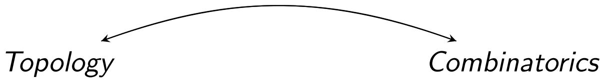
Topology of braid arrangement via counting polynomials.

Weiyan Chen

University of Chicago

AMS Special Session on Topology and Combinatorics of Arrangements
(in honor of Mike Falk)
March 20, 2016

Theme



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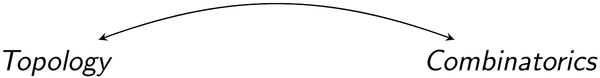

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Topology *Combinatorics*

- Cohomology of the braid arrangement complement with an action of S_n

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- **Polynomials over $\mathbb{F}_q$.** $\mathrm{UConf}_n(\mathbb{C})$ and $\mathrm{Conf}_n(\mathbb{C})$ are algebraic varieties. Their cohomology groups contain information about counting polynomials over $\mathbb{F}_q$ with weighting.

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Theorem (Church-Ellenberg-Farb)

For any character polynomial P, for each fixed i, the multiplicity

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will be eventually independent of n when $n >> i$.

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Theorem (C-)

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ be any sequence of nonnegative integers. Let μ be the classical Möbius function, and let $M_k(z^{-1}) := \frac{1}{k}\sum_{j|k} \mu(\frac{k}{j}) z^{-j}$. Abbreviate $b_{i,n}(\lambda) := \langle H^i(\mathrm{Conf}_n(\mathbb{C})), \binom{X}{\lambda}\rangle_{S_n}$.

$$\sum_{n=0}^{\infty}\sum_{i=0}^{\infty} b_{i,n}(\lambda)(-z)^i t^n = \frac{1-zt^2}{1-t}\prod_{k=1}^{l}\binom{M_k(z^{-1})}{\lambda_k}\left(\frac{(tz)^k}{1+(tz)^k}\right)^{\lambda_k}$$

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For any character polynomial P, the i-th stable multiplicity

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will eventually satisfy a linear recurrence relation in i. There exist $c_1, \cdots, c_N$ such that $b_{i+N,\infty} = c_1 b_{i,\infty} + \cdots + c_N b_{i+N-1,\infty}$ for all $i \geq 2$.

Example: $b_{i,n} = \langle H^i(\mathrm{Conf}_n(\mathbb{C})); \bigwedge^2 \mathbb{Q}^{n-1} \rangle_{S_n}$

$b_{i,n}$	$n = 3$	4	5	6	7	8	9	10	11	12	13	14
$i = 0$	**0**	0	0	0	0	0	0	0	0	0	0	0
1	0	**0**	0	0	0	0	0	0	0	0	0	0
2	0	1	**2**	2	2	2	2	2	2	2	2	2
3		1	3	**5**	5	5	5	5	5	5	5	5
4			1	4	**6**	6	6	6	6	6	6	6
5				1	5	**7**	7	7	7	7	7	7
6					2	7	**10**	10	10	10	10	10
7						3	9	**13**	13	13	13	13
8							3	10	**14**	14	14	14
9								3	11	**15**	15	15
10									4	13	**18**	18
11										5	15	**21**
12											5	16
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    when M is a manifold?
  - Is there a topological proof/explanation for the recurrence of stable multiplicities for $\mathrm{Conf}_n(\mathbb{C})$?

Thank you!
Happy birthday, Mike!

Reference

W. Chen.
Twisted cohomology of configuration spaces and spaces of maximal tori via point-counting.
Preprint, arXiv:1603.03931