Spring Eastern AMS-Sectional Meeting
Topology and Combinatorics of Arrangements

Representation stability and point counting

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Arithmetic	Topology
statistics/counts of polynomials over $\mathbb{F}_q$	cohomology of hyperplane complements
asymtotic counts over $\mathbb{F}_q$	representation stability

Goal: Illustrate this “bridge”

This is joint work with Jennifer Wilson.

Hyperplane complements of type $\mathcal{W}_n$

Type A_{n-1}	Symmetric group $S_n \curvearrowright \{1, \ldots, n\}$	Permutation matrices $n \times n$
Type B_n/C_n	Hyperocthahedral group $B_n \curvearrowright \{\pm 1, \ldots, \pm n\}$	Signed permutation matrices $n \times n$

$\mathcal{W}_n \curvearrowright \mathbb{R}^n$ by (signed) permutation matrices

$$\mathcal{M}_{\mathcal{W}_\mathbf{n}}(\mathbb{C}) := \mathbb{C}^n \backslash \text{complexified reflection hyperplanes}$$

Hyperplane complements and polynomials

$\mathcal{W}_{\mathbf{n}}$	S_n	B_n
$\mathcal{M}_{\mathcal{W}_{\mathbf{n}}}(\mathbb{C})$	$\mathbb{C}^n \backslash \{z_i - z_j = 0\}$ $\mathcal{M}_{S_n}(\mathbb{C}) = \mathrm{PConf}_n(\mathbb{C})$	$\mathbb{C}^n \backslash \{z_i \pm z_j = 0, z_i = 0\}$ $\mathcal{M}_{B_n}(\mathbb{C})$
$\mathcal{M}_{\mathcal{W}_{\mathbf{n}}}/\mathcal{W}_{\mathbf{n}}(\mathbb{C})$ $\mathcal{Y}_{\mathcal{W}_n}(\mathbb{C})$	$\left\{\{z_1, \ldots, z_n\} : z_i \in \mathbb{C}\right\}$ $\mathcal{Y}_{S_n}(\mathbb{C}) = \mathrm{Conf}_n(\mathbb{C})$	$\left\{\{\pm z_1, \ldots, \pm z_n\} : z_i \in \mathbb{C}^\times\right\}$ $\mathcal{Y}_{B_n}(\mathbb{C})$
Space of polynomials	$\{(T - z_1) \cdots (T - z_n) : z_i \neq z_j\}$	$\{(T - z_1^2) \cdots (T - z_n^2) : z_i^2 \neq z_j^2, z_i \neq 0\}$

The $\mathbb{K}$-points

Classical fact:

$\mathcal{Y}_{S_n}$ and $\mathcal{Y}_{B_n}$ are algebraic varieties defined over $\mathbb{Z}$.

$$\mathcal{P}oly_n := \{\text{ monic polynomials of degree } n\} \cong \mathbb{A}^n$$

$$\mathcal{Y}_{\mathcal{W}_n} = \text{subvariety of } \mathcal{P}oly_n \text{ defined by the non-vanishing of the discriminant } (+ \text{ nonzero constant term})$$

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For a field $\mathbb{K}$: the $\mathbb{K}$-points

$$\mathcal{Y}_{\mathcal{W}_n}(\mathbb{K}) = \text{ Monic, degree } n\text{, polynomials in } \mathbb{K}[T] \text{ with no repeated roots } (\text{ and nonzero constant term})$$

$$\mathbb{K} = \mathbb{F}_q \quad \text{v.s.} \quad \mathbb{K} = \mathbb{C}$$

The "bridge" topology-arithmetic

Theorem (Grothendieck-Lefschetz, Artin, Lehrer, Kim,...)

Let q be the power of an odd prime and χ a class function of $\mathcal{W}_n$, then

$$\sum_{f \in \mathcal{Y}_{\mathcal{W}_n}(\mathbb{F}_q)} \chi(f) = \sum_{k=0}^{n} (-1)^k q^{n-k} \langle \chi, H^k(\mathcal{M}_{\mathcal{W}_n}(\mathbb{C}); \mathbb{C}) \rangle_{\mathcal{W}_n}$$

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Arithmetic:

$\mathrm{Frob}_q \curvearrowright \mathcal{Y}_{\mathcal{W}_n}(\overline{\mathbb{F}}_q)$

$$\mathrm{Fix}(\mathrm{Frob}_q) = \mathcal{Y}_{\mathcal{W}_n}(\mathbb{F}_q)$$

Frob_q fixes $f \in \mathcal{Y}_{\mathcal{W}_n}(\mathbb{F}_q)$, but permutes the roots of $f \quad \rightsquigarrow \quad \sigma_f \in \mathcal{W}_n$

$$\chi(f) := \chi(\sigma_f)$$

Topology:

$$\mathcal{W}_n \curvearrowright H^k(\mathcal{M}_{\mathcal{W}_n}(\mathbb{C}); \mathbb{C})$$

$$\langle \chi, H^k(\mathcal{M}_{\mathcal{W}_n}(\mathbb{C}); \mathbb{C}) \rangle_{\mathcal{W}_n} =$$

"multiplicity" of $\mathcal{W}_n$-representation V with character χ

Example: The trivial representation

$\chi(\sigma) = 1$ for all $\sigma \in \mathcal{W}_n$

$$\sum_{f \in \mathcal{Y}_n(\mathbb{F}_q)} 1 = \sum_{k=0}^{n} (-1)^k q^{n-k} \dim_{\mathbb{C}} H^k(\mathcal{Y}_{\mathcal{W}_n}(\mathbb{C}); \mathbb{C})$$

$$|\mathcal{Y}_{\mathcal{W}_n}(\mathbb{F}_q)| = q^n P_{\mathcal{Y}_{\mathcal{W}_n}(\mathbb{C})}(-q^{-1})$$

Point counting:

$$\#\mathrm{Conf}_n(\mathbb{F}_q) = q^n - q^{n-1}$$

$$\#\mathcal{Y}_{B_n}(\mathbb{F}_q) = q^n - 2q^{n-1} + 2q^{n-2} - \dots$$

Topology:

$$H^k(\mathrm{Conf}_n(\mathbb{C})) = \begin{cases} \mathbb{Q} & k = 0 \\ \mathbb{Q} & k = 1 \\ 0 & k \geq 2 \end{cases}$$

(Arnol'd, F.Cohen)

$$H^k(\mathcal{Y}_{B_n}(\mathbb{C})) = \begin{cases} \mathbb{Q} & k = 0 \\ \mathbb{Q}^2 & 0 < k < n \\ \mathbb{Q} & k = n \\ 0 & k > 2 \end{cases}$$

(Brieskorn, Lehrer)

Representation stability: $n \to \infty$

Sequence of $\mathcal{W}_n$-representations satisfies *representation stability*: the decomposition into irreducibles "stabilizes" for n large

$$H^1(\mathcal{M}_{S_n}(\mathbb{C});\mathbb{C}) = V(\ydiagram{4}\cdots\ydiagram{1}) \oplus V(\ydiagram{3,1}\cdots\ydiagram{1}) \oplus V(\ydiagram{2,2}\cdots\ydiagram{1}) \text{ for } n \geq 4$$

$$H^1(\mathcal{M}_{B_n}(\mathbb{C});\mathbb{C}) = V(\ydiagram{4}\cdots\ydiagram{1}, \emptyset)^{\oplus 2} \oplus V(\ydiagram{2,1,1}\cdots\ydiagram{1}, \emptyset)^{\oplus 2} \oplus V(\ydiagram{4}\cdots\ydiagram{1}, \ydiagram{2}) \oplus V(\ydiagram{2,2}\cdots\ydiagram{1}, \emptyset)^{\oplus 2} \quad \text{for } n \geq 4$$

Theorem (Chruch–Farb, Wilson)

The sequence $\left\{H^k(\mathcal{M}_{\mathcal{W}_n}(\mathbb{C});\mathbb{C})\right\}_n$ satisfies rep. stability for $n \geq 4k$.

When $n \to \infty$, *multiplicities* become constant

Character polynomials

Class function $P : \coprod_n \mathcal{W}_n \to \mathbb{Z}$ which is a polynomial in

$X_r(\sigma) = \#$ positive r-cycles of σ $\qquad$ $Y_r(\sigma) = \#$ negative r-cycles of σ

Examples:

$\chi_{H^1(\mathcal{M}_{S_n}(\mathbb{C}))}(\sigma) = \binom{X_1}{2} + X_2$

$\chi_{H^1(\mathcal{M}_{B_n}(\mathbb{C}))}(\sigma) = 2\binom{X_1}{2} + 2\binom{Y_1}{2} + 2X_2 + X_1 - Y_1$

Key consequence of representation stability:

For every character polynomial P

$$\left\langle P, H^k(\mathcal{M}_{\mathcal{W}_n}(\mathbb{C}); \mathbb{C}) \right\rangle_{\mathcal{W}_n}$$

is constant for $n \geq \deg(P) + 2k$

Representation stability and asymptotic counts

Theorem (Grothendieck-Lefschetz, Artin, Lehrer, Kim,...)

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Theorem (Church–Ellenberg–Farb, J. R.–Wilson)

Let q be the power of an odd prime and P a polynomial character of $\mathcal{W}_n$, then

$$\lim_{n\to\infty} q^{-n} \sum_{f \in \mathcal{Y}_{\mathcal{W}_n}(\mathbb{F}_q)} P(f) = \sum_{k=0}^{\infty} (-1)^k q^{-k} \lim_{n\to\infty} \langle P, H^k(\mathcal{M}_{\mathcal{W}_n}(\mathbb{C}); \mathbb{C}) \rangle_{\mathcal{W}_n}.$$

and the series converges.

Example: $P(\sigma) = X_1(\sigma) =$ number of 1-cycles in σ

$\langle X_1, H^k(\mathcal{M}_{S_n}(\mathbb{C}))\rangle_{S_n} = \begin{cases} 0 & n \leq k \\ 1 & n = k+1 \\ 2 & n \geq k+2 \end{cases}$ (using formulas by Lehrer-Solomon)

$\frac{\sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} X_1(f)}{|\mathrm{Conf}_n(\mathbb{F}_q)|} =$ Expected value of the $\#$ of linear factors of a polynomial in $\mathrm{Conf}_n(\mathbb{F}_q)$

$$\lim_{n\to\infty} \frac{\sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} X_1(f)}{|\mathrm{Conf}_n(\mathbb{F}_q)|} = 1 - \frac{1}{q} + \frac{1}{q^2} - \frac{1}{q^3} + \dots$$

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$$\langle X_1 - Y_1, H^k(\mathcal{M}_{B_n}(\mathbb{C}))\rangle_{B_n} = 0$$

$$\langle X_1 + Y_1, H^k(\mathcal{M}_{B_n}(\mathbb{C}))\rangle_{B_n} = \begin{cases} 0 & k = 0 \\ 4k & n \ge 6, n \ge 2k+1 \end{cases}$$ (using formulas by Douglass)

$\frac{\sum_{f \in \mathcal{Y}_{B_n}(\mathbb{F}_q)} X_1(f)}{|\mathcal{Y}_{B_n}(\mathbb{F}_q)|} =$ Expected value of the $\#$ of linear factors $(T - c^2)$ with $c \in \mathbb{F}_q^\times$

$$\lim_{n\to\infty} \frac{\sum_{f \in \mathcal{Y}_{B_n}(\mathbb{F}_q)} X_1(f)}{|\mathcal{Y}_{B_n}(\mathbb{F}_q)|} = \frac{1}{2} \lim_{n\to\infty} \left(1 - \frac{2}{q} + \frac{2}{q^2} - \frac{2}{q^3} + \dots\right) = \frac{1}{2}\left(\frac{q-1}{q+1}\right)$$

Stability of maximal tori statistics

Theorem (J. R.–Wilson)

Let q be an integral power of an odd prime p. For $n \geq 1$, let $\mathcal{T}_n^{Frob_q}$ denote the set of $Frob_q$-stable maximal tori corresponding to either the linear algebraic group $Sp_{2n}(\overline{\mathbb{F}_p})$ or to $SO_{2n+1}(\overline{\mathbb{F}_p})$. If P is any hyperoctahedral character polynomial, then

$$\lim_{n\to\infty} q^{-2n^2} \sum_{T\in\mathcal{T}_n^{Frob_q}} P(T) = \sum_{d=0}^{\infty} \frac{\lim_{m\to\infty}\langle P, R_m^d\rangle_{B_m}}{q^d},$$

and the series in the right hand side converges. R_n^d is the d^{th}-graded piece of the complex coinvariant algebra R_n in type B/C.

- (Church-Ellenberg-Farb) Stability of maximal tori statistics for GL_n
- (Lehrer) The “bridge” formula

Reasons for convergence:

- The cohomology rings of the topological spaces considered have additional structure:

  finitely generated $\mathbf{FI}_{\mathcal{W}}$-algebras

- Finite generation/Representation stability is NOT enough
- Generators in degree at most one $\Longrightarrow$ convergence
- Generators in degree two or more $\not\Longrightarrow$ convergence
- Convergence results for hyperplane arrangements uses relations in the cohomology