Support of Sabbah's specialization functor

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Introduction

Let X be a smooth complex algebraic variety. Let

$$F = (f_1, \dots, f_r) : X \to \mathbb{C}^r$$

be a collection of regular functions on X and let $f = f_1 \cdot f_2 \cdots f_r$. We are interested in the algebraic and topological properties of the singularities of the hypersurface $f = 0$.

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Given X and F as above, we introduce the following algebraic/topological invariants.

introduction

1. Bernstein-Sato ideals;
2. support of Sabbah's specialization complex;
3. (local) cohomology support loci;
4. non-simple loci;
5. monodromy Zeta function.

Among these invariants, (1) is algebraic, (2), (3), (4) are topological and (5) is both algebraic and topological. Our main motivation is to understand the relation between (1) and (2).

Bernstein-Sato ideals

Let $\mathcal{D}_X$ be the sheaf of algebraic differential operators on X. The Bernstein-Sato ideal associated to F is the ideal

$$B_F \subset \mathbb{C}[s_1, \ldots, s_r]$$

of all polynomials $b(s_1, \ldots, s_r)$ such that

$$b(s_1, \ldots, s_r) \prod_{1 \leq i \leq r} f_i^{s_i} = P \prod_{1 \leq i \leq r} f_i^{s_i + 1}$$

for some global algebraic differential operator P, i.e., a global section of $\mathcal{D}_X[s_1, \ldots, s_r]$.

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When $r = 1$, the monic generator of B_F is the Bernstein-Sato polynomial b_f.

Bernstein-Sato ideals

Example

Let $X = \mathbb{C}^2$ and let $F = (x, y, 1 - x - y)$. Then
$B_F =< (s_1 + 1)(s_2 + 1)(s_3 + 1) >$.

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$(s_1 + 1)(s_2 + 1)^2(5s_2 + 2)(5s_2 + 3)(5s_2 + 4)(5s_2 + 6)(s_1 + 2)(s_1 + 3)(s_1 + 4)(s_1 + 5)$,
$(s_1+1)(s_2+1)^2(5s_2+2)(5s_2+3)(5s_2+4)(5s_2+6)(5s_2+7)(s_1+2)$,
$(s_1+1)(s_2+1)^2(5s_2+2)(5s_2+3)(5s_2+4)(5s_2+6)(5s_2+7)(5s_2+8)$.

Bernstein-Sato ideals

Conjecture (Budur)

Given any X, F, B_F is always generated by products of linear polynomials of the form

$$\alpha_1 s_1 + \cdots + \alpha_r s_r + \alpha$$

where $\alpha_i \in \mathbb{Q}_{\geq 0}$ and $\alpha \in \mathbb{Q}_{>0}$.

Sabbah's specialization complex

Denote the zero locus of $f = \prod_{i=1}^{r} f_i$ by Y. Then Y is a hypersurface in X. Denote the complement of Y in X by U. Consider the following diagram,

$$\begin{tikzcd}
Y \arrow[r, hook, "i"] & X \arrow[d, "F"] & U \arrow[l, hook', "j"'] \arrow[d, "F"] & \tilde{U} \arrow[l, "\pi"'] \arrow[d, "\tilde{F}"] \\
 & \mathbb{C}^r & (\mathbb{C}^*)^r \arrow[l, hook'] & \mathbb{C}^r \arrow[l, "Exp"']
\end{tikzcd}.$$

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Sabbah's specialization complex functor of F is defined by

$$\psi_F = i^{-1} Rj_* R\pi_! (j \circ \pi)^* : \mathbf{D}_c^b(X, \mathbb{C}) \to \mathbf{D}_c^b(Y, A)$$

where $A = \mathbb{C}[t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}]$.

Sabbah's specialization complex

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We define the support of Sabbah's specialization complex to be

$$\mathcal{S}(F) = \bigcup_{x \in Y} \bigcup_{i} Supp\left(\mathcal{H}^i\left(\psi(\underline{\mathbb{C}}_X)\right)_x\right)$$

a Zariski closed subset of $(\mathbb{C}^*)^r = Spec(\mathbb{C}[t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}])$.

Sabbah's specialization complex

Conjecture (Budur)

Let $Exp : \mathbb{C}^r \to (\mathbb{C}^*)^r$ *be the map sending* $(z_i)_{1 \le i \le r}$ *to* $(\exp(2\pi\sqrt{-1}z_i))_{1 \le i \le r}$. *Then*

$$Exp(V(B_F)) = \mathcal{S}(F).$$

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Theorem (Budur)

Under the above notations,

$$Exp(V(B_F)) \supset \mathcal{S}(F).$$

Sabbah's specialization complex

By relating the support of Sabbah's specialization complex with the pole and zero locus of the monodromy Zeta function, we proved the following.

Theorem (Budur-Liu-Saumell-W)

Under the above notations, $\mathcal{S}(F) \subset (\mathbb{C}^*)^r$ is a finite union of torsion translated tori of codimension one.

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In general, $V(B_F)$ may have irreducible components of higher codimension. However, in all the examples we know, any of the higher codimensional components is contained in a translate of another codimension one component by some lattice point in $\mathbb{C}^r$. Therefore, in all the examples we know, $Exp(V(B_F))$ is of pure codimension one in $(\mathbb{C}^*)^r$.

Hyperplane arrangement

Recall that the support of Sabbah's specialization complex is the union of support of local germs.

$$\mathcal{S}(F) = \bigcup_{x \in Y} \bigcup_{i} Supp \left(\mathcal{H}^i \left(\psi(\underline{\mathbb{C}}_X) \right)_x \right)$$

Hyperplane arrangement

Recall that the support of Sabbah's specialization complex is the union of support of local germs.

$$\mathcal{S}(F) = \bigcup_{x \in Y} \bigcup_{i} Supp \left(\mathcal{H}^i \left(\psi(\underline{\mathbb{C}}_X) \right)_x \right)$$

At any $x \in Y$, the support

$$\bigcup_{i} Supp \left(\mathcal{H}^i \left(\psi(\underline{\mathbb{C}}_X) \right)_x \right)$$

is determined by the cohomology jump support loci of the small ball complement $B \setminus Y$, where B is a small ball in X centered at x.

Hyperplane arrangement

Since the cohomology support loci of the small ball complement is well-understood for a hyperplane arrangement, we have a formula for the support of Sabbah's specialization complex for a hyperplane arrangement.

Theorem

Let $F = (f_1, \ldots, f_r)$ be a collection of linear functions on $X = \mathbb{C}^n$ defining mutually distinct hyperplanes. Then

$$\mathcal{S}(F) = Z\left(\prod_{W}\left(\prod_{i:f_i(W)=0} t_i - 1\right)\right)$$

where the product is over all dense edges W.

Length function and non-simple locus

For any object E in an Artinian abelian category, there exists a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$$

such that E_{i+1}/E_i are simple objects. The number l is independent of the choice of the filtration. We call l the length of E, denoted by $l(E)$.

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The category of perverse sheaves on X is an Artinian abelian category. Let $M(U)$ be the moduli space of rank one local systems on U. Since the inclusion map $j : U \to X$ is affine, Rj_* maps perverse sheaves on U to perverse sheaves on X. So we can define a length function on $l_{(X,F)} : M(U) \to \mathbb{Z}_{>0}$ by

$$l_{(X,F)}(L) = l\,(Rj_*(L[n]))$$

where $l\,(Rj_*(L[n]))$ is the length of $Rj_*(L[n])$ as a perverse sheaf.

Length function and non-simple locus

When $X = \mathbb{C}^n$, there is a natural isomorphism $M(U) \cong (\mathbb{C}^*)^r$, which maps every local system to the monodromy through meridians of each divisor $f_i = 0$.

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When $X = \mathbb{C}^n$, under the above isomorphism

$$\{L \in (\mathbb{C}^*)^r | l_{(X,F)}(L) \geq 2\} = \mathcal{S}(F).$$

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When $X = \mathbb{C}^n$, under the above isomorphism

$$\left\{L \in (\mathbb{C}^*)^r | l_{(X,F)}(L) \geq 2\right\} = \mathcal{S}(F).$$

In particular, when F is a collection of linear polynomials, $\left\{L \in (\mathbb{C}^*)^r | l_{(X,F)}(L) \geq 2\right\}$ is combinatorial invariant.

Length function and non-simple locus

Question

When $X = \mathbb{C}^n$, the length jump loci

$$W^i(X,F) = \{L \in (\mathbb{C}^*)^r | l_{(X,F)}(L) \geq i\}$$

is combinatorial invariant.

Length function and non-simple locus

Question

When $X = \mathbb{C}^n$, the length jump loci

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is combinatorial invariant.

Conjecture

For any smooth complex variety X and any collection of regular functions F,

$$W^i(X,F) = \{L \in M(U) | l_{(X,F)}(L) \geq i\}$$

is a finite union of torsion translated subtori in $M(U)$.

Happy birthday Mike!