Flag incidence algebras

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Outline

- Partial flag incidence algebras
- Multi-indexed Whitney numbers
- Kazhdan-Lusztig polynomials for matroids
- Characteristic polynomials

Incidence algebras

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Multiplication (Convolution):

$$(f * g)(X, Y) = \sum_{X \leq Z \leq Y} f(X, Z)g(Z, Y)$$

example

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$\mathcal{P} =$ 

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$\mathcal{I}^2(\mathcal{P}, R) =$ upper triangular 2×2 R-matrices

Important elements in $\mathcal{I}^2(\mathcal{P}, \mathbb{Z})$

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$$\chi_1(X, Y) = \sum_{Z\in[X,Y]} \mu(X, Z) t^{r-\mathrm{rk}(Z)} \in \mathcal{I}^2(\mathcal{P}, \mathbb{Z}[t])$$

Partial flags

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$$(f * g)(X_1, \ldots, X_n) =$$

$$\sum_{X_i \leq Y_i \leq X_{i+1}} f(X_1, Y_1, Y_2, \ldots, Y_{n-1}) g(Y_1, Y_2, \ldots, Y_{n-1}, X_n)$$

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Little bit of good news:

Proposition: If P and Q are finite posets then

$$\mathcal{I}^n(P \times Q, R) \cong \mathcal{I}^n(P, R) \otimes_R \mathcal{I}^n(Q, R).$$

ζ_k and μ_k

The *k^{th}-zeta* function ζ_k on $\mathcal{P}$ is the constant function 1 on $\mathrm{Fl}^n(\mathcal{P})$, so for all $(X_1, \ldots, X_k) \in \mathrm{Fl}^n(\mathcal{P})$

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The **left** *k^{th}-Möbius function* on $\mathcal{P}$ is $\mu_k : \mathrm{Fl}^k(\mathcal{P}) \to \mathbb{Z}$ recursively defined by $\mu_k(X_1, \ldots, X_k) = 1$ if $X_1 = \cdots = X_k$ and

$$\sum \mu_k(X_1, Y_1, \ldots, Y_{k-1}) = 0$$

where the sum is over all k-tuples where $X_1 \leq Y_1 \leq X_2 \leq Y_2 \leq X_3 \leq \cdots \leq Y_{k-1} \leq X_k$.

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2. The multi-indexed Whitney numbers of the second kind are

$$W_I(\mathcal{P}) = \sum_{\mathrm{rk} X_j = i_j} \zeta_k(X_1, X_2, \ldots, X_k)$$

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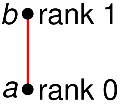

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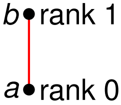

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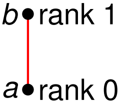
$\mathcal{P} =$ 

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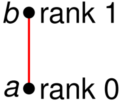

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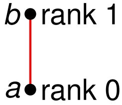
$\mathcal{P} =$ 

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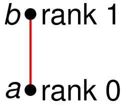
$\mathcal{P} =$ 

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$W_{0,0} = 1$, $W_{0,1} = 1$, and $W_{1,1} = 1$

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$$\sum_{F \in L_k} W_I(\mathcal{P}_F) W_J(\mathcal{P}^F) = W_{I \cup \{k\} \cup J[k]}(\mathcal{P}).$$

must have existed before?

Theorem

If $\mathcal{P}$ is a locally finite, ranked poset and $1 \leq n \leq \mathrm{rk}\mathcal{P}$ then

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Theorem (Elias, Proudfoot, W)

Let $\mathcal{P}$ be a finite ranked lattice. The Kazhdan-Lusztig polynomial of $\mathcal{P}$, $P(\mathcal{P}, t)$ is the polynomial recursively defined which satisfies

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3. *For all* $\mathcal{P}$,

$$t^{\mathrm{rk}(\mathcal{P})} P(\mathcal{P}, t^{-1}) = \sum_{F \in \mathcal{P}} \chi_1(\mathcal{P}_F, t) P(\mathcal{P}^F, t)$$

where $\chi_1(\mathcal{P}, t)$ *is the usual characteristic polynomial.*

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They are some special basis for the Möbius algebra of $L(\mathcal{A})$.

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- $c_2 = -(W_{1,r-1} - W_{1,2}) + (W_{r-3,r-1} - W_{r-3,r-2}) + (W_{r-2} - W_2)$

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where S_k and s_k are recursively defined and I and $t(I)$ make a "top heavy pair".

Multivariable characteristic polynomials

$$\chi_k(\mathcal{P}, \mathbf{t}) = \sum_{|I|=k} w_{\{0\}\cup I}\mathbf{t}^I$$

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$w_{0,0,0} = 1$
$w_{0,0,1} = -1$
$w_{0,1,1} = 1$

$$\chi_2(\mathcal{P}; t_1, t_2) = t_1 t_2 - t_1 + 1 = t_1(t_2 - 1) - 1$$

Boolean lattice

$B_n =$ Boolean lattice.

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$$\chi_k(\mathcal{B}_n; t_1, \ldots, t_k) = \left(\sum_{i=0}^{k} (-1)^i \prod_{j=1}^{k-i} t_j \right)^n$$

$$= \left(t_1(t_2(\cdots(t_{k-1}(t_k - 1) + 1) \cdots + (-1)^{k-1}) + (-1)^k \right)^n$$

Happy Birthday Mike Falk!!!!