Towards a new resonance-Chen ranks formula: the case of welded braids

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Associated graded Lie algebras

- G: a finitely generated group.
- The *lower central series* of G: $\Gamma_1 G = G$, $\Gamma_2 G = [G, G]$, $\Gamma_{k+1} G = [\Gamma_k G, G]$, $k \geq 1$.
- The *associated graded Lie algebra* of G is defined to be

$$\mathrm{gr}(G; \mathbb{C}) := \bigoplus_{k \geq 1} (\Gamma_k(G)/\Gamma_{k+1}(G)) \otimes_{\mathbb{Z}} \mathbb{C}.$$

- The *LCS ranks* of G are defined as $\phi_k(G) := \dim(\mathrm{gr}_k(G; \mathbb{C}))$.

Example (Free group F_n with n generators)

$\mathrm{gr}(F_n; \mathbb{C})$ is isomorphic to the free Lie algebra.

$$\phi_k(F_n) = \frac{1}{k} \sum_{d|k} \mu(k/d) n^d,$$

where μ is the Möbius function.

Pure braid groups

- Let P_n be the pure braid group on n strings.
- A classifying space for P_n is the configuration space

$$\mathrm{Conf}_n(\mathbb{C}) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}.$$

- Moreover,

$$P_n = F_{n-1} \rtimes_{\alpha_{n-1}} P_{n-1} = F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_1,$$

where $\alpha_n \colon P_n \subset B_n \hookrightarrow \mathrm{Aut}(F_n)$. In fact,

$$B_n = \{\beta \in \mathrm{Aut}(F_n) \mid \beta(x_i) = wx_{\tau(i)}w^{-1}, \beta(x_1 \cdots x_n) = x_1 \cdots x_n\}.$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{IA}(n) & \longrightarrow & \mathrm{Aut}(F_n) & \longrightarrow & \mathrm{GL}_n(\mathbb{Z}) & \longrightarrow & 0\,. \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P(n) & \longrightarrow & B(n) & \longrightarrow & S_n & \longrightarrow & 0
\end{array}$$

Theorem (Falk–Randell 85)

Let $G = H \rtimes Q$ be a semi-direct product of groups, and suppose Q acts trivially on H_{ab}. Then there is a split exact sequence of graded Lie algebras,

$$0 \longrightarrow \mathrm{gr}(H) \longrightarrow \mathrm{gr}(G) \longrightarrow \mathrm{gr}(Q) \longrightarrow 0 .$$

Theorem (Falk–Randell 85, Kohno85)

The graded Lie algebra $\mathrm{gr}(P_n;\mathbb{C})$ is generated by degree 1 elements s_{ij} for $1 \le i \ne j \le n$, subjects to the relations

$$s_{ij} = s_{ji},\ [s_{jk}, s_{ik} + s_{ij}] = 0,\ [s_{ij}, s_{kl}] = 0 \text{ for } i \ne j \ne l.$$

In particular, the LCS ranks of P_n are given by

$$\phi_k(P_n) = \sum_{s=1}^{n-1} \phi_k(F_s).$$

Chen Lie algebras

- The *Chen Lie algebra* of a finitely generated group G is defined to be the graded Lie algebra, $\mathrm{gr}(G/G''; \mathbb{C})$, of the maximal metabelian quotient of G.
- The quotient map $h\colon G \twoheadrightarrow G/G''$ induces $\mathrm{gr}(G; \mathbb{C}) \twoheadrightarrow \mathrm{gr}(G/G''; \mathbb{C})$.
- The *Chen ranks* of G are defined as $\theta_k(G) := \dim(\mathrm{gr}_k(G/G''; \mathbb{C}))$.
- $\phi_k(G) \geq \theta_k(G)$. Equality holds for $k \leq 3$.
- $\theta_k(F_n) = (k-1)\binom{n+k-2}{k}$, $k \geq 2$. [Chen (51)]

Proposition (Cohen–Suciu 95)

The Chen ranks of the pure braid groups P_n are given by

$$\theta_1(P_n) = \binom{n}{2}, \ \theta_2(P_n) = \binom{n}{3}, \ \text{and} \ \theta_k(P_n) = (k-1)\binom{n+1}{4} \ \text{for } k \geq 3.$$

Notice that $\theta_k(P_n) \neq \sum_{s=1}^{n-1} \theta_k(F_s)$. $\Longrightarrow P_n \not\cong F_{n-1} \times \cdots \times F_1$.

Resonance varieties

- Suppose $A^* := H^*(G, \mathbb{C})$ has finite dimension in each degree.
- For each $a \in A^1$, define a cochain complex of finite-dimensional $\mathbb{C}$-vector spaces,

$$(A, a) : A^0 \xrightarrow{a\cup -} A^1 \xrightarrow{a\cup -} A^2 \xrightarrow{a\cup -} \cdots ,$$

  with differentials given by left-multiplication by a.
- The *resonance varieties* of G are the homogeneous subvarieties of A^1

$$\mathcal{R}_i(G, \mathbb{C}) = \{a \in A^1 \mid \dim_{\mathbb{C}} H^1(A^*; a) \geq i\}.$$

- The resonance varieties were introduced by Michael Falk in the context of hyperplane arrangements (97).
- $\mathcal{R}_1(\mathbb{Z}^n, \mathbb{C}) = \{0\}$; $\mathcal{R}_1(\pi_1(\Sigma_g), \mathbb{C}) = \mathbb{C}^{2g},\ g \geq 2$.

Pure braid groups P_n

Based on Falk's work (97), Cohen and Suciu (99) computed the first resonance variety of the pure braid group P_n, which has decomposition into irreducible components given by

$$\mathcal{R}_1(P_n) = \bigcup_{1\le i<j<k\le n} L_{ijk} \cup \bigcup_{1\le i<j<k<l\le n} L_{ijkl},$$

where

$$L_{ijk} = \left\{x_{ij} + x_{ik} + x_{jk} = 0 \text{ and } x_{st} = 0 \text{ if } \{s,t\} \not\subset \{i,j,k\}\right\},$$

$$L_{ijkl} = \left\{ \begin{array}{l} \sum_{\{p,q\}\subset\{i,j,k,l\}} x_{pq} = 0,\ x_{ij} = x_{kl},\ x_{jk} = x_{il},\ x_{ik} = x_{jl}, \\ x_{st} = 0 \text{ if } \{s,t\} \not\subset \{i,j,k,l\} \end{array} \right\}.$$

In particular, $\dim(L_{ijk}) = \dim(L_{ijkl}) = 2$.

Conjecture (Chen ranks conjecture, Suciu 01)

Let G be an arrangement group.
Let h_n be the number of n-dimensional irreducible components of $\mathcal{R}_1(G)$.

$$\theta_k(G) = \sum_{n\geq 2} h_n \cdot \theta_k(F_n), \quad \text{for} \quad k \gg 1. \tag{1}$$

Theorem (Cohen-Schenck 15)

- *Formula (1) holds if G is a 1-formal, commutator-relators group, such that the resonance variety $\mathcal{R}_1(G)$ is 0-isotropic, projectively disjoint, and reduced as a scheme.*
- *The hyperplane arrangement groups $PA_n = P_{n+1}$, PB_n and PD_n associated to the Coxeter groups A_n, B_n and D_n satisfy the above conditions. Furthermore, for $k \gg 1$,*
  $\theta_k(PB_n) = (k-1)[16\binom{n}{3} + 9\binom{n}{4}] + (k^2-1)\binom{n}{2}$ *and*
  $\theta_k(PD_n) = (k-1)[5\binom{n}{3} + 9\binom{n}{4}]$.

Pure welded braid groups wP_n (McCool groups)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{IA}(n) & \longrightarrow & \mathrm{Aut}(F_n) & \longrightarrow & \mathrm{GL}_n(\mathbb{Z}) & \longrightarrow & 0\,. \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & wP_n & \longrightarrow & wB_n & \longrightarrow & S_n & \longrightarrow & 0 \\
wP_n^+ \nearrow & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & P_n & \longrightarrow & B_n & \longrightarrow & S_n & \longrightarrow & 0
\end{array}
$$

- $wB_n = \{\beta \in \mathrm{Aut}(F_n) \mid \beta(x_i) = w x_{\tau(i)} w^{-1}\}$ and $wP_n = wB_n \cap IA_n$.
- The pure welded braid group wP_n has presentation [McCool 86]

$$
\left\langle x_{ij}, (1 \le i \ne j \le n) \;\middle|\; \begin{array}{l} x_{ij}x_{ik}x_{jk} = x_{jk}x_{ik}x_{ij}, \\ [x_{ij}, x_{kl}] = 1, \\ [x_{ij}, x_{kj}] = 1, \ \text{for } i, j, k, l \text{ distinct} \end{array} \right\rangle .
$$

- wP_n^+ is the subgroup of wP_n generated by $\{x_{ij} \mid 1 \le i < j \le n\}$.

McCool groups

- wP_n and wP_n^+ are 1-formal.[Berceanu-Papadima 09]
- $H^*(wP_n, \mathbb{C})$ was computed by Jensen, McCammond, and Meier (06), proving a conjecture of Brownstein and Lee (93).
- D. Cohen (09) computed the first resonance variety of the group wP_n:

$$\mathcal{R}_1(wP_n, \mathbb{C}) = \bigcup_{1 \le i < j \le n} C_{ij} \cup \bigcup_{1 \le i < j < k \le n} C_{ijk},$$

  where $C_{ij} = \mathbb{C}^2$ and $C_{ijk} = \mathbb{C}^3$.
- D. Cohen and Schenck (15) showed that the Chen ranks of wP_n are given by the 'Chen ranks formula'

$$\theta_k(wP_n) = (k-1)\binom{n}{2} + (k^2-1)\binom{n}{3}$$

  for $k \gg 1$.

Upper upper McCool groups

F. Cohen, Pakhianathan, Vershinin, and Wu (07) computed the following:

- The cohomology algebra $H^*(wP_n^+; \mathbb{C})$ and the Betti numbers $b_k(wP_n^+) = b_k(P_n)$.
- $wP_n^+ \cong F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_1$ satisfies the assumptions of the theorem of Falk–Randell.
- The graded Lie algebra $\mathrm{gr}(wP_n^+; \mathbb{C})$ and the LCS ranks $\phi_k(wP_n^+) = \phi_k(P_n) = \sum_{s=1}^{n-1} \phi_k(F_s)$.

Both wP_n^+ and P_n are subgroups of wP_n.
Question: are wP_n^+ and P_n isomorphic for $n \geq 4$?

For $n = 4$, an incomplete argument was given by Bardakov and Mikhailov (08) using single-variable Alexander polynomials.

Theorem (Suciu–W. 15)

The Chen ranks θ_k of wP_n^+ are given by $\theta_1 = \binom{n}{2}, \theta_2 = \binom{n}{3}, \theta_3 = 2\binom{n+1}{4}$,

$$\theta_k = \binom{n+k-2}{k+1} + \theta_{k-1} = \sum_{i=3}^{k} \binom{n+i-2}{i+1} + \binom{n+1}{4}, \ k \geq 4.$$

Corollary

The pure braid group P_n, the upper McCool groups wP_n^+, and the product group $\Pi_n := \prod_{i=1}^{n-1} F_i$ are not isomorphic for $n \geq 4$.

Proof: $\theta_4(P\Sigma_n^+) = 2\binom{n+1}{4} + \binom{n+2}{5}, \theta_4(P_n) = 3\binom{n+1}{4}, \theta_4(\Pi_n) = 3\binom{n+2}{5}$.

Theorem (Suciu–W. 15)

The first resonance variety of the upper McCool groups wP_n^+ is

$$\mathcal{R}_1(wP_n^+, \mathbb{C}) = \bigcup_{n \geq i > j \geq 2} L_{i,j},$$

where $L_{i,j} = \mathbb{C}^j$ the linear subspace defined by the equations

$$\begin{cases} x_{i,l} + x_{j,l} = 0 & \text{for } 1 \leq l \leq j-1, \\ x_{i,l} = 0 & \text{for } j+1 \leq l \leq i-1, \\ x_{s,t} = 0 & \text{for } s \neq i,\ s \neq j, \text{ and } 1 \leq t < s. \end{cases}$$

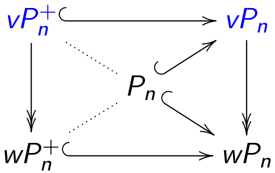
In particular, L_{ij} is not 0-isotropic if $j \geq 3$.

Corollary

- *There is no epimorphism from wP_n to wP_n^+ for $n \geq 4$. In particular, the inclusion $\iota\colon wP_n^+ \to wP_n$ admits no splitting for $n \geq 4$.*
- *The Chen ranks formula does not hold for wP_n^+ for $n \geq 4$, i.e., $\theta_k(wP_n^+) \neq \sum_{j=2}^{n-1}(n-j)\theta_k(F_j)$.*

Pure virtual braid groups

- The presentations of vP_n and vP_n^+ were given by Bardakov (04).



- The cohomology algebras $H^*(vP_n; \mathbb{C})$ and $H^*(vP_n^+; \mathbb{C})$ were computed by Bartholdi, Enriquez, Etingof, and Rains 06, and Lee 13.

Theorem (Suciu, W. 16)

The pure virtual braid groups vP_n and vP_n^+ are 1-formal if and only if $n \leq 3$.

Proposition (Suciu, W. 16)

If both G_1 and G_2 satisfy the Chen ranks formula, then $G_1 \times G_2$ also satisfies the Chen ranks formula, but $G_1 * G_2$ may not.

- Examples of free products do not satisfy the Chen ranks formula include free product $\mathbb{Z} * \mathbb{Z}^n$ and the groups $vP_3 = \mathbb{Z} * \overline{P}_4$.
- The groups vP_4^+ and vP_5^+ do not satisfy the Chen ranks formula.

The pure braid groups on Riemann surfaces

- $P_{g,n} = \pi_1(\mathrm{Conf}(\Sigma_g, n))$, where Σ_g is a compact Riemann surface of genus g ($g \geq 1$).
- Dimca–Papadima–Suciu (09) computed the (first) resonance variety of $P_{1,n}$, which is not linear. Consequently, $P_{1,n}$ is not 1-formal for $n \geq 3$.

The Chen ranks formula is not satisfied. However, $P_{1,n}$ is "filtered-formal". A generalized Chen ranks formula is satisfied by replacing the resonance varieties by the tangent cone of the "characteristic varieties".

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Thank You! Happy Birthday Mike Falk!