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# Linial arrangements and Eulerian polynomials

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# Contents

# 1. Eulerian Polynomials.

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$$1 + x + x^2 + \cdots + x^k + \cdots = \frac{1}{1-x}$$

$$\Big\downarrow \; x\frac{d}{dx}$$

$$0 + 1\cdot x + 2\cdot x^2 + \cdots + k\cdot x^k + \cdots = \frac{x}{(1-x)^2}$$

$$\Big\downarrow \; x\frac{d}{dx}$$

$$0 + 1^2\cdot x + 2^2\cdot x^2 + \cdots + k^2\cdot x^k + \cdots = \frac{x+x^2}{(1-x)^3}$$

$$\Big\downarrow \; x\frac{d}{dx}$$

$$0 + 1^3\cdot x + 2^3\cdot x^2 + \cdots + k^3\cdot x^k + \cdots = \frac{x+4x^2+x^3}{(1-x)^4}$$

<u>Def.</u> $$\sum_{k=1}^{\infty} k^{\ell}\cdot x^k = \left(x\frac{d}{dx}\right)^{\ell}\cdot\frac{1}{1-x} =: \frac{A_\ell(x)}{(1-x)^{\ell+1}}$$

$\leftarrow$ "Eulerian polynomial"

# 1. Eulerian Polynomials.

$$F_\ell(x) := \sum_{k=1}^{\infty} k^\ell \cdot x^k = \left(x\frac{d}{dx}\right)^\ell \frac{1}{1-x} = \frac{A_\ell(x)}{(1-x)^{\ell+1}}$$

First some $A_\ell(x)$:

$$A_1(x) = x$$

$$A_2(x) = x + x^2$$

$$A_3(x) = x + 4x^2 + x^3$$

$$A_4(x) = x + 11x^2 + 11x^3 + x^4$$

$$A_5(x) = x + 26x^2 + 66x^3 + 26x^4 + x^5$$

$$A_6(x) = x + 57x^2 + 302x^3 + 302x^4 + 57x^5 + x^6$$

From Euler's book (1755)

ubi quilibet coefficiens 16800 oritur, i
periorum 1560 + 1800 per exponente
eſt 5, multiplicetur.

173. Reſtituamus autem loco $q$ v

$$\alpha = \frac{1}{1(p-1)}$$

$$\beta = \frac{p+1}{1.2(p-1)^2}$$

$$\gamma = \frac{pp+4p+1}{1.2.3(p-1)^3}$$

$$\delta = \frac{p^3+11p^2+11p+1}{1.2.3.4(p-1)^4}$$

$$\varepsilon = \frac{p^4+26p^3+66p^2+26p+1}{1.2.3.4.5(p-1)^5}$$

# 1. Eulerian Polynomials.

$$F_\ell(x) := \sum_{k=1}^{\infty} k^\ell \cdot x^k = \left(x\frac{d}{dx}\right)^\ell \frac{1}{1-x} = \frac{A_\ell(x)}{(1-x)^{\ell+1}}$$

Euler's motivation: Special values of $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

One of the properties that Euler used is:

$$F_\ell(x) = 1^\ell \cdot x + 2^\ell \cdot x^2 + 3^\ell \cdot x^3 + 4^\ell \cdot x^4 + \cdots$$

$$F_\ell(x^2) = x^2 + 2^\ell \cdot x^4 + \cdots$$

$$\therefore F_\ell(x) - 2^{\ell+1} \cdot F_\ell(x^2) = 1^\ell \cdot x - 2^\ell \cdot x^2 + 3^\ell \cdot x^3 - 4^\ell \cdot x^4 + \cdots$$

$$= -F_\ell(-x).$$

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$$F_\ell(x) := \sum_{k=1}^{\infty} k^\ell \cdot x^k = \left(x\frac{d}{dx}\right)^\ell \frac{1}{1-x} = \frac{A_\ell(x)}{(1-x)^{\ell+1}}$$

$$F_\ell(x) - 2^{\ell+1} \cdot F_\ell(x^2) = -F_\ell(-x).$$

Using these formulae, Euler obtained fundamental results on

$$F_\ell(1) = 1^\ell + 2^\ell + 3^\ell + \cdots = \zeta(-\ell)$$

e.g.

- $\zeta(-1) = -\frac{1}{12}$, $\zeta(-2) = 0$, $\zeta(-3) = \frac{1}{120}$, $\cdots$
- $\zeta(s) = 0$ for $s$: negative even number
- functional eq. $\zeta(s) \longleftrightarrow \zeta(1-s)$ for $s \in \mathbb{Z}$.

# 2. Linial arrangements

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Let $\Phi$ be a root system of rank $\ell$ (i.e. $\Phi \subseteq \mathbb{R}^\ell$ finite set, $\cdots$)

Fix a postive system $\Phi^+ \subset \Phi$.

For $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$, define

$$H_{\alpha,k} := \{ x \in \mathbb{R}^\ell \mid (\alpha, x) = k \}.$$

<u>Def.</u> ($n$-th Linial arr. of $\Phi$)

$$\mathcal{L}_\Phi^n := \{ H_{\alpha,k} \mid \alpha \in \Phi^+, k \in \mathbb{Z}, 1 \leq k \leq n \}.$$

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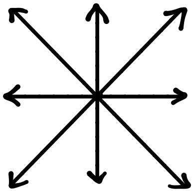
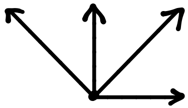
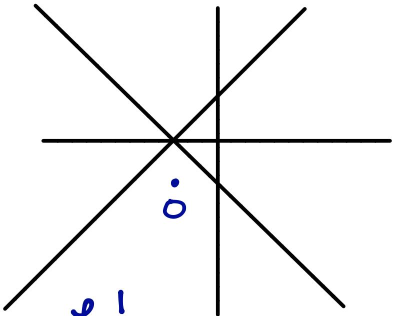
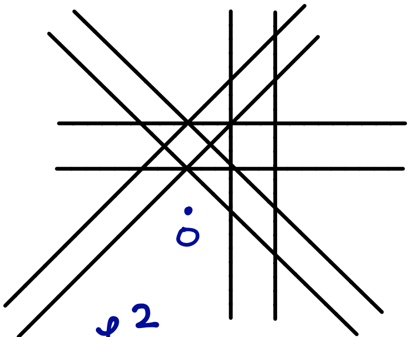
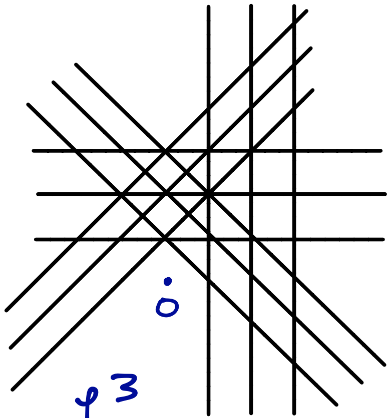
<u>Ex</u> $\Phi = B_2$ :  $\supset$ $\Phi^+$ : 




$\mathcal{L}^1_{B_2}$




$\mathcal{L}^2_{B_2}$




$\mathcal{L}^3_{B_2}$

# 2. Linial arrangements

Recall the characteristic polynomial of $\mathcal{A}=\{H_1,\cdots,H_n\}$ in $V$.

$L(\mathcal{A}):=\{\cap S \mid S\subset\mathcal{A}\}$ intersection poset.

$\mu: L(\mathcal{A})\longrightarrow\mathbb{Z}$ : the Möbius function defined by

$$\begin{cases}\mu(V)=1\\ \mu(X)=-\sum_{X\subsetneq Y\subseteq V}\mu(Y), & \text{if } Y\in L(\mathcal{A})\setminus\{V\}.\end{cases}$$

<u>Def.</u> (characteristic poly.)

$$\chi(\mathcal{A},t):=\sum_{X\in L(\mathcal{A})}\mu(X)\cdot t^{\dim X}.$$

# 2. Linial arrangements

<u>Conjecture</u> ("Riemann hypothesis" for $\mathcal{L}^n_\Phi$ by Postnikov - Stanley)

The root $z \in \mathbb{C}$ of the equation

$$\chi(\mathcal{L}^n_\Phi, t) = 0$$

satisfies $\mathrm{Re}\, z = \frac{nh}{2}$, where $h$ is the Coxeter # of $\Phi$.

<u>Partial results</u> "RH" is true for

- $\Phi = A_\ell$ (Postnikov - Stanley 1996)
- $\Phi = ABCD$ (Athanasiadis 1999)
- $\Phi = E_{6,7,8}$, $F_4$ with $n \gg 0$ (Y. in preparation)

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Close looking at $\Phi = A_\ell$.

$\chi(\mathcal{L}^n_{A_\ell}, t)$ can be expressed by using

- the Eulerian poly. $A_\ell(x)$, and
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**Thm** (Y. 2015)

the Eulerian polynomial.

$$\chi(\mathcal{L}^n_{A_\ell}, t) = A_\ell(S^{n+1}) \cdot \binom{t+\ell}{\ell},$$

where $\displaystyle \binom{t+\ell}{\ell} = \frac{(t+1)(t+2)\cdots(t+\ell)}{\ell!}$.

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where $\binom{t+\ell}{\ell} = \dfrac{(t+1)(t+2)\cdots(t+\ell)}{\ell!}$. If $A_\ell(x) = \sum_{i=1}^{\ell} a_i x^i$, then

$$\chi(\mathcal{L}^n_{A_\ell}, t) = \sum_{i=1}^{\ell} a_i S^{(n+1)i} \binom{t+\ell}{\ell} = \sum_{i=1}^{\ell} a_i \binom{t-(n+1)i+\ell}{\ell}.$$

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$$\chi(\mathcal{L}^n_{A_\ell}, t) = A_\ell(S^{n+1}) \cdot \binom{t+\ell}{\ell}.$$

Another formula:

<u>Thm</u> (Postnikov-Stanley, Athanasiadis)

$$\chi(\mathcal{L}^n_{A_\ell}, t) = \left(\frac{1+S+S^2+\cdots+S^n}{n+1}\right)^{\ell+1} \cdot t^\ell$$

<u>Q</u>. Why does there exist different expressions?

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<u>Cor.</u> (Y. 2015) $\ell > 0$, $m \geq 1$

The Eulerian polynomial $A_\ell(x)$ satisfies

$$A_\ell(x^m) \equiv \left(\frac{1+x+\cdots+x^{m-1}}{m}\right)^{\ell+1} A_\ell(x) \mod (1-x)^{\ell+1}.$$

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We work around the above congruence.

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Thm. $\ell \geq 1$, $m \geq 2$. Then

$$A_\ell(x^m) \equiv \left(\frac{1+x+\cdots+x^{m-1}}{m}\right)^{\ell+1} A_\ell(x) \begin{cases} \mod (1-x)^{\ell+1}, & \ell : \text{odd} \\ \mod (1-x)^{\ell+2}, & \ell : \text{even}. \end{cases}$$

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Rem. (New) proof does not use $\chi(\mathcal{L}^n_{A_\ell}, t)$, but uses "Euler's way."

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Recall, $F_\ell(x) = \sum_{k=1}^{\infty} k^\ell \cdot x^k = \frac{A_\ell(x)}{(1-x)^{\ell+1}}$ satisfies

$$2^{\ell+1} \cdot F_\ell(x^2) - F_\ell(x) = F_\ell(-x).$$

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$$2^{\ell+1} \cdot F_\ell(x^2) - F_\ell(x) = F_\ell(-x).$$

Just rewrite using $A_\ell(x)$:

$$A_\ell(x^2) - \left(\frac{1+x}{2}\right)^{\ell+1} \cdot A_\ell(x) = \left(\frac{1-x}{2}\right)^{\ell+1} \cdot A_\ell(-x) \equiv 0 \mod (1-x)^{\ell+1}.$$

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Rem This theorem generalizes Euler's computation.

Moreover, the extra $(1-x)$ for $m=2$ and $\ell$: even

is related to $\zeta(s)=0$ for $s$: negative even.

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Moreover, the extra $(1-x)$ for $m=2$ and $\ell$: even
is related to $\zeta(s)=0$ for $s$: negative even.

**Conj.** The extra $(1-x)$ for $m>2$ and $\ell$: even
may be related to zeros of certain Dirichlet L-fcn.

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Main result 2

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Thm. Let $\ell \geq 1$, $f(x) \in \mathbb{C}[x]$ a monic of $\deg f = \ell$.
Then TFAE.

(i)

(ii)

(iii)

(iv)

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(i) $f(x) = A_\ell(x)$.

(ii) $\forall m \in \{2,3,4,\dots\}$, $f(x^m) \equiv \left(\dfrac{1+x+\cdots+x^{m-1}}{m}\right)^{\ell+1} \cdot f(x) \bmod (1-x)^{\ell+1}$.

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Rem. "(iv) ⇒ (i)" indicates "Euler's choice was canonical" (?)

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# Summary

- Two expressions (P-S-A, Y.) of $\chi(\mathcal{L}^n_{A_\ell}, t)$ lead to congruences of the Eulerian polynomial $A_\ell(x)$.
- Special cases of the congruences implicitly appeared in Euler's computations for $\zeta(s)$.
- The congruences characterizes the Eulerian polynomial.

References:


- Y. Worpitzky partition ⋯ (arXiv:1501.04955) to appear.
- Y. Zeros of $\chi(\mathcal{L}^n_{\Phi}, t)$ ⋯, in preparation.
- Iijima-Sasaki-Takahashi-Y. Eulerian poly. ⋯, in preparation.