On local systems cohomology of arrangements and of Artin groups.

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and let

$$\mathcal{A} := \{\ell_i\}_{i=1,\dots,n} \cup \ell_0$$

be the so called *conified* arrangement in $\mathbb{C}^3$ (so $dim_{\mathbb{C}}\ \tilde{\ell}_i = 1,\ dim_{\mathbb{C}}\ \ell_i = 2$)

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and

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be the complements to the arrangements.

It is easy to pass from one to the other since it holds:

$$\mathcal{M}(\mathcal{A}) \; = \; \mathcal{M}(\tilde{\mathcal{A}}) \times \mathbb{C}^*$$

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If Γ *is* **connected** *then* $\mathcal{A}$ *is a-monodromic.*

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with *Milnor fibre*

$$\mathbf{F} = Q^{-1}(1)$$

and *geometric monodromy*

$$\pi_1(\mathbb{C}^*, 1) \to Aut(F)$$

induced by $x \to e^{\frac{2\pi i}{n+1}} \cdot x$

Let A be any unitary commutative ring and

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Consider the abelian representation

$$\pi_1(\mathcal{M}(\tilde{\mathcal{A}})) \to H_1(\mathcal{M}(\tilde{\mathcal{A}}); \mathbb{Z}) \to Aut(R) : \ \beta_j \to q\cdot$$

taking an elementary generator β_j into q-multiplication.

Let R_q be the ring R endowed with this $\pi_1(\mathcal{M}(\mathcal{A}))-$module structure.

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One has an R-module isomorphism

$$H_*(\mathcal{M}(\mathcal{A}), R_q) \cong H_*(F, A)$$

where $q-$multiplication on the left corresponds to the monodromy action on the right.

In particular for $R = \mathbb{Q}[q, q^{-1}]$, which is a PID, one has

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$$H_*(\mathcal{M}(\mathcal{A}), \mathbb{Q}[q^{\pm 1}]) \cong H_*(F, \mathbb{Q}).$$

Since the monodromy operator has order dividing $n+1$ then in this case $H_*(\mathcal{M}(\mathcal{A}); R_q)$ decomposes into cyclic modules either free, isomorphic to R, or torsion, isomorphic to $\frac{R}{(\varphi_d)}$, where φ_d is a cyclotomic polynomial, with $d|n+1$.

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One can express $b_1(F)$ in terms of a certain abelian group ([Sal.-Serventi, to appear])

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Therefore there is a bound

Corollary

$$n \leq b_1(F) \leq n + rk(\frac{[\mathbf{G},\mathbf{G}]}{[[\mathbf{G},\mathbf{G}],[\mathbf{G},\mathbf{G}]]}) \; = \; n \; + \; rk(\frac{\mathbf{G}^{(2)}}{\mathbf{G}^{(3)}})$$

From the associated spectral sequence one has

$$n+1 = dim(H_1(\mathcal{M}(\mathcal{A});\mathbb{Q})) = 1 + dim\frac{H_1(F;\mathbb{Q})}{(\mu-1)}$$

on the right one has the coinvariants w.r.t. the monodromy action.

Therefore

$$b_1(F) = n \quad \Leftrightarrow \quad \mu = 1$$

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Definition

An arrangement with trivial monodromy will be called a-monodromic.

Define the (first) characteristic varieties as

$$V(\tilde{A}) = \{\tilde{t} \in (\mathbb{C}^*)^{n+1} : \dim_{\mathbb{C}} H_1(\mathcal{M}(\mathcal{A})); \mathbb{C}_{\tilde{t}}) \geq 1\}$$

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Remark: If some $t \in \mathbb{C}^*$ gives non-trivial monodromy then the point $(t, \ldots, t)$ $(n+1$ times) is in the characteristic variety of $\mathcal{A}$

By the

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local components: contained into a coordinate hyperplane

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global components of dim $k-1 \longleftrightarrow (k,d)$-multinets.

Definition. A *multi-arrangement* is a pair $(\overline{\mathcal{A}}, m)$ where $\overline{\mathcal{A}}$ is a projective line arrangement and m is a function $m : \overline{\mathcal{A}} \longrightarrow \mathbb{Z}_{>0}$ assigning a positive integer to each line in $\overline{\mathcal{A}}$.

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Definition. A (k, d)-*multinet* on a multi-arrangement $(\overline{\mathcal{A}}, m)$ is a pair $(\mathcal{N}, \mathcal{X})$ where $\mathcal{N}$ is a partition of $\overline{\mathcal{A}}$ into $k \geq 3$ classes $\overline{\mathcal{A}_1}, \ldots, \overline{\mathcal{A}_k}$ and $\mathcal{X}$ is a set of multiple points with multiplicity greater than or equal to 3 such that

- $\sum_{l \in \overline{\mathcal{A}_i}} m(l) = d$ and is independent of i;
- for every $l \in \overline{\mathcal{A}_i}$ and $l' \in \overline{\mathcal{A}_j}$ with $i \neq j$, the point $l \cap l' \in \mathcal{X}$;
- for each $p \in \mathcal{X}$, $\sum_{l \in \overline{\mathcal{A}_i},\ p \in l} m(l)$ is constant and independent of i;
- for $1 \leq i \leq k$, for any $l, l' \in \overline{\mathcal{A}_i}$, there is a sequence $l = l_0, l_1, \ldots, l_r = l'$ such that $l_{j-1} \cap l_j \notin \mathcal{X}$ for $1 \leq j \leq r$.

Remark. Let $(\mathcal{N}, \mathcal{X})$ be a multinet on a line arrangement $\overline{A}$, then $\mathcal{X}$ determines $\mathcal{N}$: construct a graph $\Gamma' = \Gamma'(\mathcal{X})$ with $\overline{A}$ as vertex set and an edge from l to l' if $l \cap l' \notin \mathcal{X}$. Then, by definition, the connected components of Γ' are the blocks of the partition $\mathcal{N}$.

Proposition

Let $\overline{\mathcal{A}}$ be a projective line arrangement and suppose that there exists an affine part $\mathcal{A} = d\overline{\mathcal{A}}$ of $\overline{\mathcal{A}}$ such that the graph $\Gamma(\mathcal{A})$ is connected. Then $\overline{\mathcal{A}}$ does not support any multinet structure.

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Let $\overline{\mathcal{A}}$ be a projective line arrangement and suppose that there exists an affine part $\mathcal{A} = d\overline{\mathcal{A}}$ of $\overline{\mathcal{A}}$ such that the graph $\Gamma(\mathcal{A})$ is connected. Then $\overline{\mathcal{A}}$ does not support any multinet structure.

Choose a set $\mathcal{X}$ of points of multiplicity greater than or equal to 3 and build $\Gamma'(\mathcal{X})$ as in the above remark. This graph has $\overline{\mathcal{A}} = \mathcal{A} \cup \{\ell_0\}$ as set of vertices and the set of edges of $\Gamma(\mathcal{A})$ is contained in the set of edges of $\Gamma'(\mathcal{X})$. Since $\Gamma(\mathcal{A})$ is connected by hypothesis we have that $\Gamma'(\mathcal{X})$ has at most two connected components and so $\mathcal{X}$ cannot give a multinet structure an $\mathcal{A}$.

Therefore: *the resonance variety has no global components, so the monodromy can come only from points of the kind* $(t,\ldots,t)$ *belonging to some (if any) translated component of the characteristic variety.*

Therefore a natural stronger conjecture is:

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Conjecture

Assume that Γ is connected: then there are no translated components in the (first) characteristic variety of the arrangement.

Computations (with [M. Serventi, S-]) support both these conjectures:

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we use an algebraic complex which is the *discrete Morse complex* for a *minimal cell structure of the complement* $\mathcal{M}(\mathcal{A})$.

[S. Settepanella, "Combinatorial Morse Theory and minimality of hyperplane arrangements", Geometry & Topology, 2007];

[Gaiffi, S., "The Morse complex of a line arrangement", Journal of Algebra, 2009]

Now take a *reflection arrangement* $\mathcal{A} \in \mathbb{R}^n$, where the hyperplanes in $\mathcal{A}$ are the mirror reflections of a reflection group $\mathbf{W}$.

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Then one has an *orbit space*

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$$\mathbf{Y_W} \ := \ \mathcal{M}(\mathcal{A})/\mathbf{W}$$

and a covering

$$\mathcal{M}(A) \to \mathbf{Y_W}$$

For finite groups one still has a Milnor fibration with Milnor fiber $F_{\mathbf{W}}$ and again

$$H_*(\mathbf{Y}_{\mathbf{W}}; A[q^{\pm 1}]) \cong H_*(F_{\mathbf{W}}; A)$$

as $A[q^{\pm 1}]-$modules.

Complete computations were done in this case for $A = \mathbb{Q}$ for $\mathbf{W}$ a Coxeter group of finite type

over $\mathbb{Q}$ see for ex.:

- Frenkel, Func. Anal. Appl. (1988) ; De Concini, Procesi, Sal., Topology (2000) (case A_n,)
- De Concini, Procesi, Sal., and Stumbo, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (1999) (all other cases);

over $\mathbb{Z}$ see for ex:

- Callegaro and Sal., C. R. Acad. Sci. Paris, Ser. I (2004) (for all exceptional cases see);
- Callegaro, Alg. Geom. Top. (2006) (for case A_n);

Computations are essentially based on a construction given in [S. 94] "The homotopy type of Artin groups," M.R.L. which holds also for all infinite type Artin groups.

so we computed some affine cases

- Callegaro, Moroni, and Sal., Trans. Amer. Math. Soc. (2008); (cohomology in case $\tilde{A}_n$, including the cohomology with coefficients in some geometric higher rank representations),

- Callegaro, Moroni and Sal., Jour. of the European Math. Soc. (2010). ($k(\pi, 1)$ problem and cohomology for $\tilde{B}_n$).

An alternative combinatorial method based on computations for case A_n [Sal, Proceedings CoMeTa 2013]

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[Sal.- Villa, "Combinatorial methods for the twisted cohomology of Artin groups," Math. Res. Lett. 18 (2013)]

[Moroni, Sal., Villa, "Some topological problems on the configuration spaces of Artin and Coxeter groups", in Proceedings Configuration Spaces (Geometry, Combinatorics and Topology), Vol. 14 of CRM series, Edizioni della Scuola Normale Superiore (2012)]

We get the following "unespected" relation

Theorem [S]

$$H_*(Br_{n+1}; R)_{(\varphi_d)} = \widetilde{H}_{*+(d-1)}(Ind_{d-2}(A_{n-d}); \frac{R}{(\varphi_d)})$$

($R = \mathbb{Q}[q^{\pm 1}]$, trivial coefficients in the right side)

Here

$$Ind_k(A_m) = \{\text{full-subgraphs } \Gamma \subset A_m \text{ s.t. } |\Gamma'| \leq k$$

$$\text{for each connected component } \Gamma' \text{ of } \Gamma.\}$$

This result is obtained by applying a variation of discrete Morse theory to the algebraic complex (resulting from [S. 94]) which computes the local homology of any Artin group $G_{\mathbf{W}}$ associated to the reflection group $\mathbf{W}$.

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be the representation of $G_{\mathbf{W}}$ onto $R = \mathbb{Q}[q^{\pm 1}]$.

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Then $H_*(\mathbf{X_W}; R_q) = H_*(\mathbf{Y_W}; R_q)$ is computed by the algebraic complex

$$C_k := \bigoplus_{\substack{J \subset S \\ |J| = k \\ \mathbf{W}_J \text{ finite}}} R\, e_J$$

with

$$\partial(e_J) = \sum_{\substack{I \subset J \\ |I| = k-1}} [I : J] \frac{\mathbf{W}_J(q)}{\mathbf{W}_I(q)} e_I$$

To this situation one associates a triple (K, R, ψ_d) called a *weighted sheaf* and which is a very special kind of diagram over a poset:

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$K = \{T \subset S$ which generate a finite parabolic subgroup of $\mathbf{W}\}$

ψ_d : *weight function* with values $\{\varphi_d^n : \ n \in \mathbb{Z}\}$ where φ_d is the dth-cyclotomic polynomial: it measure the φ_d factor of the associated Poincaré polynomial.

The homology of this sheaf is computed by a *weighted complex* $L_* \ := \ L_*(K)$

$$L_k \ := \ \bigoplus_{|\sigma|=k} \frac{R}{(\psi_d(\sigma))} \ \bar{e}_\sigma$$

and boundary $\partial : \ L_k \ \to \ L_{k-1}$ induced by ∂^0 :

$$\partial(a_\sigma \ \bar{e}_\sigma) \ = \ \sum_{\tau \prec \sigma} [\tau : \sigma] \ i_{\tau,\sigma}(a_\sigma) \ \bar{e}_\tau$$

with map $i_{\tau,\sigma}$ induced by inclusion.

Then one makes a variation of discrete Morse theory and apply it to this complex:

Then one makes a variation of discrete Morse theory and apply it to this complex: one can produce a *Morse complex* computing the homology by acyclic matchings, where one can only match elements of the same weight.

In case of classical braid group (case A_n) the weight of a subgraph with components of cardinalities $n_1, \ldots, n_k$ is given by

In case of classical braid group (case A_n) the weight of a subgraph with components of cardinalities $n_1, \ldots, n_k$ is given by

$$\psi_d(\sigma) = \varphi_d^{\sum_{i=1}^k \lfloor \frac{n_i+1}{d} \rfloor}$$

so one can find special matchings such that the critical cells reduce to

$$\mathit{Crit} := \{\Gamma_1 \sqcup \cdots \sqcup \Gamma_r \sqcup \ A_{d-1}, \quad |\Gamma_i| \leq d-2\}$$

where $A_{d-1} = \{n-d+2, \ldots, n\}$. So

$$\mathit{Crit} \ \cong \ Ind_{d-2}(A_{n-d}) \ \sqcup \ A_{d-1}.$$

In all cases this method is very effective when considered over a PID, giving some nice combinatorics-arithmetic:

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almost finished the (1-parameter) affine cases, including some not known cases like $\widetilde{C}_n$ [G. Paolini, S.]

Cases $\widetilde{A}_n,\ \widetilde{B}_n$ were computed in general (2-parameters for the case $\widetilde{B}_n$) in

[Callegaro, Moroni, Sal., Trans. Amer. Math. Soc., '08] and

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For ex.:

$$\widetilde{H}_i(\widetilde{A}_n; R)) \cong \begin{cases} R & \text{for } i = n \\ \bigoplus_{d|n+1,\ d \leq \frac{n+1}{j+1}} \left(\frac{R}{(\varphi_d)}\right)^{d-1} & \text{for } i = n-2j-1 \\ \bigoplus_{d\not| n+1,\ d \leq \frac{n+1}{j+1}} \left(\frac{R}{(\varphi_d)}\right) & \text{for } i = n-2j-2 \end{cases}$$

This was obtained indirectly by first computing

$$H_*(G_{B_n}; \mathbb{Q}[q^{\pm 1}, t^{\pm 1}])$$

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$$H_*(G_{B_n}; \mathbb{Q}[q^{\pm 1}, t^{\pm 1}])$$

and then using the inclusion

$$G_{\widetilde{A}_n} \to G_{B_{n+1}}$$

of the Artin group of type $\widetilde{A}_n$ into the "annular braid group" in $n+1$ strands.

By discrete Morse theory one can directly perform computations and find a relation with independent graphs of the kind

$$Ind_{\{1,l\},h}(A_k)$$

where the cardinality of the component containing 1 is bounded by l.

Theorem

Let $K := Ind_{\{1,l\},d-2}(A_n),\ l \leq d-2$. Let $n = qd + r$, $r \leq d-1$, be the euclidean division of n by d. Then

1. if $l \geq r$ then $K \cong Ind_{d-2}(A_n)$;
2. if $l < r-1$ then K is contractible;
3. if $l = r-1$ then $K \cong (S^{d-3})^{*q} * S^{r-2} \ = \ S^{(d-2)q+r-2}$.

Thank you

Thank you

and

Happy Birthday,

Mike !