & \xrightarrow{\cap \mathfrak{v}_{g-1}^{2n}} & I(\pi_{g-1};\mathbb{Z}_2)^{\otimes 2n}_{\pi_{g-1} \times \pi_{g-1}} \end{array}$$

where the $\mathbb{Z}_2$ coefficients in homology are suppressed. Therefore, if the bottom horizontal map does not annihilate the generator, then neither does the top horizontal map. In other words, as in [3], the calculation can be reduced to the "genus" $g = 2$ case. Thus, for $n \geq 3$, Theorem 1.3 will follow from the following proposition which will be proved in the next section.

**Proposition 3.1.** *For $n \geq 3$, $\mathfrak{v}^{2n}([\mathcal{P}_2^n \times \mathcal{P}_2^n]_{\mathbb{Z}_2}) \neq 0$.*

**Remark 3.2.** We note that, for $M$ and $N$ closed $n$-manifolds, a similar argument to the one above permits one to conclude that $\mathsf{TC}(M\#N) = \mathsf{TC}(M) = 2n$ as soon as $\mathfrak{v}^{2n}([M \times M]_{\mathbb{Z}_2})$ is nonzero. Actually, using [8, Lemma 7] (and $\mathbb{Z}$-fundamental classes instead of $\mathbb{Z}_2$ top classes), we can see that $\mathsf{TC}(M\#N)$ is maximal as soon as $\mathsf{TC}(M)$ is maximal whenever $N$ is orientable. Note also that, for simply-connected orientable manifolds, Dranishnikov and Rustam Sadykov [8] establish the more general result that $\mathsf{TC}(M\#N) \geq \mathsf{TC}(M)$.

## 4. The Case $g = 2$

In this section, we prove Proposition 3.1.

### 4.1. Algebraic preliminaries.

Refer to Kenneth S. Brown [2] and Charles A. Weibel [18] as standard references for cohomology of groups and homological algebra. We will use the normalized bar resolution $\bar{B}_*(\pi)$ of $\mathbb{Z}$ as a trivial $\mathbb{Z}[\pi]$-module:

$$\cdots \longrightarrow \bar{B}_n(\pi) \xrightarrow{\partial_n} \cdots \longrightarrow \bar{B}_1(\pi) \xrightarrow{\partial_1} \bar{B}_0(\pi) = \mathbb{Z}[\pi] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

Here $\bar{B}_n(\pi)$ is the free $\mathbb{Z}[\pi]$-module with basis

$$\{[g_1|\cdots|g_n], (g_1,\ldots,g_n) \in \bar{\pi}^n\},$$

where $\bar{\pi} = \{g \in \pi \mid g \neq 1\}$ and $\partial_n$ is the $\mathbb{Z}[\pi]$ morphism given by

$$\begin{aligned}\partial_n([g_1|\cdots|g_n]) = g_1 \cdot [g_2|\cdots|g_n] \\ &+ \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_{i-1}|g_i g_{i+1}|g_{i+2}|\cdots|g_n] \\ &+ (-1)^n[g_1|\cdots|g_{n-1}]\end{aligned}$$

(with $[h_1|\cdots|h_k] = 0$ if $h_i = 1$ for some $i$). The homology of the space $K(\pi,1)$ (or of the group $\pi$) with coefficients in $\mathbb{Z}_2$ is then the homology of the chain complex $\bar{B}_*(\pi;\mathbb{Z}_2) := \bar{B}_*(\pi) \otimes_\pi \mathbb{Z}_2 = (\bar{B}_*(\pi))_\pi \otimes \mathbb{Z}_2$ (with differential $\partial \otimes \mathrm{id}$).

We now describe a cycle representing the image of the $\mathbb{Z}_2$ top class of $\mathcal{P}_2^n = \mathbb{RP}^n \# \mathbb{RP}^n$ under the map induced by $f_2 : \mathcal{P}_2^n \to K(\pi_2, 1)$. We have $H_i(\pi_2;\mathbb{Z}_2) = H_i(\mathbb{RP}^\infty \vee \mathbb{RP}^\infty;\mathbb{Z}_2)$. Let $\mathbf{a}_i$ and $\mathbf{b}_i$ be the homology classes (with $\mathbf{a}_0 = \mathbf{b}_0$) corresponding to the two branches of the wedge. As the two projections $\mathcal{P}_2^n = \mathbb{RP}^n \# \mathbb{RP}^n \to \mathbb{RP}^n$ each induce an isomorphism $H_n(\mathcal{P}_2^n;\mathbb{Z}_2) \to H_n(\mathbb{RP}^n;\mathbb{Z}_2)$, the image of the $\mathbb{Z}_2$ top cell of $\mathbb{RP}^n \# \mathbb{RP}^n$ under the map $f_2 : \mathcal{P}_2^n \to K(\pi_2, 1)$ can be identified with the element $\mathbf{c}_n = \mathbf{a}_n + \mathbf{b}_n$ of $H_n(\pi_2;\mathbb{Z}_2)$ and we are reduced to describe cycles representing the classes $\mathbf{a}_n$ and $\mathbf{b}_n$.

Writing $\pi_2 = \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \,|\, a^2 = 1, b^2 = 1\rangle$, the classes $\mathbf{a}_i$ and $\mathbf{b}_i$ are represented by the following cycles of $\bar{B}_i(\pi_2;\mathbb{Z}_2)$:

$$\alpha_i = [a|a|\cdots|a], \qquad \beta_i = [b|b|\cdots|b].$$

As our calculation will use portions of the calculation carried out in [3], we will use the isomorphism from $\pi_2 = \langle a, b \mid a^2 = 1, b^2 = 1\rangle$ to the infinite dihedral group $D = \langle x, y \mid yxy = x, x^2 = 1\rangle$ given by $a \mapsto x$ and $b \mapsto yx$. We will then work with the following cycles of $\bar{B}_i(D;\mathbb{Z}_2)$ as representatives of the classes $\mathbf{a}_i$ and $\mathbf{b}_i$:

$$\alpha'_i = [x|x|\cdots|x], \qquad \beta'_i = [yx|yx|\cdots|yx].$$

For $X = K(\pi, 1)$, the Costa–Farber $\mathsf{TC}$ canonical cohomology class $\mathfrak{v} \in H^1(X \times X; I(\pi))$ can be described as the class of the canonical degree 1 cocycle, $\nu\colon \bar{B}_1(\pi \times \pi) \to I(\pi)$, which is well defined on the normalized bar resolution and given by

$$\nu([(g,h)]) = g\bar{h} - 1$$

for $[(g,h)] \in \bar{B}_1(\pi \times \pi)$, and $\bar{h} = h^{-1}$ as above. As in [3], we have the following explicit expression of the $n^{\text{th}}$ power of $\mathfrak{v} \in H^1(X \times X; I(\pi))$.

**Lemma 4.1.** *The $n^{th}$ power of the canonical* $\mathsf{TC}$ *cohomology class* $\mathfrak{v}$ *is the class of the cocycle* $\nu^n$ *of degree* $n$ *given by*

$$\begin{aligned} \nu^n \colon \bar{B}_n(\pi\times\pi) &\to I(\pi)^{\otimes n} \\ [(g_1,h_1)|\cdots|(g_n,h_n)] &\mapsto (-1)^{n(n-1)/2}\cdot u_1\otimes u_2\otimes\cdots\otimes u_n \end{aligned}$$

*where* $u_1 = g_1\bar{h}_1 - 1$ *and* $u_i = (g_1\cdots g_{i-1})(g_i\bar{h}_i - 1)(\bar{h}_{i-1}\cdots\bar{h}_1)$ *for each* $i$, $2\le i\le n$.

We will also use the Eilenberg–Zilber chain equivalence (well defined on normalized bar resolutions)

$$EZ\colon \bar{B}_*(\pi)\otimes\bar{B}_*(\pi)\longrightarrow \bar{B}_*(\pi\times\pi), \tag{4.1}$$

which is the $\mathbb{Z}[\pi\times\pi]\cong\mathbb{Z}[\pi]\otimes\mathbb{Z}[\pi]$ morphism given by

$$\begin{aligned} EZ_n\colon \bigoplus_{i=0}^{n}\bar{B}_i(\pi)\otimes\bar{B}_{n-i}(\pi) &\to \bar{B}_n(\pi\times\pi) \\ [g_1|\cdots|g_i]\otimes[h_{i+1}|\cdots|h_n] &\mapsto \sum_{\sigma\in\mathcal{S}_{i,n-i}}\operatorname{sgn}(\sigma)[q_{\sigma^{-1}(1)}|\cdots|q_{\sigma^{-1}(n)}] \end{aligned}$$

where $\mathcal{S}_{i,n-i}$ denotes the set of $(i,n-i)$ shuffles, $\operatorname{sgn}(\sigma)$ is the signature of the shuffle $\sigma$ (which can be omitted over $\mathbb{Z}_2$), and

$$q_k = \begin{cases} (g_k,1) & \text{if } 1\le k\le i, \\ (1,h_k) & \text{if } i+1\le k\le n. \end{cases}$$

**Example 4.2.** We find an explicit expression of $\nu^4(EZ(\alpha_2'\otimes\beta_2'))$ in $I(D;\mathbb{Z}_2)^{\otimes 4}$, which will be useful in the proof of Proposition 3.1. Since $\alpha_2' = [x|x]$ and $\beta_2' = [yx|yx]$, we have

$$\begin{aligned} EZ(\alpha_2'\otimes\beta_2') = {} & [x_1|x_1|y_2x_2|y_2x_2] + [x_1|y_2x_2|x_1|y_2x_2] \\ & + [x_1|y_2x_2|y_2x_2|x_1] + [y_2x_2|x_1|x_1|y_2x_2] \\ & + [y_2x_2|x_1|y_2x_2|x_1] + [y_2x_2|y_2x_2|x_1|x_1] \end{aligned}$$

where $x_1 = (x,1)$, $x_2 = (1,x)$, $y_1 = (y,1)$, and $y_2 = (1,y)$. Using Lemma 4.1 together with the fact that $x^2 = 1$ and $(yx)^2 = 1$, we obtain

(4.2)

$$\begin{aligned} \nu^4(EZ(\alpha_2'\otimes\beta_2')) = {} & (x-1)\otimes(1-x)\otimes(yx-1)\otimes(1-yx) \\ & + (x-1)\otimes x(yx-1)\otimes(1-x)yx\otimes(1-yx) \\ & + (x-1)\otimes x(yx-1)\otimes x(1-yx)\otimes(1-x) \\ & + (yx-1)\otimes(x-1)yx\otimes(1-x)yx\otimes(1-yx) \\ & + (yx-1)\otimes(x-1)yx\otimes x(1-yx)\otimes(1-x) \\ & + (yx-1)\otimes(1-yx)\otimes(x-1)\otimes(1-x) \end{aligned}$$

The image of this expression in the coinvariants $I(D;\mathbb{Z}_2)^{\otimes 4}_{D\times D}$ corresponds to the element $\mathfrak{v}^4(\mathbf{a}_2\times\mathbf{b}_2)\in H_0(D\times D; I(D;\mathbb{Z}_2)^{\otimes 4})\cong I(D;\mathbb{Z}_2)^{\otimes 4}_{D\times D}$.

Let $Y=\langle y\,|\,y^2=1\rangle$ and $Z=\langle z\,|\,z^2=1\rangle$. We have $I(Y;\mathbb{Z}_2)\cong\mathbb{Z}_2(y-1)$ and $I(Z;\mathbb{Z}_2)\cong\mathbb{Z}_2(z-1)$. Consider the projection $I(D;\mathbb{Z}_2)\to I(Y;\mathbb{Z}_2)$ sending $x$ to 1 and the projection $I(D;\mathbb{Z}_2)\to I(Z;\mathbb{Z}_2)$ sending both $x$ and $y$ to $z$ (and hence $yx\mapsto 1$). One can check that, after projection onto $I(Y;\mathbb{Z}_2)^{\otimes 2}\otimes I(Z;\mathbb{Z}_2)^{\otimes 2}\cong\mathbb{Z}_2$, (4.2) yields a unique non-zero term $(y-1)\otimes(y-1)\otimes(z-1)\otimes(z-1)$ which corresponds to the element $[y_2x_2|y_2x_2|x_1|x_1]\in\bar{B}_4(D\times D)$.

### 4.2. Proof of Proposition 3.1.

The statement will follow from the fact that the image of the $\mathbb{Z}_2$ top class of $\mathcal{P}^n_2$,

$$\mathbf{c}_n=(f_2)_*([\mathcal{P}^n_2]_{\mathbb{Z}_2})\in H_n(\pi_2;\mathbb{Z}_2)=H_n(D;\mathbb{Z}_2),$$

satisfies $\mathfrak{v}^{2n}(\mathbf{c}_n\times\mathbf{c}_n)=\mathfrak{v}^{2n}\cap(\mathbf{c}_n\times\mathbf{c}_n)\neq 0$.

Let $G=D\times D$ and $X=K(G,1)$, and let $\Delta\colon X\to X\times X$ be the diagonal map. For the $G$-modules $M=I(D)^{\otimes 2n}$, $M'=I(D)^{\otimes 4}$, and $M''=I(D)^{\otimes 2n-4}$, and the homology and cohomology classes $\zeta\in H_{2n}(G;\mathbb{Z}_2)$ and $\omega\in H^{2n}(G\times G;M)=H^{2n}(G\times G;M'\otimes M'')$, a cap product diagram as in [1, Ch. V, §10] yields

$$\Delta_*(\zeta\cap\Delta^*(\omega))=\Delta_*(\zeta)\cap\omega$$

in $H_0(G\times G;M\otimes\mathbb{Z}_2)$. Fixing $\omega=\mathfrak{v}^4\times\mathfrak{v}^{2n-4}\in H^{2n}(G\times G;M'\otimes M'')$, so that $\Delta^*(\omega)=\mathfrak{v}^4\cup\mathfrak{v}^{2n-4}=\mathfrak{v}^{2n}$, we obtain the commuting diagram

$$\begin{array}{ccc} H_{2n}(G;\mathbb{Z}_2) & \xrightarrow{\Delta_*} & H_{2n}(G\times G;\mathbb{Z}_2) \\ \downarrow{\scriptstyle \mathfrak{v}^{2n}} & & \downarrow{\scriptstyle \mathfrak{v}^4\times\mathfrak{v}^{2n-4}} \\ H_0(G;M\otimes\mathbb{Z}_2) & \xrightarrow{\Delta_*} & H_0(G\times G;M'\otimes M''\otimes\mathbb{Z}_2) \end{array}$$

where the vertical maps are cap products with the indicated cohomology classes.

Let $\kappa_{4,2n-4}$ denote the composition of the Künneth isomorphism and the projection indicated below

$$H_{2n}(G\times G;\mathbb{Z}_2)\to\bigoplus_{i+j=2n}H_i(G;\mathbb{Z}_2)\otimes H_j(G;\mathbb{Z}_2)\twoheadrightarrow H_4(G;\mathbb{Z}_2)\otimes H_{2n-4}(G;\mathbb{Z}_2),$$

and let $\Delta_{4,2n-4} = \kappa_{4,2n-4} \circ \Delta_*$. As $G = D \times D$ and $M = M' \otimes M'' = I(D)^{\otimes 2n}$, identifying zero-dimensional homology groups in the above commuting diagram yields

(4.3)

$$\begin{array}{ccc} H_{2n}(D \times D; \mathbb{Z}_2) & \xrightarrow{\Delta_{4,2n-4}} & H_4(D \times D; \mathbb{Z}_2) \otimes H_{2n-4}(D \times D; \mathbb{Z}_2) \\ \downarrow{\scriptstyle \mathfrak{v}^{2n}} & & \downarrow{\scriptstyle \mathfrak{v}^4 \otimes \mathfrak{v}^{2n-4}} \\ I(D; \mathbb{Z}_2)^{\otimes 2n}_{D \times D} & \longrightarrow & I(D; \mathbb{Z}_2)^{\otimes 4}_{D \times D} \otimes I(D; \mathbb{Z}_2)^{\otimes 2n-4}_{D \times D} \end{array}$$

As above, we consider $Y = \langle y \,|\, y^2 = 1 \rangle$ and $Z = \langle z \,|\, z^2 = 1 \rangle$ and the projections $I(D; \mathbb{Z}_2) \to I(Y; \mathbb{Z}_2)$ and $I(D; \mathbb{Z}_2) \to I(Z; \mathbb{Z}_2)$. We then compose the $\mathfrak{v}^{2n-4}$ portion of the right-hand vertical map in the diagram (4.3) with the projection

$$I(D; \mathbb{Z}_2)^{\otimes 2n-4}_{D \times D} \longrightarrow I(Y; \mathbb{Z}_2)^{\otimes n-2} \otimes I(Z; \mathbb{Z}_2)^{\otimes n-2}.$$

There is no need to pass to coinvariants since the $D \times D$ action on $I(Y; \mathbb{Z}_2)$ and $I(Z; \mathbb{Z}_2)$ is trivial. Observe that

$$I(Y; \mathbb{Z}_2)^{\otimes n-2} \otimes I(Z; \mathbb{Z}_2)^{\otimes n-2} \cong \mathbb{Z}_2(y-1)^{\otimes n-2} \otimes \mathbb{Z}_2(z-1)^{\otimes n-2} \cong \mathbb{Z}_2.$$

Since $\mathbf{c}_n = \mathbf{a}_n + \mathbf{b}_n$, the expression $\Delta_{4,2n-4}(\mathbf{c}_n \times \mathbf{c}_n)$ decomposes as

$$\sum_{i=0}^{4} (\mathbf{a}_i \times \mathbf{a}_{4-i}) \otimes (\mathbf{a}_{n-i} \times \mathbf{a}_{n-4+i}) + \sum_{i=0}^{4} (\mathbf{a}_i \times \mathbf{b}_{4-i}) \otimes (\mathbf{a}_{n-i} \times \mathbf{b}_{n-4+i})$$
$$+ \sum_{i=0}^{4} (\mathbf{b}_i \times \mathbf{a}_{4-i}) \otimes (\mathbf{b}_{n-i} \times \mathbf{a}_{n-4+i}) + \sum_{i=0}^{4} (\mathbf{b}_i \times \mathbf{b}_{4-i}) \otimes (\mathbf{b}_{n-i} \times \mathbf{b}_{n-4+i})$$

Now, we can check that, among the right-hand components, the only terms on which the projection of $\mathfrak{v}^{2n-4}$ on $I(Y; \mathbb{Z}_2)^{\otimes n-2} \otimes I(Z; \mathbb{Z}_2)^{\otimes n-2}$ does not vanish are $\mathbf{a}_{n-2} \times \mathbf{b}_{n-2}$ and $\mathbf{b}_{n-2} \times \mathbf{a}_{n-2}$, represented respectively by $EZ(\alpha'_{n-2} \otimes \beta'_{n-2})$ and $EZ(\beta'_{n-2} \otimes \alpha'_{n-2})$. Furthermore, calculating in $I(Y; \mathbb{Z}_2)^{\otimes n-2} \otimes I(Z; \mathbb{Z}_2)^{\otimes n-2}$, we have

$$\mathfrak{v}^{2n-4}(\mathbf{a}_{n-2} \times \mathbf{b}_{n-2}) = \mathfrak{v}^{2n-4}(\mathbf{b}_{n-2} \times \mathbf{a}_{n-2}) = (y-1)^{\otimes n-2} \otimes (z-1)^{\otimes n-2}.$$

Consequently, in $I(D; \mathbb{Z}_2)^{\otimes 4}_{D \times D} \otimes I(Y; \mathbb{Z}_2)^{\otimes n-2} \otimes I(Z; \mathbb{Z}_2)^{\otimes n-2}$, we have

$$\mathfrak{v}^{2n}(\mathbf{c}_n \times \mathbf{c}_n) = \mathfrak{v}^4(\mathbf{a}_2 \times \mathbf{b}_2 + \mathbf{b}_2 \times \mathbf{a}_2) \otimes (y-1)^{\otimes n-2} \otimes (z-1)^{\otimes n-2}.$$

We now check that $\mathfrak{v}^4(\mathbf{a}_2 \times \mathbf{b}_2 + \mathbf{b}_2 \times \mathbf{a}_2)$ does not vanish in $I(D; \mathbb{Z}_2)^{\otimes 4}_{D \times D}$. For this, recall the expression (in $I(D; \mathbb{Z}_2)^{\otimes 4}$) of $\nu^4(EZ(\alpha'_2 \otimes \beta'_2))$ which has been obtained in (4.2).

By considering, as in [3], the projection

$$I(D; \mathbb{Z}_2)^{\otimes 4} \to I(D; \mathbb{Z}_2) \otimes \bigwedge^3 (I(D; \mathbb{Z}_2)),$$

together with the relation $xyx = \bar{y}$, we see that the expression (4.2) reduces to

$$(yx - x) \otimes (1 - yx) \wedge (1 - \bar{y}) \wedge (1 - x).$$

Calculating the image of $\nu^4(EZ(\beta_2' \otimes \alpha_2'))$ in $I(D; \mathbb{Z}_2) \otimes \bigwedge^3(I(D; \mathbb{Z}_2))$ in an analogous manner yields

$$(yx - x) \otimes (1 - yx) \wedge (1 - y) \wedge (1 - x).$$

As in [3], we then send the first component to $I(Y; \mathbb{Z}_2) \cong \mathbb{Z}_2$ (through $x \mapsto 1$) and the statement follows from the fact that the sum of the two elements above is the element

$$s = (x - 1) \wedge (yx - 1) \wedge (y - \bar{y}) \in \bigwedge^3 I(D; \mathbb{Z}_2),$$

which is shown to be nonzero in [3, §3.3.2]. □

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