Arrangement groups and right-angled Artin groups

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joint work with
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Baton Rouge
March 29, 2008

§0 Notation

$\mathcal{L}$: line arrangement in $\mathbb{C}P^2$

$M := \mathbb{C}P^2 - \cup\mathcal{L}$

"the complement of $\mathcal{L}$"

$G := \pi_1(M)$

$\mathcal{N}$ = a set of points of multiplicity ≥ 3

For $x \in \mathcal{N}$,

$$M_x = \mathbb{C}P^2 - \cup\{\ell \in \mathcal{L} \mid x \in \ell\}$$

Note $M_x \supseteq M$.

$G_x := \pi_1(M_x) \cong F_{r(x)}$ (a free group)

$r(x) = \text{mult}(x) - 1$

$$G_x = \langle a_\ell,\ x \in \ell \mid \prod_{x \in \ell} a_\ell = 1 \rangle$$

§1 A "natural" homomorphism

Let $\varphi : G \longrightarrow \prod_{x \in \mathcal{N}} G_x$

$\varphi = \prod (i_x)_*$, $i_x : M \hookrightarrow M_x$

The target group has generators
$\{ a_{\ell, x} \mid x \in \ell,\ \ell \in \mathcal{A} \}$
with relations $\prod_{x \in \ell} a_{\ell, x} = 1$ for $x \in \mathcal{N}$

For $a_\ell \in G$ a generator of G $(\ell \in \mathcal{A})$

$$\varphi(a_\ell) = \prod_{\substack{x \in \mathcal{N} \\ x \in \ell}} a_{\ell, x}$$

Notes: ① The target is a right-angled Artin group, associated with the graph $\Gamma = \ast_{x \in \mathcal{N}} \overline{K}_{r(x)}$ (a complete multi-partite)

② φ is induced by $M \longrightarrow \prod_{x \in \mathcal{N}} M_x$ and $\prod_{x \in \mathcal{N}} M_x$ is an arrangement complement

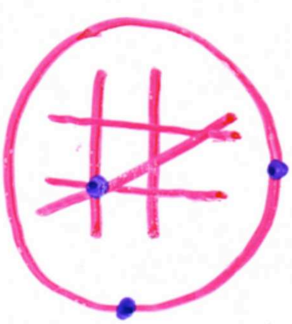
Notes, continued

3 $\mathrm{im}(\varphi)$ is torsionfree, linear, residually nilpotent, and combinatorial (determined by the underlying matroid of $\mathcal{L}$), and has cohomological dimension at most $|\eta|$.

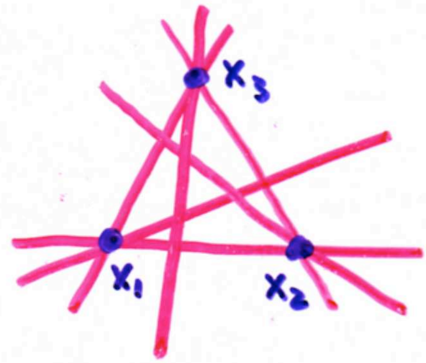
Question (A. Suciu) Can φ be injective?

[M Falk, MSRI 8/06 : "Never!"]

Ex/ ("rank-three wheel." Arvola, Terao)



a.k.a.



$$\varphi : G \longrightarrow G_{x_1} \times G_{x_2} \times G_{x_3} \cong F_2 \times F_2 \times F_2$$

Arvola : $H_3(G)$ is not finitely-generated.

Matei-Suciu : $G \cong \ker\left(F_2 \times F_2 \times F_2 \longrightarrow \mathbb{Z} \right)$
$$v_i \longmapsto 1$$

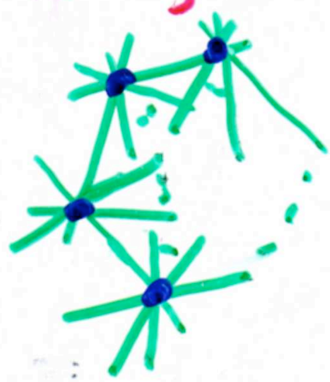
= Stallings' example

Ex/ (continued)

<u>Corollary</u> φ is injective.

proof: later

Ex/ (Artal, Cogolludo, Matei)



- $G = \ker(F_{r_1} \times \cdots \times F_{r_k} \longrightarrow \mathbb{Z})$
- these are the only arrangement groups that are Bestvina-Brady groups

$$\varphi : G \longrightarrow G_{X_1} \times \cdots \times G_{X} \cong F_{r_1} \times \cdots \times F_{r_k}$$

<u>Corollary</u> φ is injective.

§ 3 Decomposable arrangements

$G = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n \supseteq \cdots$

The lower central series of G:

$$G_{n+1} = [G, G_n] \quad n \geq 1.$$

$$G_\infty = \bigcap_{n=1}^{\infty} G_n \quad \text{"nilpotent residue"}$$

(G is residually nilpotent iff $G_\infty = 1$.)

$$\mathrm{Lie}(G) = \bigoplus_{n \geq 1} (G_n / G_{n+1}) \otimes \mathbb{Q}$$

(a graded Lie algebra under $[\,,\,]$)

$\mathrm{Lie}(G) \cong \mathfrak{h}(\mathcal{L})$ the holonomy Lie algebra of $\mathcal{L}$.

(dual to the Sullivan 1-minimal model of M)

<u>Def</u> $\mathcal{L}$ is <u>decomposable</u> if

$$\overline{\varphi}_n : \mathrm{Lie}_n(G) \longrightarrow \bigoplus_{X \in \mathcal{N}} \mathrm{Lie}_n(G_X).$$

is an isomorphism for all $n \geq 2$.

$\mathcal{N}$ = set of all points of mult. ≥ 3

- $\overline{\varphi}_2$ is always an isomorphism
- $\overline{\varphi}_n$ is always surjective for $n \geq 2$.

<u>Thm</u> (Papadima-Suciu)

$\mathcal{L}$ is decomposable iff $\overline{\varphi}_3$ is an isomorphism

(equivalently, $\dim \mathfrak{h}_3(\mathcal{L}) = \sum_{x \in \mathcal{N}} \dim \mathfrak{h}_3(\mathcal{L}_x)$)

"<u>Thm</u>" If $\mathcal{L}$ is decomposable, and $\mathcal{N}$ is the set of all points of multiplicity ≥ 3, then $\ker \varphi = G_\infty$.

proof: If $x \in \ker \varphi$ and $x \notin G_n$ (with n minimal) then x represents a nontrivial element of $\ker \overline{\varphi}_n$. If $x \in G_\infty$ then $\varphi(x) = 1$ since the target is residually nilpotent. $\square$

<u>Cor</u> φ is injective for Artal-Cogolludo-Matei examples.

pf: G is residually nilpotent.

Cor If $\mathcal{L}$ is decomposable, then G/G_∞ is torsionfree, linear, and has cohomological dimension $\leq$ the number of points of multiplicity ≥ 3.

Cor If $\mathcal{L}$ is decomposable, and $G \not\cong \mathrm{im}(\varphi)$, then G is not residually nilpotent

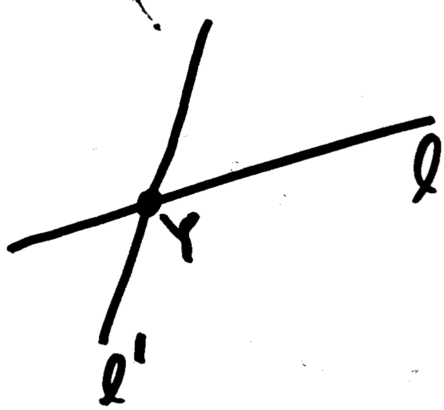
§4 The image of φ

<u>Thm 1</u> $\operatorname{im}(\varphi)$ is a normal subgroup of $\prod_{X \in \mathcal{P}} G_X$.

proof: Recall $\varphi(a_\ell) = \prod_{X \in \ell} a_{\ell, X}$.

Then

$$\varphi(a_\ell)^{a_{\ell', Y}} = \begin{cases} \varphi(a_\ell) & \text{if } Y \notin \ell \\ \varphi(a_\ell)^{\varphi(a_{\ell'})} & \text{if } Y \in \ell \end{cases}$$



<u>Thm 2</u> $\prod_{X \in \mathcal{N}} G_X / \mathrm{im}(\varphi)$ is abelian.

proof • $[a_{\ell,X}, a_{\ell',Y}] = 1$ if $X \neq Y$.

• $[a_{\ell,X}, a_{\ell',X}] \equiv [\prod_{\substack{Y \in \ell \\ Y \neq X}} a_{\ell,Y}^{-1}, a_{\ell',X}]$

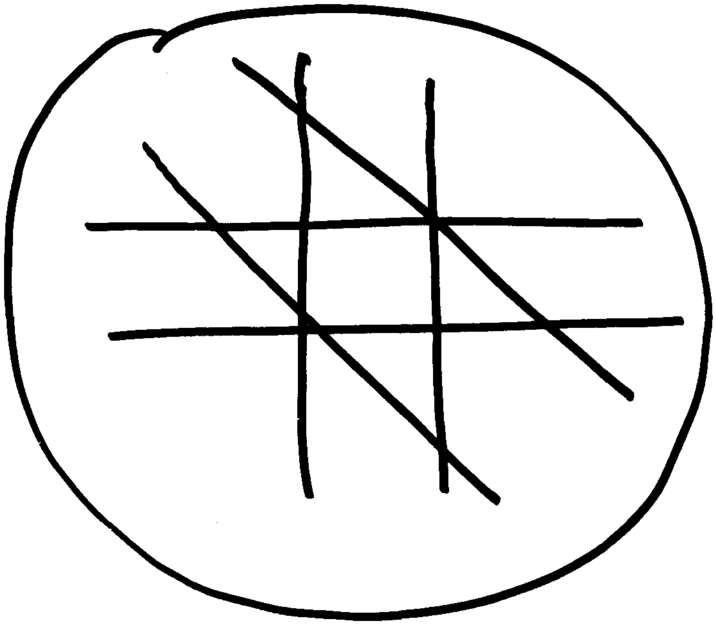
$\equiv 1 \mod \mathrm{im}(\varphi)$.

<u>Thm 3</u> $\prod_{X \in \mathcal{N}} G_X / \mathrm{im}(\varphi)$ is free abelian of rank $\sum_{X \in \mathcal{N}} r(X) - |\mathcal{L}|$.

<u>Remark</u> The Artal-Cogolludo-Matei examples are the only arrangements for which $\sum_{X \in \mathcal{N}} r(X) - |\mathcal{L}| = 1$.

<u>Summary</u> $\mathrm{im}(\varphi)$ is the kernel of a surjection from a RAAG to a free abelian group.

<u>Conjecture</u> $\exists$ decomposable arrangement whose group is not residually nilpotent.



§5| Making φ injective.

Observation : $\mathcal{L}$ admits k generating functions iff $\mathcal{L}$ is a $\binom{k}{2}$-multinet.

Replace $\mathcal{N}$ by the set of multinet subarrangements of $\mathcal{L}$.

- expect φ to become injective (mod G_∞)
- study groups of multinets :
  - torsionfree ?
  - linear ?

<u>Question</u> $\mathcal{L}$ not decomposable $\Longrightarrow$ $\mathcal{L}$ supports non-local resonance component ?