Co-arrangements and bi-arrangements

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Hyperplane arrangements

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- The *strata* (intersections of hyperplanes in $\mathcal{A}$) form a poset, graded by $|S| = \mathrm{codim}(S)$.
- The *Orlik–Solomon* algebra $A_\bullet(\mathcal{A})$ only depends on the poset of strata.
- It has a combinatorial decomposition

$$A_p(\mathcal{A}) = \bigoplus_{|S|=p} A_p^S(\mathcal{A}) \ .$$

- The mixed Hodge structure on $h^k(\mathcal{A})$ is pure of weight $2k$.

Hypersurface arrangements

Definition

A *hypersurface arrangement* $\mathcal{A}$ in a smooth complex variety X is a divisor that locally (in the analytic topology) looks like a hyperplane arrangement. We are interested in

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Simplifying assumption: all the irreducible components are smooth.

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Example

— (simple) normal crossing divisors;

— hyperplane arrangements in $\mathbb{C}^n$ or $\mathbb{P}^n(\mathbb{C})$;

— toric arrangements in $(\mathbb{C}^*)^n$, the irreducible components are of the form

$$\{z_1^{a_1} \cdots z_n^{a_n} = b\} \quad \text{with } a_i \in \mathbb{Z},\ b \in \mathbb{C}^*$$

(De Concini–Procesi);

— abelian arrangements (Bibby);

— arrangements of diagonals in C^n, for C a Riemann surface

$\rightsquigarrow$ partial configuration spaces.

The Orlik–Solomon spectral sequence

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Global version of the Brieskorn–Orlik–Solomon theorem:

Theorem (Looijenga)

Let $\mathcal{A}$ be a hypersurface arrangement, there is a spectral sequence

$$E_1^{-p,q} = \bigoplus_{|S|=p} H^{q-2p}(S)(-p) \otimes A_p^S \implies h^{-p+q}(\mathcal{A}) \ .$$

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- Application to toric arrangements (De Concini–Procesi, Callegaro–Delucchi).
- Application to abelian arrangements (Bibby).
- Application to the rational homotopy theory of $X - \mathcal{A}$ (D.).

Definition and duality

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Poincaré–Verdier duality

If X is *projective* of dimension n, then

$$h^k(\mathcal{A}^\vee) \cong (h^{2n-k}(\mathcal{A}))^\vee.$$

In general, there is no such duality, and $h^\bullet(\mathcal{A}^\vee)$ is a *new cohomological invariant*.

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The dual Orlik–Solomon spectral sequence

$$E_1^{p,q} = \bigoplus_{|S|=p} H^q(S) \otimes (A_p^S)^\vee \implies h^{p+q}(\mathcal{A}^\vee) .$$

Hyperplane co-arrangements

Proposition (D.)

Let $\mathcal{A}$ be an essential affine hyperplane arrangement in $\mathbb{C}^n$.

- *We have natural isomorphisms $h^k(\mathcal{A}^\vee) \cong (H_k(A_\bullet(\mathcal{A}), d))^\vee$.*
- *This is zero for $k \neq n$.*
- *Thus, the dimension of $h^n(\mathcal{A}^\vee)$ is*

$$(-1)^n \chi(A_\bullet(\mathcal{A})) = (-1)^n \chi(\mathcal{A}, 1) .$$

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$\chi(\mathcal{A}, q)$ is the *characteristic polynomial* of $\mathcal{A}$:

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Theorem (Zaslavsky '75)

If $\mathcal{A}$ is an essential affine real arrangement, then the number of bounded connected components of the real complement $\mathbb{R}^n - \mathcal{A}$ is $(-1)^n \chi(\mathcal{A}, 1)$.

(This is not surprising!)

Toric co-arrangements

Proposition (?)

Let $\mathcal{A}$ be an essential toric arrangement in $(\mathbb{C}^*)^n$.

- *We have $h^k(\mathcal{A}^\vee) = 0$ for $k \neq n$.*
- *The Hodge polynomial of $h^n(\mathcal{A}^\vee)$ is*

$$\sum_k \dim \left(\mathrm{gr}^W_{2k} h^n(\mathcal{A}^\vee)\right) x^k = (-1)^n \chi(\mathcal{A}, 1-x),$$

  thus its dimension is $(-1)^n \chi(\mathcal{A}, 0)$.

(Toric version of Zaslavsky's theorem: Ehrenborg–Readdy–Slone.)

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Mixed Hodge structures

- The weight-graded quotients are combinatorial invariants.
- However, the mixed Hodge structure on $h^n(\mathcal{A}^\vee)$ is an *arithmetic* invariant, not combinatorial.
- Example : for $a \neq b \in \mathbb{C}^*$, the mixed Hodge structure on $H^1(\mathbb{C}^*, \{a, b\})$ knows about the number

$$\int_a^b \frac{dz}{z} = \log(b/a).$$

Definitions

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Definition

To a bi-arrangement $\mathcal{B}$ one associates

$$h^\bullet(\mathcal{B}) := H^\bullet(\widetilde{X} - \widetilde{\mathcal{L}}, \widetilde{\mathcal{M}} - \widetilde{\mathcal{L}} \cap \widetilde{\mathcal{M}}) \text{ , where}$$

- $\widetilde{X} \to X$ is a resolution of singularities of the arrangement = iterated blow-up of strata ("wonderful compactification");
- $\widetilde{\mathcal{L}}$ is the union of the irreducible components of the total transform of the arrangement corresponding to the blow-up of strata whose color is λ;
- $\widetilde{\mathcal{M}}$ is the union of the irreducible components of the total transform of the arrangement corresponding to the blow-up of strata whose color is μ.

(does not depend on the choice of a resolution of singularities)

Properties

- Arrangements $\mathcal{A}$ are bi-arrangements for which only the color λ is used.
- Co-arrangements $\mathcal{A}^\vee$ are bi-arrangements for which only the color μ is used.
- Duality among bi-arrangements: swapping the colors λ and μ.
- Poincaré–Verdier duality: for X projective of dimension n,

$$h^k(\mathcal{B}^\vee) \cong (h^{2n-k}(\mathcal{B}))^\vee \ .$$

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Even in the case $X = \mathbb{P}^n(\mathbb{C})$ and $\mathcal{B}$ a bi-arrangement *of hyperplanes*...

there is a *highly non-trivial* mixed Hodge structure on $h^\bullet(\mathcal{B})$,

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Arithmetic content: "Aomoto polylogarithms", e.g. special values of the dilogarithm

$$\mathrm{Li}_2(t) = \sum_{k\geqslant 1} \frac{t^k}{k^2} = \iint_{0<x<y<t} \frac{dx\,dy}{(1-x)y}$$

(Aomoto, Beilinson–Goncharov–Varchenko–Schechtman).

The Orlik–Solomon bi-complex and the spectral sequence

The Orlik–Solomon bi-complex (D.)

To a bi-arrangement $\mathcal{B}$ one associates a collection of groups $A^S_{i,j}$, for S a stratum of codimension $i+j$.

- Case of arrangements: $A^S_{p,0} = A^S_p$.
- Case of co-arrangements: $A^S_{0,p} = (A^S_p)^\vee$.

One also has arrows

$$d' : A^S_{i,j} \to A^T_{i-1,j} \quad \text{and} \quad d'' : A^S_{i,j} \to A^T_{i,j+1}.$$

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All arrangements and co-arrangements are exact.

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Theorem (D.)

If $\mathcal{B}$ is exact, there is a spectral sequence

$$E_1^{-p,q} = \bigoplus_{\substack{i-j=p \\ |S|=i}} H^{q-2i}(S)(-i) \otimes A^S_{i,j} \implies h^{-p+q}(\mathcal{B})\ .$$