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Topological complexity of configuration spaces and related objects, I

Daniel C. Cohen

Investigation of the collision-free motion of n distinct ordered particles in a topological space X leads one to study the (classical) configuration space

$$F(X,n) = \{(x_1,\dots,x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\},$$

and the topological complexity of this space. For a path-connected topological space Y, and Y^I the space of all continuous paths $\gamma\colon [0,1] \to Y$ (with the compact-open topology), the *topological complexity* of Y is the sectional category (or Schwarz genus) of the fibration $\pi\colon Y^I \to Y \times Y$, $\gamma \mapsto (\gamma(0),\gamma(1))$, $\mathsf{TC}(Y) = \operatorname{secat}(\pi)$. This homotopy invariant, introduced by Farber, provides a topological approach to the motion planning problem from robotics

In this lecture, and the next, we survey results on the topological complexity of configuration spaces $F(X,n)$ in the case where X is an orientable surface, as well as related objects. The general principle is as follows:

The topological complexity is as large as possible, given natural constraits.

Throughout the discussion, we will make use of the following basic tools. For details and other relevant facts, see Farber's survey [3]. Additional references mentioned but not explicitly cited below are listed at the end of the second lecture.

$$\mathsf{TC}(Y) \leq 2\cdot \operatorname{hdim}(X)+1 \qquad \mathsf{TC}(Y\times Z) \leq \mathsf{TC}(Y)+\mathsf{TC}(Z)-1$$

$$\mathsf{TC}(Y) > \mathsf{zcl}\, H^*(Y) = \text{cup length}\big[\ker\big(H^*(Y)\otimes H^*(Y) \xrightarrow{\cup} H^*(Y)\big)\big]$$

We call the first two of these the dimension and product inequalities, and use cohomology with $\mathbb{C}$-coefficients (unless stated otherwise) in the context of the third, the zero-divisor cup length. We use the unreduced notion of topological complexity.

The plane: $X = \mathbb{R}^2 = \mathbb{C}$

Theorem 1 (Farber-Yuzvinsky [6])**.** $\mathsf{TC}(F(\mathbb{C},n)) = 2n-2$ for $n \geq 2$

We recall some relevant facts from the theory of hyperplane arrangements.

$$F(\mathbb{C},n) = \{(x_1,\dots,x_n) \in \mathbb{C}^n \mid x_i \neq x_j \text{ if } i \neq j\} \cong M_n \times \mathbb{C}, \text{ where}$$
$$M_n = \{(y_1,\dots,y_{n-1}) \in \mathbb{C}^{n-1} \mid y_i \neq 0\,\forall\, i, y_i - y_j \neq 0\,\forall\, i<j\}$$

M_n is the complement of an essential, central hyperplane arrangement in $\mathbb{C}^{n-1}$.

An *arrangement* $\mathcal{A} = \{H_1,\dots,H_m\}$ in $\mathbb{C}^\ell$ is a finite collection of affine hyperplanes, $H_i = \{f_i = 0\}$, f_i a linear polynomial. $\mathcal{A}$ is *essential* if $\exists\, \ell$ hyperplanes in $\mathcal{A}$ whose intersection is a point. $\mathcal{A}$ is *central* if $0 \in H_i$ for each $i \iff f_i$ is a linear form for each i. The complement of $\mathcal{A}$ is $M = M(\mathcal{A}) = \mathbb{C}^\ell \smallsetminus \bigcup_{i=1}^m H_i$.

$\mathcal{A}$ central $\implies$ restriction of the Hopf bundle $p\colon \mathbb{C}^\ell \smallsetminus \{0\} \to \mathbb{C}P^{\ell-1}$ to M is trivial.

$$\begin{array}{ccccc} \mathbb{C}^* & & & & \mathbb{C}^* \\ \downarrow & & & & \downarrow \\ M & \to & \mathbb{C}^\ell \smallsetminus H_1 & \to & \mathbb{C}^\ell \smallsetminus \{0\} \\ \downarrow & & \downarrow & & \downarrow \\ p(M) & \to & \mathbb{C}P^{\ell-1} \smallsetminus \mathbb{C}P^{\ell-2} & \to & \mathbb{C}P^{\ell-1} \end{array}$$

Consequently, $M \cong p(M) \times \mathbb{C}^*$
$\mathcal{A}$ essential $\Longrightarrow$ $\operatorname{hdim} p(M) = \ell - 1$
$\mathsf{TC}(M) \le \mathsf{TC}(p(M)) + \mathsf{TC}(\mathbb{C}^*) - 1$
$\mathsf{TC}(M) \le 2\ell$
(product, dimension inequalities)

In particular, $F(\mathbb{C}, n) \cong M_n \times \mathbb{C} \simeq M_n \cong p(M_n) \times \mathbb{C}^* \implies \mathsf{TC}(F(\mathbb{C}, n) \le 2n-2$. For the reverse inequality, we use the zero-divisor cup length. The cohomology ring $A = H^*(F(\mathbb{C}, n))$ (with $\mathbb{C}$ coefficients) is classically known, thanks to work of Arnold and Cohen: A is generated by degree one classes $\omega_{i,j} = d\log(x_i - x_j) \in A^1$, $i < j$, with relations consequences of $\omega_{i,j}\omega_{i,k} - \omega_{i,j}\omega_{j,k} + \omega_{i,k}\omega_{j,k} = 0$, $i < j < k$.

Proposition 2 ([6])**.** The zero-divisors $\bar{\omega}_{i,j} = 1 \otimes \omega_{i,j} - \omega_{i,j} \otimes 1 \in A^1 \otimes A^1$ satisfy $\bar{\omega}_{1,2} \cdot \bar{\omega}_{1,3} \cdots \bar{\omega}_{1,n} \cdot \bar{\omega}_{2,3} \cdots \bar{\omega}_{2,n} \neq 0$. Consequently, $\mathsf{zcl}\, H^*(F(\mathbb{C}, n)) \ge 2n-3$.

With the above considerations, this yields $\mathsf{TC}(F(\mathbb{C}, n)) = 2n-2$. Similarly:

Theorem 3 (F.-Grant-Y. [5])**.** $\mathsf{TC}(F(\mathbb{C} \smallsetminus \{m \text{ points}\}, n)) = \begin{cases} 2n & m = 1 \\ 2n+1 & m \ge 2 \end{cases}$

Remark 4. The topological complexity of the configuration space of points in a higher dimensional Euclidean space is also known:

$$\mathsf{TC}(F(\mathbb{R}^k, n)) = \begin{cases} 2n-1 & k \ge 3 \text{ odd } [6] \\ 2n-2 & k \ge 4 \text{ even } [4] \end{cases}$$

<u>Genus zero: $X = S^2$</u>

Theorem 5 (C.-F. [1])**.** $\mathsf{TC}(F(S^2, n)) = \begin{cases} 3 & n = 1,2 \\ 4 & n = 3 \\ 2n-2 & n \ge 4 \end{cases}$

$n \le 2$: $F(S^2, n) \simeq S^2$, and $\mathsf{TC}(S^2) = 3$.
$n = 3$: $F(S^2, 3) \cong \mathrm{PSL}(2, \mathbb{C}) \simeq \mathrm{SO}(3)$, and $\mathsf{TC}(\mathrm{SO}(3)) = \mathrm{cat}(\mathrm{SO}(3)) = 4$, as $\mathrm{SO}(3)$ is a connected Lie group, see [3].
$n \ge 4$: $F(S^2, n) \simeq \mathrm{SO}(3) \times F(S^2 \smallsetminus \{3 \text{ points}\}, n-3)$. The results of [5] apply to $F(S^2 \smallsetminus \{3 \text{ points}\}, n-3) \cong F(\mathbb{C} \smallsetminus \{2 \text{ points}\}, n-3)$. The product inequality gives $\mathsf{TC}(F(S^2, n)) \le 2n-2$. Then one checks that $\mathsf{zcl}\, H^*(F(S^2, n); \mathbb{Z}_2) \ge 2n-3$.

<u>Genus one: $X = T = S^1 \times S^1$</u>

Theorem 6 (C.-F. [1])**.** $\mathsf{TC}(F(T, n)) = 2n+1$

$n = 1$: $F(T, 1) = T$, and $\mathsf{TC}(T) = \mathsf{TC}(S^1 \times S^1) = 3 = 2+1$.
$n \ge 2$: Since T is a group, we have $F(T, n) \cong T \times F(T \smallsetminus \{1 \text{ point}\}, n-1)$ via
$\big((u,v), (uz_1, vw_1), \ldots, (uz_{n-1}, vw_{n-1})\big) \leftarrow\!\shortmid \big((u,v), \big((z_1, w_1), \ldots, (z_{n-1}, w_{n-1})\big)\big)$.

Recall the classical Fadell-Neuwirth theorem: for X a manifold with $\dim X \geq 2$, and $\ell < n$, the map $F(X,n) \to F(X,\ell)$, $(x_1,\ldots,x_n) \mapsto (x_1,\ldots,x_\ell)$, is a bundle, with fiber $F(X \smallsetminus \{\ell \text{ points}\}, n-\ell)$. These bundles often admit sections. Use this result repeatedly, $F(T \smallsetminus \{1 \text{ point}\}, n-1) \to F(T \smallsetminus \{1 \text{ point}\}, n-2) \to \ldots$ each bundle with fiber homotopy equivalent to a wedge of circles and section, to see that $F(T \smallsetminus \{1 \text{ point}\}, n-1)$ is a $K(G,1)$-space. As G is an iterated semidirect product of free groups, the cohomological and geometric dimensions of G are both equal to $n-1$, $\mathrm{cd}(G) = \mathrm{gd}(G) = n-1$, see [2]. Thus, $\mathrm{hdim}(F(T \smallsetminus \{1 \text{ point}\}, n-1)) = n-1$. Then the dimension and product inequalities yield $\mathsf{TC}(F(T,n)) \leq 2n+1$.

The proof of the theorem is completed by showing that $\mathsf{zcl}\, H^*(F(T,n)) \geq 2n$. The tool here is the Cohen-Taylor/Totaro spectral sequence. For X a closed m-manifold, let $p_i \colon X^n \to X$ and $p_{i,j} \colon X^n \to X^2$ be the obvious projections. The inclusion $F(X,n) \to X^n$ yields a Leray spectral sequence converging to $H^*(F(X,n))$. The initial term is the quotient of the algebra $H^*(X^n) \otimes H^*(F(\mathbb{R}^m,n))$ by the relations $(p_i^*(u) - p_j^*(u)) \otimes \omega_{i,j}$ for $i \neq j$, $u \in H^*(X)$, and $\omega_{i,j}$ the generators of $H^*(F(\mathbb{R}^m,n))$ (from the Arnold/Cohen result noted previously in the case $m=2$). The first nontrivial differential is given by $d(\omega_{i,j}) = p_{i,j}^*(\Delta)$, where $\Delta \in H^m(X \times X)$ is the cohomology class dual to the diagonal.

As shown by Totaro, for X a smooth complex projective variety, this spectral sequence degenerates immediately, d above is the *only* nontrivial differential.

Proposition 7 ([1])**.** For X a smooth complex projective variety, let $H = H^*(X)$, and let I be the ideal in H generated by $\{p_{i,j}^*(\Delta) \mid i < j\}$. Then H/I is a subalgebra of $H^*(F(X,n))$. Thus, $\mathsf{zcl}\, H^*(F(X,n)) \geq \mathsf{zcl}\, H/I$ and $\mathsf{TC}(F(X,n)) \geq \mathsf{zcl}\, H/I + 1$.

In the case $X = T$, these considerations may be used to obtain the needed lower bound on $\mathsf{zcl}\, H^*(F(T,n))$. In this instance, the algebra $A = H/I$ may be described as follows: A is generated by degree one classes x_i, y_i, $1 \leq i \leq n$, with relations $x_i y_i = 0$, $2 \leq i \leq n$, $x_j y_k + x_k y_j$, $2 \leq j < k \leq n$, and their consequences.

Proposition 8 ([1])**.** The zero-divisors $\bar{x}_i = 1 \otimes x_i - x_i \otimes 1$ and $\bar{y}_i = 1 \otimes y_i - y_i \otimes 1$ in $A^1 \otimes A^1$ satisfy $\bar{x}_1 \cdot \bar{y}_1 \cdot \bar{x}_2 \cdot \bar{y}_2 \cdots \bar{x}_n \cdot \bar{y}_n \neq 0$. Consequently, $\mathsf{zcl}\, H^*(F(T,n)) \geq 2n$.

Higher genus: $X = \Sigma_g$, $g \geq 2$

Theorem 9 (C.-F. [1])**.** $\mathsf{TC}(F(\Sigma_g,n)) = 2n+3$

$n=1$: $F(\Sigma_g,1) = \Sigma_g$, and $\mathsf{TC}(\Sigma_g) = 5$.
$n \geq 2$: $F(\Sigma_g,n)$ is a $K(G,1)$, $G =$ pure braid group of Σ_g. Fadell-Neuwirth bundle $F(\Sigma_g,n) \to \Sigma_g$ has a section $\Longrightarrow G \cong \pi_1(F(\Sigma_g \smallsetminus \{1 \text{ point}\}, n-1)) \rtimes \pi_1(\Sigma_g) \Longrightarrow \mathrm{cd}(G) = \mathrm{gd}(G) = n+1$. Thus, $\mathrm{hdim}(F(\Sigma_g,n)) = n+1$ and $\mathsf{TC}(F(\Sigma_g,n)) \leq 2n+3$. The reverse inequality is obtained in a manner analogous to the genus 1 case above.

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Topological complexity of configuration spaces and related objects, II

Daniel C. Cohen

We continue our discussion of the topological complexity of classical configuration spaces and related objects, now focusing primarily on the latter.

Punctured surfaces

Theorem 1 (C.-F. [3])**.** $\mathsf{TC}(F(\Sigma_g \smallsetminus \{m \text{ points}\}, n)) = 2n+1$ for $g \geq 1$ and $m \geq 1$

$n = 1$: $F(\Sigma_g \smallsetminus \{m \text{ points}\}, 1)$ is a wedge of circles, with topological complexity 3.
$n \geq 2$: Fadell-Neuwirth bundles can be used to show that $F(\Sigma_g \smallsetminus \{m \text{ points}\}, n)$ is a $K(G, 1)$, where G is an iterated semidirect product of free groups with $\mathrm{cd}(G) = \mathrm{gd}(G) = n$. It follows that $\mathsf{TC}(F(\Sigma_g \smallsetminus \{m \text{ points}\}, n)) \leq 2n + 1$.

Establishing the reverse inequality in this context is substantially more involved. As $\Sigma_g \smallsetminus \{m \text{ points}\}$ is *not* a projective variety, Totaro's theorem does not apply directly. Here, the inequality $\mathsf{zcl}\, H^*(F(\Sigma_g \smallsetminus \{m \text{ points}\}, n)) \geq 2n$ is obtained by using mixed Hodge structures (on the cohomology of the quasi-projective variety $F(\Sigma_g, n)$, etc.) in conjunction with Totaro's theorem and its consequences recorded in the previous lecture.

Orbit configuration spaces

Let X be a manifold without boundary, and Γ a (finite) group acting freely on X. The *orbit configuration space* $F_\Gamma(X, n)$ is the space of all ordered n-tuples of points in X which lie in distinct Γ-orbits,

$$F_\Gamma(X, n) = \{(x_1, \ldots, x_n) \mid \Gamma \cdot x_i \cap \Gamma \cdot x_j = \emptyset \text{ if } i \neq j\}.$$

If $\Gamma = \{1\}$ is trivial, $F_{\{1\}}(X, n) = F(X, n)$ is the classical configuration space.

For this discussion, we focus on the case $X = \mathbb{C}^*$, with $\Gamma = \mathbb{Z}_r$ acting by multiplication by $\zeta = \exp(2\pi i/r)$. The associated orbit configuration space is

$$\begin{aligned} F_{\mathbb{Z}_r}(\mathbb{C}^*, n) &= \{(x_1, \dots, x_n) \in (\mathbb{C}^*)^n \mid x_j \neq \zeta^k x_i,\ i \neq j,\ 1 \leq k \leq n\} \\ &= \mathbb{C}^n \smallsetminus \bigcup_{H \in \mathcal{A}_{r,n}} H, \text{ where } \mathcal{A}_{r,n} = \{x_i = 0\}_{1 \leq i \leq n} \cup \{x_j - \zeta^k x_i = 0\}_{\substack{i<j \\ 1 \leq k \leq n}} \end{aligned}$$

The arrangement $\mathcal{A}_{r,n}$ consists of the reflecting hyperplanes of the complex reflection group $G(r,n)$, the full monomial group. For instance, when $r = 2$, this the type B Coxeter group, and $\pi_1(F_{\mathbb{Z}_2}(\mathbb{C}^*, n))$ is the type B pure braid group.

Theorem 2. $\mathsf{TC}(F_{\mathbb{Z}_r}(\mathbb{C}^*, n)) = 2n$

This may be obtained from work of Farber-Yuzvinsky. As shown by Brieskorn (conjectured by Arnold), for any arrangement $\mathcal{A} = \{f_j = 0\}$ with complement M, $H^*(M;\mathbb{Z})$ is torsion free, and is generated by degree one classes $\frac{1}{2\pi i} d \log f_j$ ($\Longrightarrow M$ is $\mathbb{Q}$-formal). The conditions below insure that $\mathsf{zcl}\, H^*(M)$ is as large as possible.

Proposition 3 ([4])**.** Suppose $\mathcal{A}$ is central and essential in $\mathbb{C}^n$. If $\exists H_1, ..., H_{2n-1} \in \mathcal{A}$ with $\{f_1, \dots, f_n\}$ and $\{f_j, f_{n+1}, \dots, f_{2n-1}\}$, $1 \leq j \leq n$, all linearly independent, then $\mathsf{TC}(M) = 2n$.

This result applies to the reflection arrangements $\mathcal{A}_{r,n}$.

Another perspective

For a discrete group G, define $\mathsf{TC}(G) := \mathsf{TC}(Y)$, where Y is a $K(G,1)$-space. It is natural to ask for $\mathsf{TC}(G)$ in terms of algebraic properties of G.

Example 4. Associated to a simple graph Γ on n vertices is a right-angled Artin group G_Γ with generators corresponding to the vertices of Γ, and commutator relators corresponding to the edges. As discussed in the lecture of B. Gutiérrez [6], one has $\mathsf{TC}(G_\Gamma) = z(\Gamma) + 1$, where $z(\Gamma)$ is the maximal number of vertices of Γ covered by two (disjoint) cliques in Γ.

Many of the configuration spaces discussed previously are $K(G,1)$-spaces, for surface pure braid groups, for pure braid groups associated to reflection groups… For example, $\pi_1(F(\mathbb{C}, n)) = P_n$ is the Artin pure braid group. From the homotopy exact sequence of the Fadell-Neuwirth bundle $F(\mathbb{C}, m) \to F(\mathbb{C}, m-1)$, with fiber $\mathbb{C} \smallsetminus \{m-1 \text{ points}\}$ and section, we see (inductively) that $F(\mathbb{C}, n)$ is a $K(P_n, 1)$-space, and obtain a split, short exact sequence $1 \to F_{n-1} \to P_n \to P_{n-1} \to 1$, where F_k is the free group on k generators. Thus,

$$P_n = F_{n-1} \rtimes P_{n-1} = F_{n-1} \rtimes (F_{n-2} \rtimes P_{n-2}) = \cdots = F_{n-1} \rtimes (\cdots \rtimes (F_3 \rtimes (F_2 \rtimes F_1)))$$

is an iterated semidirect product of free groups. Further, the action of P_{n-1} on $H_*(F_{n-1},;\mathbb{Z})$ (via the Artin representation $P_{n-1} \to \mathrm{Aut}(F_n)$) is trivial.

An *almost-direct product of free groups* is an iterated semidirect product $G = F_{d_n} \rtimes \cdots \rtimes F_{d_1}$ of finitely generated free groups for which F_{d_i} acts trivially on $H_*(F_{d_j};\mathbb{Z})$ for $i < j$. Thus, P_n is an almost-direct product of free groups.

The pure braid group $P_{r,n} = \pi_1(F_{\mathbb{Z}_r}(\mathbb{C}^*, n))$ associated to the full monomial group $G(r, n)$ also admits this structure. As first shown by Xicoténcatl [7], the map $F_{\mathbb{Z}_r}(\mathbb{C}^*, n) \to F_{\mathbb{Z}_r}(\mathbb{C}^*, n-1)$ defined by forgetting the last coordinate is a bundle, with fiber $\mathbb{C}^* \smallsetminus \{n-1 \text{ orbits}\} = \mathbb{C} \smallsetminus \{r(n-1)+1 \text{ points}\}$. This bundle may be realized as a pullback of the classical configuration space bundle $F(\mathbb{C}, N+1) \to F(\mathbb{C}, N)$ where $N = r(n-1)+1$, see [1]. It follows that this bundle admits a section, and the fundamental group of the base acts trivially on the homology of the fiber. Hence, $P_{r,n}$ is an almost-direct product of free groups.

Theorem 5 (C. [2])**.** *If $G = F_{d_n} \rtimes \cdots \rtimes F_{d_1}$ is an almost-direct product of free groups with $d_j \geq 2$ for each j, and $m \geq 0$, then $\mathsf{TC}(G \times \mathbb{Z}^m) = 2n + m + 1$.*

For an almost-direct product of n free groups G, $\operatorname{cd}(G) = \operatorname{gd}(G) = n$, the homology $H_*(G; \mathbb{Z})$ is torsion free, and the betti numbers are given by $\sum_{k \geq 0} b_k(G) \cdot t^k = (1+d_1 t)(1+d_2 t)\cdots(1+d_n t)$. Let $N = b_1(G) = d_1 + d_2 + \cdots + d_n$, and let $\mathfrak{a}\colon G \to \mathbb{Z}^N$ be the abelianization. The induced homomorphism $\mathfrak{a}^2\colon H^2(\mathbb{Z}^N) \to H^2(G)$ in integral cohomology is surjective, denote the kernel by $J = \ker(\mathfrak{a}^2)$, an ideal in the exterior algebra $H^*(\mathbb{Z}^N)$. The integral cohomology ring of G is then given by $H^*(G) \cong H^*(\mathbb{Z}^N)/J$. If $d_j \geq 2$ for each j, one can produce $2n$ zero-divisors in $H^1(G) \otimes H^1(G)$ with nonzero product. These considerations yield $\mathsf{TC}(G) = 2n+1$ for G as in the statement of the theorem. The general case $\mathsf{TC}(G \times \mathbb{Z}^N) = 2n+m+1$ may be obtained from this, the product inequality, and a straightforward analysis of the zero-divisor cup length of $H^*(G \times \mathbb{Z}^m)$.

Several of the results on the topological complexity of discrete groups mentioned above may also be obtained by other (group-theoretic) means.

Theorem 6 (Grant-Lupton-Oprea [5])**.** *If H and K are subgroups of G which satisfy $gHg^{-1} \cap K = \{1\}$ for all $g \in G$, then $\mathsf{TC}(G) \geq \operatorname{cd}(H \times K) + 1$.*

This may be used to recover the topological complexity of the pure braid group, $\mathsf{TC}(P_n) = \mathsf{TC}(F(\mathbb{C}, n)) = 2n-2$. As noted by, for instance, Birman, P_n has a free abelian subgroup $H \cong \mathbb{Z}^{n-1}$, generated in terms of the standard generators $A_{i,j}$ of P_n by $A_{j,j+1}A_{j,j+2}\cdots A_{j,n}$, $1 \leq j \leq n-1$. Let $K < P_n$ be the image of the (right) splitting in the split exact sequence $1 \to F_{n-1} \to P_n \to P_{n-1} \to 1$. The subgroup K consists of pure braids with trivial last strand, and is generated by $A_{i,j}$ with $j < n$. It can be shown geometrically [5], or algebraically, that $gHg^{-1} \cap K = \{1\}$ $\forall\, g \in P_n$. Consequently, $\mathsf{TC}(P_n) \geq \operatorname{cd}(H \times K) + 1 = (n-1) + (n-2) + 1 = 2n-2$.

We anticipate that this result may be used to recover the topological complexity of other almost-direct products of free groups, such as the groups $P_{r,n}$.

This result is also used in [5] to find the topological complexity of right-angled Artin groups, and strikingly, to show that $\mathsf{TC}(\mathcal{H}) = 5$ for Higman's acyclic group $\mathcal{H}$.

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Topological Complexity of Non-Generic Arrangement Complements

Nathan Fieldsteel

The goal of this talk is to discuss some progress towards computing the higher topological complexity of central complex arrangement complements.

In what follows, an *arrangement of hyperplanes* is a finite set $\mathcal{A} = \{H_1, \ldots, H_n\}$ of codimension 1 linear subspaces of complex affine space $\mathbb{A}^r_{\mathbb{C}}$. The *complement* of $\mathcal{A}$ is the space

$$X_{\mathcal{A}} := \mathbb{A}^r_{\mathbb{C}} \setminus \bigcup_{i=1}^{n} H_i.$$

We will work only with central arrangements, although this is not a serious restriction. The (reduced) s^{th} topological complexity $TC_s(X)$ of a space X is the smallest integer m such that there exists an open cover $\{U_0, \ldots U_m\}$ of X^s satisfying that the restriction of the standard path fibration $PX \to X^s$ to each U_i admits a continuous section. The goal of this talk is to discuss progress towards a combinatorial formula for $TC_s(X_{\mathcal{A}})$.

Let $A = H^*(X_{\mathcal{A}}, \mathbb{C})$ and let K be the kernel of the multiplication map

$$A \otimes_{\mathbb{C}} A \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} A \longrightarrow A$$