Bernstein–Sato polynomials of determinantal varieties (with A. Lörincz, C. Raicu, J. Weyman)

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1. The D-module generated by f^s
2. Bernstein–Sato polynomials
3. Zeta functions

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$R = \mathbb{C}[x_1, \ldots, x_n]$	polynomial ring
$f \in R$	polynomial, not constant
$D = R\langle \partial_1, \ldots, \partial_n \rangle$	Weyl algebra
$D[s]$	adjoining a new variable

$D[s]$ acts on $R[f^{-1}, s] \cdot f^s$ by the formal chain rule:

$$\partial_i \bullet (g(x,s)f^s) = (f\partial_i(g(x,s)) + g(x,s)s\partial_i(f))f^{s-1}$$

for all $g \in R[f^{-1}, s]$.

Some remarks

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1. $R[f^{-1}, s] \supsetneq D[s] \bullet f^s \supsetneq D[s] \bullet f^{s+1}$ even if $f = x$: $x^s \notin D[s] \bullet x^{s+1}$.
2. Example (Cayley?, Kimura, Raicu): $f = \det(A)$, $A = ((x_{i,j}))_1^n$. Then

$$\det((\partial_{i,j})) \bullet f^{s+1} = (s+1)\dots(s+n)f^s$$

and

$$D[s] \bullet 1 \subsetneq D[s] \bullet \frac{1}{f} \subsetneq D[s] \bullet \frac{1}{f^2} \subsetneq \dots \subsetneq D[s] \bullet \frac{1}{f^n} = R[1/f]$$

Bernstein–Sato polynomial

Theorem ((Sato,) Bernstein (, Björk), Malgrange, Kashiwara)

$\exists\, P \in D[s]$, $\exists 0 \neq b_P \in \mathbb{Q}[s]$,

$$P(x, \partial, s) \bullet f^{s+1} = b_P(s) f^s.$$

(Means: $D[s] \bullet f^s / D[s] \bullet f^{s+1}$ is killed by $b_P(s)$).

Definition

The (monic) generator of the ideal $\{b_P\}$ is the Bernstein–Sato polynomial $b_f(s)$.

Classical example: $f = x_1 * x_4 - x_2 * x_3$, $P = \partial_1\partial_4 - \partial_2\partial_3$, $b_f(s) = (s+1)(s+2)$.

Classical results

ρ_f =root set of $b_f(s)$

Theorem

- $b_f(-1) = 0$
- $b_f(s) \in \mathbb{Q}[s]$
  in fact, $\rho_f \subseteq \mathbb{Q}_-$ *(Malgrange/Kashiwara)*
- $\rho_f \subseteq (-n, 0)$ *(Saito)*
- *isol sing:* ρ_f *=e-vals of some operator on* R/J *(Malgrange)*
- *isolated and w-homogeneous:* $-\rho_f = \deg_w((R/J)\frac{dx}{f}) \cup \{1\}$

Also relates to: Milnor fibers, lct, multiplier ideals, periods, p-adic stuff, . . .

Zeta functions

We are over $\mathbb{C}$, so have embedded resolution of singularities:

$$\pi\colon (Y \supseteq \tilde{X}) \to (\mathbb{C}^n \supseteq X = \mathrm{Var}(f)).$$

Let $\tilde{X} = \bigcup_{i \in S} E_i$, and for $I \subseteq S$,

$$E_I^* := \bigcap_{i \in I} E_i \setminus \bigcup_{j \notin I} E_j.$$

Definition

The topological zeta function of f is

$$Z_f(s) = \sum_I \chi(E_I^*) \prod \frac{1}{N_i s + \nu_i}$$

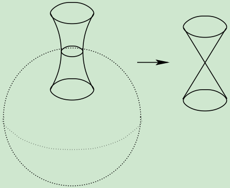
where N_i is the multiplicity of E_i in $\mathrm{Div}(f \circ \pi)$ and $\nu_i - 1$ is the multiplicity of E_i in $\mathrm{Div}(\pi^*(dx_1 \wedge \ldots \wedge dx_n))$.

Denef, Loeser: this is independent of the resolution.

A zeta function example

Example (A zeta function)

- Let $f = x_1x_4 - x_2x_3$ in $\mathbb{C}[x_1, x_2, x_3, x_4]$.
- Then $S = \{1, 2\}$, E_1 is the strict transform and $E_2 = \mathbb{P}^3$.
- $E_1^* = \mathrm{Var}(f) \setminus \{0\}$,
  $E_{1,2}^* = \mathrm{Proj}(f) \cong \mathbb{P}^1 \times \mathbb{P}^1$,
  $E_2^* = \mathbb{P}^3 \setminus \mathrm{Proj}(f)$



- $\chi(E_1^*) = 1 - 1 = 0, \qquad \chi(E_{1,2}^*) = 4, \qquad \chi(E_2^*) = 4 - 4 = 0$
- $N_1 = 1,\ N_2 = 2,\ \nu_1 = 1,\ \nu_2 = 4.$
- $Z_f(s) = 0 + 0 + 4\dfrac{1}{1 \cdot s + 1} \cdot \dfrac{1}{2 \cdot s + 4}.$

The monodromy conjecture

Conjecture

(SMC) *If* α *is a pole of* $Z_f(s)$ *then* $b_f(\alpha) = 0$.

(WMC) *If* α *is a pole of* $Z_f(s)$ *then* $\exp(2\pi i\alpha)$ *is a monodromy eigenvalue along* $f = 0$.

Higher codimension

Let $\underline{f} = f_1, \ldots, f_r$ and $\underline{s} = s_1, \ldots, s_r$,

$$D[\underline{s}]\text{-acts on } R[1/\underline{f}][\underline{s}] \cdot \underline{f}^{\underline{s}} \text{ as expected.}$$

Let $s = s_1 + \ldots + s_r$.

Theorem (Budur–Mustata–Saito)

There exists $P_{\underline{c}}(\underline{s}) \in D[\underline{s}]$, $b(s) \in \mathbb{Q}[s]$, $\underline{c} \in \mathbb{Z}^r$ with $|\underline{c}| = 1$:

$$\sum_{\underline{c}} P_{\underline{c}} \cdot \underline{\hat{c}}_{\underline{s}} \bullet \underline{f}^{\underline{s}+\underline{c}} = b(s) \cdot \underline{f}^{\underline{s}}$$

where $\underline{\hat{c}}_{\underline{s}} = \prod_{c_i<0} s_i \cdots (s_i + c_i + 1)$.
The generator $b_{\underline{f}}(s)$ of the ideal of all $b(s)$ is a function of the ideal $R \cdot \underline{f}$.

Facts and conjectures

1. Roots of $b_{\underline{f}}(s)$ are rational.
2. For monomial ideals, algorithms/formulas exist (Budur–Mustata–Saito via reduction mod p).
3. Conjecturally, $b_{\underline{f}}(s)$ relates to Zeta-function as in original case.
4. $b_{\underline{f}}(s)$ relates to Sabbah's specialization functor (takes the role of the Milnor fiber).
5. Most Bernstein–Sato polynomials are unknown.
6. Not combinatorial for arrangements.

Determinantal ideals

Set-up

$R = \mathbb{C}[\{x_{1,1}, \ldots, x_{m,n}\}]$, $\underline{f}$ =all $n \times n$-minors.

Theorem

$$b_{\underline{f}}(s) = (s+n-m+1)\cdots(s+n)$$

For $n = m$: Kimura, based on pre-homogeneous vector spaces.

Outline of method

- Make the right guess: look at zeta function. (Note that singular locus of a determinant is cut out by its minors).
- Upper bound: find explicit $\{P_{\underline{c}}(s)\}$ and a functional equation.
- Lower bound (certify roots):
  - $n = m$: show $\left(D \bullet f^{-i} \subsetneq D \bullet f^{-i+1}\right)_1^n$ via representations of $Sl(n)$ (Raicu).
  - $n = m + 1$: Induction plus local cohomology.
  - $n = n$: Statement on inequality of inclusions remains a conjecture. Lörincz: consider intermediate polynomial defined over D^{sl}.

Local cohomology and roots

For $r = 1$, $b_f(-1) = 0$ since $R[1/f] \neq R$.

Theorem

If $H^r_{\underline{f}}(R) \neq 0$ then one of $-r - \mathbb{N}$ is a root of $b_{\underline{f}}(s)$.

Idea: If no such root, Čech complex gives a contradiction.