Freeness and flags of hyperplane arrangements

Takuro ABE

(Institute of Mathematics for Industry, Kyushu University)

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Flags of arrangements

A flag $\{X_i\}_{i=0}^{\ell}$ of $\mathcal{A}$ is the set of flats $X_i \in L(\mathcal{A})$ with $\operatorname{codim} X_i = i$ and that

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(3) Heavy flags.

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Definition-Theorem (SS flags, Stanley, A-?)

For any flag $\{X_i\}$ and their localizations

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An flag $\{X_i\}$ is an SS flag if the above inequality is the equality. In this case $\mathcal{A}$ is called SS.

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DF generalizes SS by replacing localization by restriction, e.g., Weyl arr. of type D_ℓ ($\ell \geq 4$) are not SS but DF ($SS \subsetneq IF \subsetneq DF$).

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DF arrangement have any good topology? $\Rightarrow$ one possibility is representation of cohomology ring by flags!

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Flag and homology groups (Schechtman-Varchenko)

Let

$$Fl_p(\mathcal{A}) := \langle \{X_0 \supset \cdots \supset X_p \mid X_i \in L(\mathcal{A}), \operatorname{codim} X_i = i\rangle_{\mathbb{Z}} / \ (\textit{equiv. relation}).$$

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Then $Fl_p(\mathcal{A}) \simeq H_p(M(\mathcal{A}) \otimes \mathbb{C})$.

Hence $\exists\{X_i\}$:divisional flag $\iff$ $H_*(\mathcal{A})$ contains some special diff. form $\Rightarrow$ We may interpret DF in terms of homology.

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Remark

When $\ell = 2$ and $H \in \mathcal{A}$ is heavy for $(\mathcal{A}, m)$, then $\exp(\mathcal{A}, m) = (m(H), |m| - m(H))$, i.p., exponents are combinatorial in this case. The above generalizes this case.

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Heavy hyperplane

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Results on heavy hyperplanes

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Theorem (A- and Kühne)

Let $H \in \mathcal{A}$ be heavy for $(\mathcal{A}, m)$. Then

$$b_2(\mathcal{A}, m) - m(H)(|m| - m(H)) \geq b_2(\mathcal{A}^H, m^H).$$

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The case $m \equiv 1$ was due to A-Yoshinaga

The quantity $b_2(\mathcal{A}, m) - m(H)(|m| - m(H))$ corresponds to $b_2(d\mathcal{A})$ for a simple arrangement case.

A_3-case and the heavy flag

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Corollary

The freeness of multi-A_3-arrangement with the heavy plane depends only on $L(\mathcal{A})$ and m.

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Heavy flag

$\{X_i\}$ is a heavy flag of $(\mathcal{A}, m)$ if $X_i \in \mathcal{A}^{X_{i-1}}$ is heavy for $(\mathcal{A}^{X_{i-1}}, m^{X_{i-1}})$ for $i = 1, \ldots, \ell - 1$.

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Purely algebraic multi-freeness is related to geometry of $\mathcal{A}$!

We stop here

Thank you for your attention!