The $\mathcal{G}$-invariant and catenary data of a matroid

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* Some results are joint with J. Bonin; preliminary versions available on arxiv.

$\mathcal{G}$-invariant (Derksen)

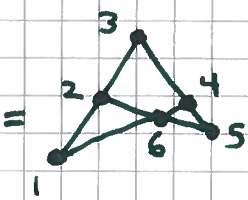
rank sequence of a permutation $\quad r(\pi) = r_1 r_2 \cdots r_n$

$$r_i = \text{rank}\{\pi(1), \ldots, \pi(i)\} - \text{rank}\{\pi(1), \ldots, \pi(i-1)\}$$

$$\mathcal{G}(M) = \sum_{\pi} [r(\pi)]$$

all permutations

a formal symbol

$M =$ 

$r(123456) = [110100]$
$r(124356) = [111000]$

$\mathcal{G}(M) = 576\,[111000] + 144\,[110100]$

$$\begin{cases} \nu(M; 0,1,1,4) = 6 \\ \nu(M; 0,1,2,3) = 12 \end{cases}$$

Catenary data 

$\nu(M; a_0, a_1, \cdots, a_r)$

$= \#$ flags $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_r$,

$$|X_i - X_{i-1}| = a_i$$

maximal chains of flags

Theorem: $\mathcal{G} \Leftrightarrow$ 

Quasisymmetric functions / algebra of partitions

$\mathcal{G}(n,r)$ = vector space of formal linear combinations $[r]$, where

r is a 01-sequence, r 1's, $n-r$ 0's.

① $[r]$ $\longleftrightarrow$ $\boxed{\text{symbol basis}}$

② $\gamma(a_0, a_1, \cdots, a_r) = \sum [r(\pi)]$ $\leftrightarrow$ $\boxed{\gamma\text{-basis}}$

flag$(\pi) = (X_i)$, $|X_{i+1} - X_i| = a_i$

$a_0 \geq 0$
$a_i \geq 1$
$a_0 + a_1 + \cdots + a_r = n$

$X_0 = \overline{\emptyset}$ = set of loops

$X_i = \overline{\{\pi(1), \pi(2), \cdots, \pi(i)\}}$

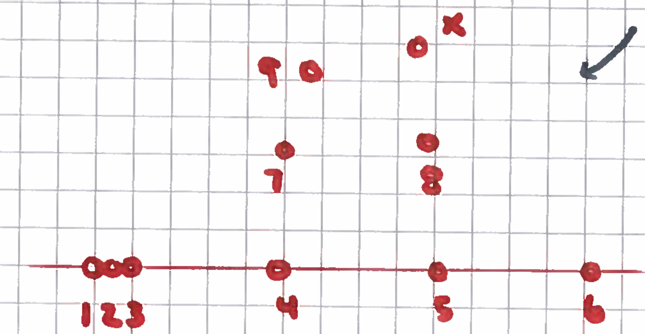
flag$(\pi) = X_0, X_1, \cdots, X_r$, with duplicates removed

③ $\mathcal{G}(F(r))$ $\leftrightarrow$ $\boxed{\text{Freedom-matroid basis}}$

Freedom Matroids

F(1 0 0 1 0 0 1 1 0 0)
1 2 3 4 5 6 7 8 9 x

3-dimensional space



③ Theorem. $\{\mathcal{G}(F(r))\}$ is a basis for $\mathcal{G}(n,r)$.

The $\trianglerighteq$-partial order.

1011000010 $\trianglerighteq$ 1001100001

1's are moved left

$\trianglerighteq$ = weak order on freedom matroids
= a sublattice of Young's partition lattice

Theorem (Derksen). The Tutte polynomial is a specialization of $\mathcal{G}$

$$\mathrm{Sp}: [r_1 r_2 \cdots r_n] \mapsto \sum_{m=0}^{n} \frac{(x-1)^{r-\mathrm{wt}(r_1,\cdots r_m)}\,(y-1)^{m-\mathrm{wt}(r_1\cdots r_m)}}{m!\,(n-m)!}$$

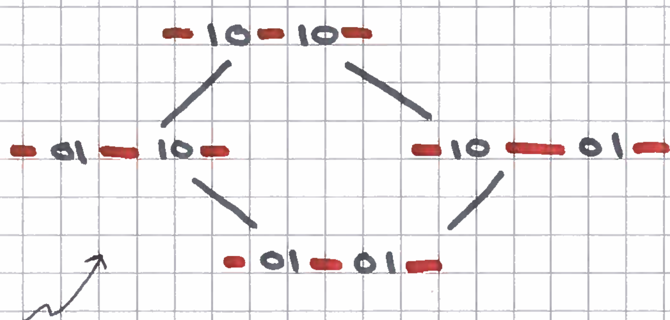
Syzygies: a spanning set for $\ker \mathrm{Sp}$

$\mathrm{Sp}: \mathcal{G}(n,r) \to$ subspace in $\mathbb{Q}[x,y]$ spanned by the Tutte polynomials of rank-r matroids on an n-set

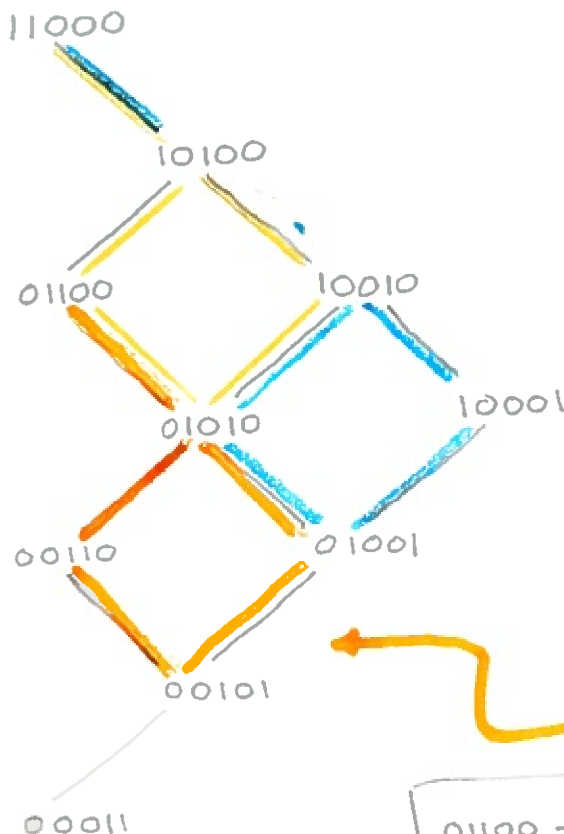
Theorem: $\ker \mathrm{Sp}$ is spanned by

[— 01 — 01 —] − [— 01 — 10 —]

− [— 10 — 01 —] + [— 10 — 10 —]



4-element interval in Young's lattice



linear relation on TP's

$$\boxed{\begin{aligned} &01100 - 01010 \\ &\quad = 01010 - 00110 - 01001 + 00101 \end{aligned}}$$

Weak order on freedom matroids, $r=2$, $n=5$

$\mathcal{G}$-invariant versus the Tutte polynomial:

1. dim space spanned by $\mathcal{G}$-invariants of matroids, rank r, size n

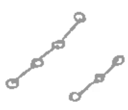
$$= \dim \mathcal{G}(n,r) = \binom{n}{r}$$

dim space spanned by Tutte polynomials of (n,r)-matroids

$$= r(n-r)+1$$

"Almost all matroids are paving."

2. $\mathcal{G}$, T have the same power to distinguish paving matroids

Example: $\frac{\mathcal{G}}{T}($  $) = \frac{\mathcal{G}}{T}($  $)$.

3. $\mathcal{G}$ contains much more information about the matroid than T

4. Both $\mathcal{G}$ and T are reconstructible from copoint decks

Many open problems ...

Find combinatorial algorithms to express $\mathcal{G}(M)$ as a linear combination of $\mathcal{G}(F(r))$ of $\mathcal{G}$-invariants of freedom matroids ....