Representation stability for the cohomology of arrangements

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Representation Stability

Example. $S_n \circlearrowright \mathrm{Conf}_n(\mathbb{C}) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n \mid x_i \neq x_j \text{ for } i \neq j\}$

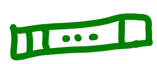
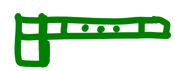
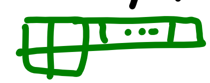
(Arnold '69) $H_i(\mathrm{Conf}_n(\mathbb{C})/S_n; \mathbb{Q}) \xrightarrow{\cong} H_i(\mathrm{Conf}_{n+1}(\mathbb{C})/S_{n+1}; \mathbb{Q})$, $n \gg 0$

That is, the unordered config. space is homologically stable

The ordered config space is not: $\dim H^1(\mathrm{Conf}_n(\mathbb{C}); \mathbb{Q}) = \binom{n}{2}$

but as an S_n-representation, for $n \geq 4$:

$$H^1(\mathrm{Conf}_n(\mathbb{C}); \mathbb{Q}) = V_{(n)} \oplus V_{(n-1,1)} \oplus V_{(n-2,2)}$$

Recall: Irreducible reps of S_n are indexed by partitions of n.

def. (Church - Farb '13)

A sequence $\{V_n\}$ of S_n-representations with S_n-equivariant maps $\phi_n : V_n \to V_{n+1}$ is uniformly representation stable with stable range $n \geq N$ if for $n \geq N$ ...

(I) ϕ_n is injective

(II) $S_{n+1} \cdot \phi_n(V_n) = V_{n+1}$

(III) $V_n = \bigoplus_{\lambda} V(\lambda)_n^{\oplus c_\lambda}$ where c_λ doesn't depend on n.

$\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash k$

irred rep of S_n corresp. to $\lambda[n] := (n-k, \lambda_1, \lambda_2, \ldots, \lambda_\ell)$

ex: $V(0)_n = V_{(n)}$ trivial rep.

$V(1)_n = V_{(n-1,1)}$ standard rep.

Example (continued) $H^1(\mathrm{Conf}_n(\mathbb{C});\mathbb{Q}) = V_{(n)} \oplus V_{(n-1,1)} \oplus V_{(n-2,2)}$

$= V(0) \oplus V(1) \oplus V(2)$

Maps $\phi_n : H^i(\mathrm{Conf}_n(\mathbb{C});\mathbb{Q}) \longrightarrow H^i(\mathrm{Conf}_{n+1}(\mathbb{C});\mathbb{Q})$

induced by $\mathrm{Conf}_{n+1}(\mathbb{C}) \to \mathrm{Conf}_n(\mathbb{C})$

"forget the last point"

Theorem (Church '12) Let X be a connected, orientable manifold. Then $\{H^i(\mathrm{Conf}_n(X);\mathbb{Q})\}$ is uniformly representation stable with stable range $n \geq 4i$.

Corollary. $\mathrm{Conf}_n(X)/S_n$ is rationally homologically stable.

$W_n = \mathbb{Z}_2 \wr S_n = \mathbb{Z}_2^n \rtimes S_n$ hyperoctahedral group

irred. rep.s indexed by pairs of partitions $\lambda = (\lambda^+, \lambda^-)$
with $|\lambda^+| + |\lambda^-| = n$.

def. (Wilson '14, Church-Farb '13)

A sequence $\{V_n\}$ of W_n-representations with W_n-equivariant maps $\phi_n : V_n \to V_{n+1}$ is uniformly representation stable with stable range $n \geq N$ if for $n \geq N$ ...

(I) ϕ_n is injective

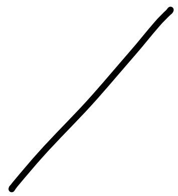
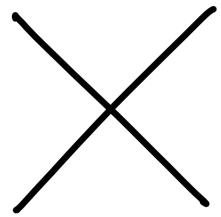
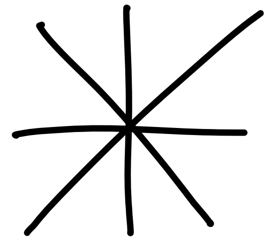
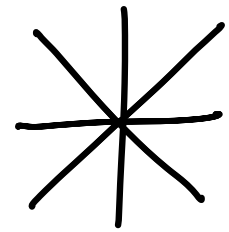
(II) $W_{n+1} \cdot \phi_n(V_n) = V_{n+1}$

(III) $V_n = \bigoplus_{\lambda} V(\lambda)_n^{\oplus c_\lambda}$ where c_λ doesn't depend on n.

$\lambda = (\lambda^+, \lambda^-)$
with $\lambda^- \vdash k$.

irred rep of W_n corresp. to
$(\lambda^+[n-k], \lambda^-)$

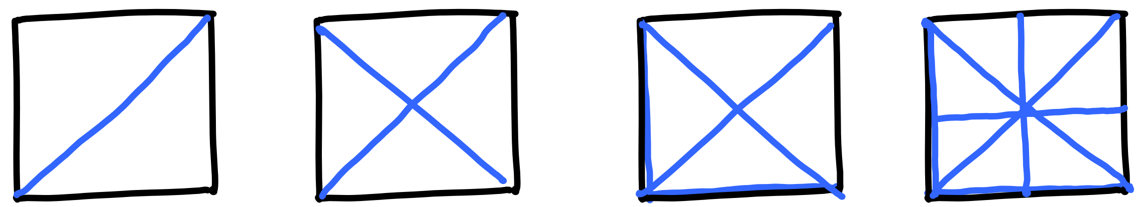
Arrangements Associated to Root Systems

		A_{n-1}	D_n	B_n	C_n	
Type:		A_{n-1}	D_n	B_n	C_n	
Weyl group:		S_n	$D_n \leq W_n$	W_n	W_n	
Reflecting hyperplanes in $\mathbb{R}^n$	$n=2$:					
	general n:	$x_i = x_j$	$x_i = x_j$ $x_i = -x_j$	$x_i = x_j$ $x_i = -x_j$ $x_k = 0$	$x_i = x_j$ $x_i = -x_j$ $2x_k = 0$	$1 \leq i < j \leq n$ $1 \leq k \leq n$

Note: These equations make sense over $\mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve E using the group operation and define Linear, toric, or elliptic arrangements (resp.)

Type:	A_{n-1}	D_n	B_n	C_n
Weyl group:	S_n	$D_n \leq W_n$	W_n	W_n

Reflecting hyperplanes in $(\mathbb{C}^\times)^n$ n=2: $(S^1 \times S^1)$



general n:	A_{n-1}	D_n	B_n	C_n	
	$x_i = x_j$	$x_i = x_j$	$x_i = x_j$	H_{ij}: $x_i = x_j$	$1 \leq i < j \leq n$
		$x_i = x_j^{-1}$	$x_i = x_j^{-1}$	H'_{ij}: $x_i = x_j^{-1}$	
			$x_k = 1$	H_k: $x_k^2 = 1$	$1 \leq k \leq n$

What's new? Over $\mathbb{C}^\times$, H_k has 2 connected components (indexed by 1 & -1) and so does $H_{ij} \cap H'_{ij}$.

Over E, H_k and $H_{ij} \cap H'_{ij}$ each have 4 connected components (indexed by the 2-torsion points)

Let $X = \mathbb{C}, \mathbb{C}^\times$, or E and let $\mathcal{A}_n$ be one of the above arrangements in X^n with complement $M(\mathcal{A}_n) = X^n \setminus \bigcup_{H \in \mathcal{A}_n} H$

Type A: $S_n \circlearrowright M(\mathcal{A}_n) = \mathrm{Conf}_n(X)$

Type B/C/D: $W_n \circlearrowright M(\mathcal{A}_n)$

Theorem. $\{H^i(M(\mathcal{A}_n); \mathbb{Q})\}$ is uniformly representation stable with stable range $n \geq 4i$.

Type A: Church '12

$X = \mathbb{C}$: Wilson '15

Thank you