Abstracts

Chen ranks and resonance

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(joint work with Henry K. Schenck)

Chen Ranks. Let G be a finitely presented group, with commutator subgroup $G' = [G, G]$, and second commutator subgroup $G'' = [G', G']$. The Chen groups of G are the lower central series quotients $\mathrm{gr}_k(G/G'')$ of G/G''. These groups were introduced by K.T. Chen in [1], so as to provide accessible approximations of the lower central series quotients of a link group. For example, if $G = F_n$ is the free group of rank n (the fundamental group of the n-component unlink), the Chen groups are free abelian, and their ranks, $\theta_k(G) = \operatorname{rank} \mathrm{gr}_k(G/G'')$, are given by $\theta_k(F_n) = (k-1)\binom{k+n-2}{k}$ for $k \geq 2$. In particular, $\theta_k(F_2) = k-1$ for $k \geq 2$.

Let P_n be the Artin pure braid group on n strands, the fundamental group of the configuration space of n ordered points in $\mathbb{C}$. The Chen groups of P_n are free abelian, and their ranks are given by $\theta_k(P_n) = \binom{n+1}{4}(k-1)$ for $k \geq 3$, see [3].

Resonance. Let $A = H^*(G; \mathbb{C})$. For $a \in A^1$, since $a \cup a = 0$, multiplication by a provides A with the structure of a (cochain) complex: $A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{a} \dots$ The (first) resonance variety of G is $\mathcal{R}^1(G) = \{a \in A^1 \mid H^1(A, a) \neq 0\}$, a homogeneous algebraic subvariety in $A^1 = H^1(G; \mathbb{C})$.

The group G is said to be 1-formal if the Malcev Lie algebra of G is quadratic (see [9] for details). For any finitely generated 1-formal group, Dimca-Papadima-Suciu [5] show that all irreducible components of the resonance variety $\mathcal{R}^1(G)$ are linear subspaces of A^1. In particular, this holds for an arrangement group, the fundamental group $G(\mathcal{A}) = \pi_1(M(\mathcal{A}))$ of the complement of a complex hyperplane arrangement, as previously shown by a number of authors, including Libgober-Yuzvinsky [8], who additionally show that the irreducible components of $\mathcal{R}^1(G(\mathcal{A}))$ are "projectively disjoint" – they meet only at the origin in A^1.

For example, if $G = P_n$ is the pure braid group, the fundamental group of the complement of the braid arrangement, $\mathcal{R}^1(P_n)$ is a union of $\binom{n+1}{4}$ 2-dimensional linear subspaces of $A^1 = H^1(P_n; \mathbb{C})$, see for instance [4]. Note that, for $k \geq 3$, $\theta_k(P_n) = \binom{n+1}{4}\theta_k(F_2)$. This, and many other examples, led to the following.

Conjecture 1 (Suciu [12])**.** *If $G = G(\mathcal{A})$ is an arrangement group, then, for $k \gg 0$, $\theta_k(G) = \sum_{r\geq 2} h_r \theta_k(F_r)$, where h_r is the number of irreducible components of dimension r in $R^1(G)$.*

In this talk, we announce a positive resolution of this conjecture. Some additional terminology is required to state the general result.

Recall that $A = H^*(G; \mathbb{C})$, and let $\mu \colon A^1 \wedge A^1 \to A^2$ be the cup product map, $\mu(a \wedge b) = a \cup b$. A non-zero subspace $U \subseteq A^1$ is said to be p-isotropic with respect to the cup product map if the restriction of μ to $U \wedge U$ has rank p.

A group G is said to be a commutator-relators group if it admits a presentation $G = F/R$, where F is a finitely generated free group and R is the normal closure of a finite subset of $[F,F]$. For such a group, the resonance variety $\mathcal{R}^1(G)$ may be realized as the variety defined by the annihilator of the linearized Alexander invariant B of G, a module over the polynomial ring $S = \mathrm{Sym}(H_1(G;\mathbb{C})) = \mathbb{C}[x_1,\dots,x_n]$, $\mathcal{R}^1(G) = V(\mathrm{ann}(B))$. We can thus view $\mathcal{R}^1(G)$ as a scheme.

Theorem 1. *Let G be a finitely presented, 1-formal, commutator-relators group. Assume that the components of $\mathcal{R}^1(G)$ are (i) 0-isotropic, (ii) projectively disjoint, and (iii) reduced (viewing $\mathcal{R}^1(G)$ as a scheme). Then, for $k \gg 0$,*

$$\theta_k(G) = \sum_{r\geq 2} h_r (k-1)\binom{r+k-2}{k} = \sum_{r\geq 2} h_r \theta_k(F_r),$$

where h_r is the number of irreducible components of dimension r in $R^1(G)$.

Examples illustrating the necessity of the hypotheses in the theorem include:

(1) The Heisenberg group $G = \langle a, b \mid [a,[a,b]], [b,[a,b]]\rangle$ is not 1-formal. Here, $\mathcal{R}^1(G) = H^1(G;\mathbb{C})$ is 0-isotropic since the cup product is trivial. But $\theta_k(G) \neq \theta_k(F_2)$. Since G is nilpotent, $\theta_k(G) = 0$. See [5].

(2) The fundamental group G of a closed, orientable surface of genus $g \geq 2$ is 1-formal. But $\mathcal{R}^1(G) = H^1(G;\mathbb{C})$ is not 0-isotropic, and $\theta_k(G) \neq \theta_k(F_{2g})$. See [9].

(3) Let Γ be the graph with vertex set $\mathcal{V} = \{1,2,3,4,5\}$ and edge set $\mathcal{E} = \{12, 13, 24, 34, 45\}$, and let $G = G_\Gamma$ be the corresponding right angled Artin group. The resonance variety is the union of two 3-dimensional subspaces in $H^1(G;\mathbb{C})$ which are not projectively disjoint, and $\theta_k(G) \neq 2\theta_k(F_3)$. See [10].

(4) Let $G = \langle a,b,c,d \mid [b,c],[a,d],[c,d],[a,c][d,b]\rangle$. As a variety, $\mathcal{R}^1(G)$ is a 2-dimensional subspace of $H^1(G;\mathbb{C})$. But $\mathcal{R}^1(G)$ is not reduced, and $\theta_k(G) \neq \theta_k(F_2)$. We do not know if this group is 1-formal.

Arrangement groups satisfy the hypotheses of the theorem, as does the "group of loops," see below. Examples illustrating the utility of the theorem include:

(1) Let PB_n denote the type B pure Artin group, the fundamental group of the complement of the type B Coxeter arrangement in $\mathbb{C}^n$. Analysis of $\mathcal{R}^1(PB_n)$ yields $\theta_k(PB_n) = \left[16\binom{n}{3} + 9\binom{n}{2}\right](k-1) + \binom{n}{2}(k^2-1)$ for $k \gg 0$.

(2) Let $P\Sigma_n$ be the McCool group of basis conjugating automorphisms of the free group of rank n, also known as the group of loops. Analysis of $\mathcal{R}^1(P\Sigma_n)$ (see [2]) yields $\theta_k(P\Sigma_n) = \binom{n}{2}(k-1) + \binom{n}{3}(k^2-1)$ for $k \gg 0$.

Discussion. We discuss some elements of the proof of the theorem.

As noted previously, since G is 1-formal, work of Dimca-Papadima-Suciu [5] implies that $\mathcal{R}^1(G)$ is a union of linear subspaces in $A^1 = H^1(G;\mathbb{C})$. Since G is additionally a commutator-relators group, work of Papadima-Suciu [9] implies that the Chen ranks of G are given by the Hilbert series of the linearized Alexander invariant B of G, $\sum_{k\geq 2}\theta_k(G)t^k = \mathrm{Hilb}(B,t)$ (with appropriate degree conventions).

Assume that G is minimally generated by n elements, so that $A^1 = H^1(G;\mathbb{C}) \cong \mathbb{C}^n$, generated by $e_1, \ldots, e_n$. Let $E = \bigwedge A^1$ be the exterior algebra on A^1, and let I be the ideal in E generated by $\ker(\mu\colon A^1 \wedge A^1 \to A^2)$, the kernel of the cup product map in degree 2. The linearized Alexander invariant B of G admits a presentation $S \otimes I^3 \xrightarrow{\partial} S \otimes I^2 \to B \to 0$, where the map ∂ is dual to the map $S \otimes I^2 \to S \otimes I^3$ given by multiplication by $x = \sum_{i=1}^n x_i \otimes e_i \in S \otimes E^1$.

If L is an irreducible component of $\mathcal{R}^1(G)$, let I_L be the ideal in E generated by $\bigwedge^2 L$, a subideal of I. Associated to I_L, we have a "local" linearized Alexander invariant B_L and a surjection $B \to B_L$. This yields an exact sequence of S-modules

$$0 \longrightarrow K \longrightarrow B \longrightarrow \oplus B_L \longrightarrow C \longrightarrow 0,$$

the direct sum over all irreducible components L of $\mathcal{R}^1(G)$. One can check that $\mathrm{Hilb}(B_L, t) = \sum_{k\geq 2} \theta_k(F_r)$ if $\dim(L) = r$. To prove the theorem, it suffices to show that the kernel K and cokernel C above have finite length.

If $G = G(\mathcal{A})$ is an arrangement group, Schenck-Suciu [11] show that C has finite length. This argument extends. Showing that K has finite length is more involved. One part of this is the following. Given $L \subset \mathcal{R}^1(G)$ as above, let J_L be the ideal in E generated by $\{q \in I \mid \ell \wedge q \in I_L \forall \ell \in L\}$.

Lemma 1. *$I_L = J_L$ if and only if L is reduced.*

Arrangement groups and the basis-conjugating automorphism group are known to satisfy all the hypotheses of the theorem, except possibly the condition that all components of the resonance variety are reduced. For this, by the lemma, it suffices to show that $I_L = J_L$ for each (irreducible) $L \subset \mathcal{R}^1(G)$. This can be done directly in the case where $G = P\Sigma_n$. For $G = G(\mathcal{A})$ an arrangement group, this can be done using the structure of resonance varieties of arrangement groups uncovered by work of Falk [6], Libgober-Yuzvinsky [8], and Falk-Yuzvinsky [7].

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An introduction to Tropical Geometry and Tropical Compactifications

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The field of tropical geometry began as a framework to link amoebas, logarithmic limit sets [1], and (real) algebraic geometry. It synthesized and boosted the pioneering work of Bieri–Groves [2] and Viro's "patchworking" techniques to construct real algebraic varieties by "cutting and pasting" [3, 7]. In its ten years of existence, it has brought on truly explosive development, establishing deep connections with enumerative algebraic geometry, symplectic and analytic geometry, number theory, dynamical systems, mathematical biology, statistical physics, random matrix theory, and mathematical physics.

Tropical geometry can be considered as algebraic geometry over the semifield $(\mathbb{R}, \min, +)$. It is a polyhedral version of classical algebraic geometry: algebraic varieties are replaced by weighted, balanced polyhedral complexes, in order to answer open questions or to derive simpler proofs of classical results. These objects preserve just enough data about the original varieties to remain meaningful, while discarding much of their complexity. There are many approaches to this subject: valuation theory, logarithmic limits sets in the sense of Bergman, and Gröbner theory. Here, we choose the first perspective.

Throughout this talk, we consider an algebraically closed field K with a non-trivial valuation $\mathrm{val}\colon K^* = K \smallsetminus \{0\} \to \mathbb{R}$. Here, by valuation we mean a function that satisfies the following properties:

(1) $\mathrm{val}(f\,g) = \mathrm{val}(f) + \mathrm{val}(g)$, for any pair $f, g \in K^*$,
(2) $\mathrm{val}(f+g) \geq \min\{\mathrm{val}(f), \mathrm{val}(g)\}$.

It is not hard to show that if $\mathrm{val}(f) \neq \mathrm{val}(g)$, then the second condition above is an equality, i.e. $\mathrm{val}(f+g) = \min\{\mathrm{val}(f), \mathrm{val}(g)\}$. By declaring $\mathrm{val}(0) = \infty$, we can extend the valuation to all K.

Our favorite example of a valued field (K, val) as above is given by the Puiseux series $K = \mathbb{C}\{\{t\}\}$, whose elements are Laurent polynomials in $t^{1/n}$, where we let $n \in \mathbb{N}$. The valuation of a series is given by its lowest exponent.

Definition 1. Given an algebraically closed valued field (K, val) as above, and a subvariety $Y \subset (K^*)^n$, we define:

$$\mathcal{T}Y = \mathrm{closure}\{(\mathrm{val}(y_1), \ldots, \mathrm{val}(y_n)) : \underline{y} = (y_1, \ldots, y_n) \in Y\} \subset \mathbb{R}^n,$$

where the closure is taken with respect to the Euclidean Topology in $\mathbb{R}^n$.